Discussion Notes for Calculus of Several Variables (MATH 32A)

UCLA
Fall 2016

Created by
Howard Heaton

Purpose: This document is a compilation of notes generated for discussion in MATH 32A, largely centered around the presented homework sets with reference credit due to Rogawski's and Larson's calculus texts.

Contents

Section 1: Review Material

1.1 – Notation:

Here we make note of some common symbols used in mathematics:

* The "{" and "}" are used when describing sets. For instance if we have the variable a, we say that $\{a\}$ is the set containing a.
* The "$\in$" symbol is used to mean "in". So, if S is a set and a is contained in S, e.g., $S = \{a\}$, then we write $a \in S$ and say "a is in S."
* The ":" and "|" characters are often used to say "such that." For example, let $S = \{\vec{r} \in \mathbb{R}^2 : \vec{r} \neq \vec{0}\}$. Then, using the above, we say "S is the set of all points in the plane such that $\vec{r}$ is nonzero. The underlined parts come from the $\{\}$, $\vec{r} \in \mathbb{R}^2$, :, and $\vec{r} \neq \vec{0}$, respectively.
* You may sometimes see "$\wedge$" used for "and" and you may see "$\vee$" used for "or".

1.2 – Examples with Limits, Derivatives, and Integrals:

Example: Let $f(x) = e^{x^2}$. Compute $f'(x)$.

Solution:

Let $u = x^2$ so that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$. Using the chain rule, we get

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}u}\left[e^u\right]\frac{\mathrm{d}u}{\mathrm{d}x} = e^u \cdot 2x = \boxed{2xe^{x^2}.}$$

□

Example: Let $f(x) = 12^{e^{x^2}}$. Compute $f'(x)$.

Solution:

Here we have repeated substitutions.

$$f(x) = 12^{e^{x^2}} = \left(e^{\log(12)}\right)^{e^{x^2}} = e^{\log(12)\cdot e^{x^2}}.$$

Let $u = \log(12) \cdot e^{x^2}$ so that $f(x) = e^u$. Differentiating u is not easy, so further substitute $w = x^2$.

This gives $u = \log(12)e^w$. Then, using the chain rule,

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}x} \\ &= \frac{\mathrm{d}}{\mathrm{d}u}\left[e^u\right] \cdot \frac{\mathrm{d}}{\mathrm{d}w}\left[\log(12)e^w\right] \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] \\ &= e^u \cdot \log(12)e^w \cdot 2x \\ &= \boxed{12^{e^{x^2}} \cdot \log(12)e^{x^2} \cdot 2x.} \end{aligned}$$

□

Example: Compute $\lim\limits_{x\to 3^+} \frac{4(x-3)^{-1}}{e^{1/(x-3)}}$.

Solution:

We have something of the form ∞/∞. So, we use L'Hópital's rule to get

$$4 \cdot \lim_{x\to 3^+} \frac{(x-3)^{-1}}{e^{(x-3)^{-1}}} = 4 \cdot \lim_{x\to 3^+} \frac{-(x-3)^{-2}}{e^{(x-3)^{-1}} \cdot -(x-3)^{-2}} = 4 \cdot \lim_{x\to 3^+} e^{-(x-3)^{-1}}.$$

If $u = 1/(x-3)$. Then the limit as $x \to 3+$ is the same as the limit as $u \to +\infty$, which gives

$$\lim_{x\to 3-} \frac{4(x-3)^{-1}}{e^{1/(3-x)}} = 4 \cdot \lim_{u\to+\infty} e^{-u} = 4 \cdot 0 = 0.$$

□

Remark: Sometimes we want to take derivatives or integrals of things that have several variables. If something is a variable and not a function of the variable we are interested in, we often treat it as a constant. We illustrate this with the following examples.

Example: Let $f(x) = x^2 + 5y + z + a$. Compute $f'(x)$.

Solution:

Unless we are told that y or z or a are functions of x, we often assume everything other than x as a constant. In this case, doing so gives $f'(x) = 2 \cdot x + 5 \cdot 0 + 0 + 0 = \boxed{2x.}$ □

Example: Let $f(x) = x^2 + 5y + z + a$ with $z = 3x^2$ and $y = e^x$. Compute $f'(x)$.

Solution:

Here we are told that y and z are functions of x. This means we have to use the chain rule to get

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + 5y + z + a\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] + 5\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} + \overbrace{\frac{\mathrm{d}a}{\mathrm{d}x}}^{0} = \boxed{2x + 5 \cdot e^x + 6x.}$$

$\square$

Example: Let $f(y) = x^2 + y^2$. Compute $\int_0^1 f(y)\ \mathrm{d}y$.

Solution:

Since we are not told that x is a function of y, x is a constant as far as we are concerned in this problem. So,

$$\int_0^1 f(y)\ \mathrm{d}y = \int_0^1 x^2 + y^2\ \mathrm{d}y = \left[x^2 y + \frac{y^3}{3}\right]_0^1 = \left(x^2 \cdot 1 + \frac{1}{3}\right) - (x^2 \cdot 0 + 0/3) = \boxed{x^2 + \frac{1}{3}.}$$

$\square$

Section 2: Vectors

2.1 – Basic Properties and Operations:

Objective: Understand what vectors are along their basic properties and uses.

Remark: Vectors are essentially arrows that point some distance in a given direction.

Definition: The components of a a vector $\vec{v} = \overrightarrow{PQ}$ where $P = (a_1, b_1)$ and $Q = (a_2, b_2)$ are given by $v = \langle v_1, v_2 \rangle$ where $v_1 = a_2 - a_1$ and $v_2 = b_2 - b_1$. △

Example 649/9: Find the components of $\vec{v} = \overrightarrow{PQ}$ where $P = (3, 2)$ and $Q = (2, 7)$.

Solution:

We start with the final coordinates (corresponding to Q) and subtract the initial ones (corresponding to P). This gives $v_1 = a_2 - a_1 = 2 - 3 = -1$ and $v_2 = b_2 - b_1 = 7 - 2 = 5$. Thus, $\boxed{\vec{v} = \langle -1, 5 \rangle.}$ □

Example 649/12: Find the components of $\vec{v} = \overrightarrow{PQ}$ where $P = (0, 2)$ and $Q = (5, 0)$.

Solution:

We start with the final coordinates (corresponding to Q) and subtract the initial ones (corresponding to P). This gives $v_1 = a_2 - a_1 = 5 - 0 = 5$ and $v_2 = b_2 - b_1 = 0 - 2 = -2$. Thus, $\boxed{\vec{v} = \langle 5, -2 \rangle.}$ □

Remark: The labels of our points are arbitrary; that is, they can be exchanged for any letter, which we now illustrate.

Example: Find the components of $\vec{v} = \overrightarrow{ST}$ where $S = (10, 7)$ and $T = (2, 4)$.

Solution:

We have $v_1 = 2 - 10 = -8$ for our first component and $v_2 = 4 - 7 = -3$ for our second.

Thus, $\boxed{\vec{v} = \langle -8, -3 \rangle.}$ □

Remark: Two vectors are equivalent if and only if they have the same components. This means that even if the start and end at different points they could be the same.

Example: Find the components of $\vec{v} = \overrightarrow{ST}$ where $S = (10, 7)$ and $T = (2, 4)$ and of $\vec{w} = \overrightarrow{RP}$ where $R = (6, 10)$ and $P = (-2, 7)$.

Solution:

From above, $\vec{v} = \langle -8, -3 \rangle$. We write $w = \langle w_1, w_2 \rangle$. Then we have $w_1 = -2 - 6 = -8$ for our first component and $w_2 = 7 - 10 = -3$ for our second. This gives $\vec{w} = \langle -8, -3 \rangle = \vec{v}$. □

Definition: The **magnitude** or **norm** of a vector $\vec{v} = \langle v_1, v_2 \rangle$ is given by $\sqrt{v_1^2 + v_2^2}$ and we denote this by $\|\vec{v}\|$. That is, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$. (Think of this in relation to the Pythagorean Theorem.) △

Example: If $\vec{v} = \langle 4, 3 \rangle$, then we have $\|v\| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

Definition: If $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, then

a) $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$

b) $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2 \rangle$

c) $\lambda\vec{v} = \langle \lambda v_1, \lambda v_2 \rangle$

d) $\vec{v} + \vec{0} = \vec{v}$

△

Definition: Further properties of vectors $\vec{v}, \vec{w}, \vec{u}$ include

a) Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

b) Associativity: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

c) Distributivity: $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$

△

Definition: A vector $\vec{v}$ is a **unit vector** if $\|\vec{v}\| = 1$. △

Example: Consider $\vec{v} = \langle \cos\theta, \sin\theta \rangle$ for some θ. Then $\|\vec{v}\| = \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1^2} = 1$.

Example: We can make any vector $\vec{v} = \langle v_1, v_2 \rangle$ into a unit vector by multiplying by $1/\|\vec{v}\|$. That is,

$$\frac{1}{\|v\|}\vec{v} = \langle \frac{v_1}{\|\vec{v}\|}, \frac{v_2}{\|\vec{v}\|} \rangle$$

is a unit vector. For instance, suppose $\vec{v} = \langle 3, 4 \rangle$. Then $\|v\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. Then

$$\left\| \frac{v}{\|v\|} \right\| = \left\| \langle \frac{3}{5}, \frac{4}{5} \rangle \right\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1.$$

Remark: In general, for each scalar λ, $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$ where $|\lambda|$ denotes the absolute value of λ.

Remark: If we have a vector $\vec{v}$ in the direction of an angle θ, then

$$\frac{\vec{v}}{\|\vec{v}\|} = \langle \cos\theta, \sin\theta \rangle .$$

Example 651/65: A plane is flying due east at 200 km/h encounters a 40 km/h wind blowing northeast. The angle between the velocity vector of the wind and that of the plane is $\pi/4$. Determine the resultant speed of the plane.

Solution:

First draw a picture. From this, we see the resultant velocity of the plane is $\vec{v} = \vec{p} + \vec{w}$ where $\vec{p}$ is the initial velocity of the plane and $\vec{w}$ is the velocity of the wind. From the problem statement we have that $\|\vec{p}\| = 200$ and is pointed to the right. So, $\vec{p} = \langle 200, 0\rangle$. Now we know $\|\vec{w}\| = 40$ and it points at the angle $\pi/4$. We also know that $\langle \cos(\pi/4), \sin(\pi/4)\rangle$ is a unit vector that points at an angle of $\pi/4$. Since $\vec{w}/\|\vec{w}\|$ is also a unit vector, this tells us that

$$\frac{\vec{w}}{\|\vec{w}\|} = \langle \cos(\pi/4), \sin(\pi/4)\rangle .$$

Then

$$\vec{w} = \|\vec{w}\| \langle \cos(\pi/4), \sin(\pi/4)\rangle = 40 \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = \langle 40/\sqrt{2}, 40/\sqrt{2}\rangle .$$

We can now compute $\vec{v}$:

$$\vec{v} = \vec{w} + \vec{p} = \langle 40/\sqrt{2}, 40/\sqrt{2}\rangle + 20\vec{0,0} = \langle 200 + 40/\sqrt{2}, 40/\sqrt{2}\rangle .$$

We want the speed of the plane, which is given by

$$\|v\| = \sqrt{v_1^2 + v_2^2} = \sqrt{(200 + 40/\sqrt{2})^2 + (40/\sqrt{2})^2} \approx \boxed{230.03 \text{ km/h.}}$$

□

Solution:

(This presentation that may be more useful for a student write-up.)

Problem type: add vectors and compute vector norm

(Draw Picture.)

What we know:

- plane velocity $= \vec{p}$, $\|p\| = 200$, $\vec{p}$ points east.
- wind velocity $= \vec{w}$, $\|w\| = 40$, $\vec{w}$ points at $\pi/4$ north of east.
- net velocity $= \vec{v} = \vec{p} + \vec{w}$.

What we want: $\|v\|$.

By inspection, $\vec{p} = \langle 200, 0 \rangle$. Also,

$$\frac{\vec{w}}{\|\vec{w}\|} = \langle \cos(\pi/4), \sin(\pi/4) \rangle \qquad \Rightarrow \qquad \vec{w} = 40 \left\langle 1/\sqrt{2}, 1/\sqrt{2} \right\rangle = \left\langle 40/\sqrt{2}, 40/\sqrt{2} \right\rangle.$$

Thus,

$$\vec{v} = \vec{w} + \vec{p} = \langle 40/\sqrt{2}, 40/\sqrt{2} \rangle + 20\vec{0}, 0 = \left\langle 200 + 40/\sqrt{2}, 40/\sqrt{2} \right\rangle.$$

and

$$\|v\| = \sqrt{v_1^2 + v_2^2} = \sqrt{(200 + 40/\sqrt{2})^2 + (40/\sqrt{2})^2} \approx \boxed{230.03 \text{ km/h.}}$$

$\square$

Example 651/64: Determine the magnitude of the forces $\vec{f}$ and $\vec{g}$ in Figure 29 (replacing F_2 with f and F_1 with g) in assuming there is not net force on the object.

Solution:

Problem type: add vectors and compute vector norm.

(Draw Picture.)

What we know:

- $\vec{f}$ points at $-45°$
- $\vec{g}$ points at $90°$
- $\vec{h}$ points at $180° + 30° = 210°$, $\|h\| = 20\ N$
- net force is zero, i.e., $\vec{f} + \vec{g} + \vec{h} = 0$.

What we want: $\|f\|$ and $\|g\|$.

Then

$$\frac{\vec{g}}{\|\vec{g}\|} = \langle \cos(90°), \sin(90°) \rangle \quad \Rightarrow \quad \vec{g} = \|\vec{g}\| \langle \cos(90°), \sin(90°) \rangle = \langle 0, \|\vec{g}\| \rangle ,$$

$$\frac{\vec{h}}{\|\vec{h}\|} = \langle \cos(210°), \sin(210°) \rangle = \langle -\sqrt{3}/2, -1/2 \rangle \quad \Rightarrow \quad \vec{h} = 20 \langle -\sqrt{3}/2, -1/2 \rangle = \langle -10\sqrt{3}, -10 \rangle ,$$

and

$$\frac{\vec{f}}{\|\vec{f}\|} = \langle \cos(-45°), \sin(-45°) \rangle = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle \quad \Rightarrow \quad \vec{f} = \langle \|\vec{f}\|/\sqrt{2}, -\|\vec{f}\|/\sqrt{2} \rangle .$$

Adding these gives

$$\begin{aligned} \vec{0} &= \vec{f} + \vec{g} + \vec{h} \\ &= \langle \|\vec{f}\|/\sqrt{2}, -\|\vec{f}\|/\sqrt{2} \rangle + \langle 0, \|\vec{g}\| \rangle + \langle -10\sqrt{3}, -10 \rangle \\ &= \langle \|\vec{f}\| - 10\sqrt{3}, -\|\vec{f}\|/\sqrt{2} + \|\vec{g}\| - 10 \rangle , \end{aligned}$$

which implies $0 = \|\vec{f}\| - 10\sqrt{3} \Rightarrow \boxed{\|\vec{f}\| = 10\sqrt{3}\ N.}$ and so

$$0 = -(10\sqrt{3})/\sqrt{2} + \|\vec{g}\| - 10 \quad \Rightarrow \quad \boxed{\|\vec{g}\| = 10(\sqrt{3/2} + 1)\ N.}$$

□

Example 651/63: Calculate the magnitude of the force on the cables in Figure 28.

Solution:

Problem type: add vectors and compute vector norm.

(Draw Picture.)

What we know:

- Cable 1: $\vec{u}$ points at $(90-65)+90 = 115°$
- Cable 2: $\vec{v}$ points at $25°$
- Weight: $\vec{w}$ points downward, i.e., at $-90°$ and $\|w\| = 50\ N$.
- net force is zero, i.e., $\vec{u}+\vec{v}+\vec{w} = 0$.

What we want: $\|\vec{u}\|$ and $\|\vec{v}\|$.

Then

$$\frac{\vec{w}}{\|\vec{w}\|} = \langle \cos(-90°), \sin(-90°)\rangle \quad \Rightarrow \quad \vec{w} = \langle \|\vec{w}\|\cos(-90°), \|\vec{w}\|\sin(-90°)\rangle = \langle 0, -50\rangle\,,$$

$$\frac{\vec{u}}{\|\vec{u}\|} = \langle \cos(115°), \sin(115°)\rangle \quad \Rightarrow \quad \vec{u} = \langle \|\vec{u}\|\cos(115°), \|\vec{u}\|\sin(115°)\rangle\,,$$

and

$$\frac{\vec{v}}{\|\vec{v}\|} = \langle \cos(25°), \sin(25°)\rangle \quad \Rightarrow \quad \vec{v} = \langle \|\vec{v}\|\cos(25°), \|\vec{v}\|\sin(25°)\rangle\,.$$

Since $\vec{0} = \vec{v}+\vec{w}+\vec{h}$,

$$0 = 0 + \|\vec{u}\|\cos(115°) + \|\vec{v}\|\cos(25°) \quad \text{and} \quad 0 = -50 + \|\vec{u}\|\sin(115°) + \|\vec{v}\|\sin(25°).$$

Then $\|u\| = -\|v\|\cos(25°)/\cos(115°)$. Substituting,

$$0 = -50 - \frac{\|\vec{v}\|\cos(25°)\sin(115°)}{\cos(115°)} + \|\vec{v}\|\sin(25°) = -50 + \|v\|\left(-\cos(25°)\tan(115°) + \sin(25°)\right).$$

This implies

$$\|v\| = \frac{50}{-\cos(25°)\tan(115°) + \sin(25°)} \approx \boxed{21.13\ N}$$

and

$$\|u\| = -\frac{\cos(25°)}{\cos(115°)} \cdot \frac{50}{-\cos(25°)\tan(115°) + \sin(25°)} \approx \boxed{45.32\ N.}$$

□

2.2 – Spheres and Cylinders and Parametric Equations:

Example: Find the equation of the cylinder centered at $(3, 4, 0)$, aligned in the $\hat{z}$ direction, and including the point $(6, 8, 4)$.

Solution:

Since our cylinder points along the z axis, we have an equation of the form

$$(x-3)^2 + (y-4)^2 = r^2$$

where r is the radius of the cylinder. Now, we know that $(6, 8, 4)$ is a part of the cylinder. Plugging this in, we find

$$r^2 = (6-3)^2 + (8-4)^2 = 3^2 + 4^2 = 9 + 16 = 25 \qquad \Rightarrow \qquad r = 5.$$

Thus, the equation of our cylinder is $\boxed{(x-3)^2 + (y-4)^2 = 5^2.}$ □

Example: Find the equation of a cylinder in $\mathbb{R}^3$ with central axis through the $(2, 2, 2)$ pointing along the y axis and containing the $(2, 4, 5)$.

Solution:

First note that our axis is **not** vertical. Since it is along the y axis, we know our equation will be of the form

$$(x-a)^2 + (z-c)^2 = r^2$$

for some a, c, r. Since the central axis passes through $(2, 2, 2)$, we have

$$(x-2)^2 + (z-2)^2 = r^2.$$

Because the cylinder includes the point $(2, 4, 5)$ we know $r^2 = (2-2)^2 + (5-2)^2 = 3^2$. Hence our equation is $\boxed{(x-2)^2 + (z-2)^2 = 3^2.}$ □

Example: Suppose we have the equation of a sphere $(x-7)^2+(y-5)^2+(z-c)^2 = r^2$ and that the sphere passes through $(7,5,6)$ and $(7,5,2)$. What are c and r?

Solution:

Using symmetry, from these two points $(7,5,6)$ and $(7,5,2)$ we know that the z coordinate of the sphere's center $(7,5,c)$ must be contained in the middle between 2 and 6. That is, $\boxed{c=4.}$ Then plugging in our point (7,5,6) into the resulting equation, we get

$$r^2 = (7-7)^2 + (5-5)^2 + (6-4)^2 = 0+0+2^2 \qquad \Rightarrow \qquad \boxed{r=2.}$$

□

Example: Suppose we have a circle of radius $\sqrt{5}$ centered at $(3,4,5)$. Write the equation of the circle with the left hand side as $x^2+y^2+z^2$.

Solution:

Using the equation of a circle, we have

$$(x-3)^2 + (y-4)^2 + (z-5)^2 = (\sqrt{5})^2 = 5.$$

Expanding this out, we get

$$(x^2-6x+9) + (y^2-8y+16) + (z^2-10z+25) = 5.$$

Rearranging, we get

$$x^2+y^2+z^2 = 6x+8y+10z+(5-9-16-25) = 6x+8y+10z-45.$$

□

Remark: If we are given the final form of the equation, we can then find the center and radius of the circle by the reverse process, i.e., by completing the square.

2.3 – Dot Products:

Example 667/64: Suppose we have $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 8, 2\rangle$. Find $\|\vec{u}_{\perp\vec{v}}\|$.

Solution:

We know $\vec{u} = \vec{u}_{\|\vec{v}} + \vec{u}_{\perp\vec{v}}$. And

$$\vec{u}_{\|\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{3\cdot 8 + 5\cdot 2}{8^2+2^2}\langle 8,2\rangle = \frac{34}{68}\langle 8,2\rangle = \frac{1}{2}\langle 8,2\rangle = \langle 4,1\rangle.$$

Then

$$\vec{u}_{\perp\vec{v}} = \vec{u} - \vec{u}_{\|\vec{v}} = \langle 3,5\rangle - \langle 4,1\rangle = \langle -1,4\rangle.$$

Thus, $\|\vec{u}_{\perp\vec{v}}\| = \sqrt{(-1)^2+4^2} = \boxed{\sqrt{17}.}$ □

2.4 – Cross Products:

Remark: Geometrically, if $\vec{v}$ and $\vec{u}$ are vectors, then the vector $\vec{v}\times\vec{w}$

i) is perpendicular to $\vec{v}$ and $\vec{u}$,

ii) is such that $\|\vec{v}\times\vec{u}\|$ = Area enclosed by parallelogram formed by $\vec{v}$ and $\vec{u} = \|\vec{v}\|\|\vec{u}\|\sin\theta$,

iii) and has direction following the right hand rule.

Example: Find a vector v that is perpendicular to $\langle 2,3,5\rangle$ and $\langle 4,0,1\rangle$ and verify that $\vec{v}$ is perpendicular to these two vectors using the dot product.

Solution:

Use the cross product to get

$$\vec{v} = \langle 2,3,5\rangle \times \langle 4,0,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 4 & 0 & 1 \end{vmatrix} = (3\cdot 1 - 5\cdot 0)\hat{i} - (2\cdot 1 - 4\cdot 5)\hat{j} + (2\cdot 0 - 3\cdot 4)\hat{k} = \boxed{\langle 3, 18, -12\rangle.}$$

We see that this choice of $\vec{v}$ is perpendicular to $\langle 2,3,5\rangle$ and $\langle 4,0,1\rangle$ since

$$\vec{v}\cdot\langle 2,3,5\rangle = \langle 3,18,-12\rangle\cdot\langle 2,3,5\rangle = 3\cdot 2 + 18\cdot 3 - 12\cdot 5 = 6 + 54 - 60 = 0$$

and

$$\vec{v} \cdot \langle 4, 0, 1\rangle = \langle 3, 18, -12\rangle \cdot \langle 4, 0, 1\rangle = 3 \cdot 4 + 18 \cdot 0 - 12 \cdot 1 = 12 + 0 - 12 = 0.$$

□

Example: Compute $\vec{v} \times (\vec{v} \times \vec{w})$ where $\vec{v} = \langle 1, -1, 0 \rangle$ and $\vec{w} = \langle 0, \sqrt{2}, 4 \rangle$.

Solution:

First we have

$$\begin{aligned} \vec{v} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & \sqrt{2} & 4 \end{vmatrix} \\ &= (-1 \cdot 4 - 0 \cdot \sqrt{2})\hat{i} - (1 \cdot 4 - 0 \cdot 0)\hat{j} + (1 \cdot \sqrt{2} - (-1) \cdot 0)\hat{k} \\ &= \langle -4, -4, \sqrt{2} \rangle \,. \end{aligned}$$

Then

$$\begin{aligned} \vec{v} \times (\vec{v} \times \vec{w}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ -4 & -4 & \sqrt{2} \end{vmatrix} \\ &= (-1 \cdot \sqrt{2} - 0 \cdot (-4))\hat{i} - (1 \cdot \sqrt{2} - 0 \cdot (-4))\hat{j} + (1 \cdot (-4) - (-1) \cdot (-4))\hat{k} \\ &= \boxed{\langle -\sqrt{2}, -\sqrt{2}, -8 \rangle \,.} \end{aligned}$$

□

Example 678/44: Use the cross product to find the area of the triangle with vertices $P = (1, 1, 5)$, $Q = (3, 4, 3)$, and $R = (1, 5, 7)$.

Solution:

If $\vec{v}$ and $\vec{w}$ span a parallelogram $\mathcal{P}$, then $\text{area}(\mathcal{P}) = \|\vec{v} \times \vec{w}\|$. So, if $\vec{v}$ and $\vec{w}$ span a triangle $\mathcal{T}$, then

$$\text{area}(\mathcal{T}) = \frac{\|\vec{v} \times \vec{w}\|}{2}.$$

We get $\vec{v}$ and $\vec{w}$ by

$$\vec{v} = \vec{PQ} = \langle 3 - 1, 4 - 1, 3 - 5 \rangle = \langle 2, 3, -2 \rangle \quad \text{and} \quad \vec{w} = \vec{PR} = \langle 1 - 1, 5 - 1, 7 - 5 \rangle = \langle 0, 4, 2 \rangle \,.$$

Then

$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -2 \\ 0 & 4 & 2 \end{vmatrix}$$
$$= (2\cdot 3-(-2)\cdot 4)\hat{\imath} - (2\cdot 2-(-2)\cdot 0)\hat{\jmath} + (2\cdot 4 - 3\cdot 0)\hat{k}$$
$$= 14\hat{\imath} - 4\hat{\jmath} + 8\hat{k}$$
$$= \langle 14, -4, 8\rangle .$$

Thus,

$$\text{area}(\mathcal{T}) = \frac{\|\vec{v}\times\vec{w}\|}{2} = \frac{1}{2}\|\langle 14,-4,8\rangle\| = \frac{1}{2}\sqrt{14^2+4^2+8^2} = \boxed{\frac{\sqrt{276}}{2}.}$$

□

2.5 – Planes:

Remark: The equation of a plane is given by $ax+by+cz=d$ for some $a,b,c,d\in\mathbb{R}$. Note also that the vector $\langle a,b,c\rangle$ is normal/perpendicular to the plane.

Example: What is the equation of the plane through $(1,2,3)$ perpendicular to $\langle -1,-3,-5\rangle$?

Solution:

We are given the normal vector to be $\langle -1,-3,-5\rangle$. So, the equation is $-x-3y-5z=d$. To find d, we plug in our given point into this equation to get

$$d = -(1)-3(2)-5(3) = -1-6-15 = -22.$$

Thus, our equation is $\boxed{-x-3y-5z=-22.}$ □

Example: What's the equation of the plane passing through $(0,0,1)$, $(0,-2,0)$ and $(\pi,1,-1)$?

Solution:

Step 1: Write two vectors in this plane:

$$\vec{v} = \langle 0-0, -2-0, 0-1\rangle = \langle 0,-2,-1\rangle \quad \text{and} \quad \vec{w} = \langle \pi-0, 1-0, -1-1\rangle = \langle \pi, 1, -2\rangle .$$

Step 2: Find a normal vector:

$$\vec{v} \times \vec{w} = = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -1 \\ \pi & 1 & -2 \end{vmatrix}$$
$$= \langle 5, -\pi, 2\pi \rangle .$$

Step 3: Use normal vector to say equation looks like $5x - \pi y + 2\pi z = d$.

Step 4: Find d by plugging in a point:

$$d = 5(0) - \pi(0) + 2\pi(1) = 2\pi.$$

Thus, our equation is $\boxed{-5x - \pi y + 2\pi z = 2\pi.}$ □

2.6 – Convexity:

Definition: A set S is called **convex** if whenever $u, v \in S$, we have that $(\lambda u + (1 - \lambda)v) \in S$ for any $\lambda \in [0, 1]$. (In other words, if u and v are in S, then $\lambda u + (1 - \lambda)v$ is in S for $0 \leq \lambda \leq 1$.) △

Remark: **How to show a set S is convex:**

i) Draw a picture (including u and v and $\lambda u + (1 - \lambda)v$ inside S).

ii) Let u and v be arbitrary points in S. Also let $\lambda \in (0, 1)$.

iii) Using the properties of u and v inherited from S, show that $(\lambda u + (1 - \lambda)v) \in S$.

Example: Define T to be the triangle in the plane with vertices $(0,0)$, $(a,0)$ and $(0,a)$ for some $a>0$. Show that T is convex.

Solution:

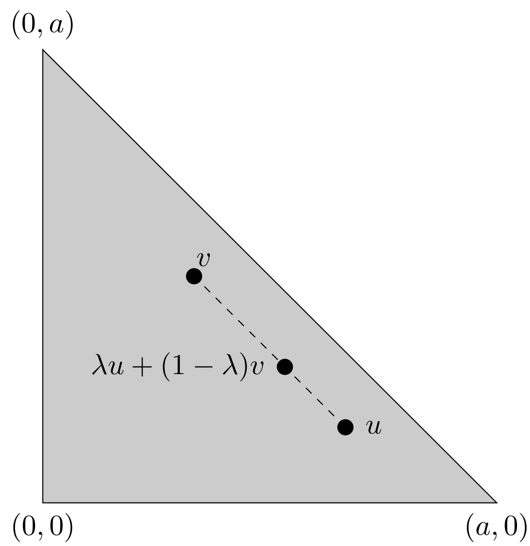

Figure 1: Triangle T in the plane with vertices $(0,0)$, $(a,0)$ and $(0,a)$

We first illustrate our problem above in Figure 1. Let $u=\langle u_1,u_2\rangle\in T$ and $v=\langle v_1,v_2\rangle\in T$ and $\lambda\in(0,1)$. We must show that $\lambda u+(1-\lambda)v\in T$. To do this, we explicitly identify T as

$$T=\{\vec{r}=\langle r_1,r_2\rangle\in\mathbb{R}^2 : 0\le r_1\le a \text{ and } 0\le r_2\le a-r_1\}.$$

Since u and v are in T, we know $0\le u_1,v_1\le a$, which implies

$$\lambda\cdot 0\le \lambda u_1\le \lambda a \quad\text{and}\quad (1-\lambda)\cdot 0\le(1-\lambda)v_1\le(1-\lambda)a.$$

Adding these equations together, we obtain

$$\lambda\cdot 0+(1-\lambda)\cdot 0\le\lambda u_1+(1-\lambda)v_1\le\lambda a+(1-\lambda)a,$$

which is equivalent to

$$0\le\lambda u_1+(1-\lambda)v_1\le a.$$

So, we have verified one of the two equations required to show that $\lambda\vec{u}+(1-\lambda)\vec{v}$ is in T. For the second equation, note that by our hypothesis that $\vec{u}, \vec{v} \in T$, we have

$$0 \leq u_2 \leq a - u_1 \quad \text{and} \quad 0 \leq v_2 \leq a - v_1,$$

which implies

$$\lambda \cdot 0 \leq \lambda u_2 \leq \lambda(a - u_1) \quad \text{and} \quad (1-\lambda) \cdot 0 \leq (1-\lambda)v_2 \leq (1-\lambda)(a - v_1).$$

Adding these equations together, we obtain

$$\lambda \cdot 0 + (1-\lambda) \cdot 0 \leq \lambda u_2 + (1-\lambda)v_2 \leq \lambda(a - u_1) + (1-\lambda)(a - v_1).$$

On the left side we have $\lambda \cdot 0 + (1-\lambda) \cdot 0 = 0$ and on the right side we have

$$\lambda(a - u_1) + (1-\lambda)(a - v_1). = (\lambda + 1 - \lambda)a - \lambda u_1 + (1-\lambda)(-v_1) = a - (\lambda u_1 + (1-\lambda)v_1)\,.$$

Thus, we obtain the second needed equation that

$$0 \leq \lambda u_2 + (1-\lambda)v_2 \leq a - (\lambda u_1 + (1-\lambda)v_1).$$

Hence $(\lambda\vec{u} + (1-\lambda)\vec{v}) \in T$. Since $\vec{u}$ and $\vec{v}$ were chosen arbitrarily, this holds for every pair of vectors in T and so T is convex. □

2.7 – Representation of Vectors:

Definition: A set of vectors $\hat{u}$, $\hat{v}$, and $\hat{w}$ is said to be **othonormal** provided that

$$\|\hat{u}\| = \|\hat{v}\| = \|\hat{w}\| = 1 \quad \text{and} \quad \hat{u}\cdot\hat{v} = \hat{u}\cdot\hat{w} = \hat{v}\cdot\hat{w} = 0.$$

△

Remark: The standard basis vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ for $\mathbb{R}^3$ are orthonormal. Also note that a set d of orthonormal vectors forms a basis for $\mathbb{R}^d$ (where we usually have either $d = 2$ or $d = 3$). This means that any vector $\vec{r} \in \mathbb{R}^d$ can be represented as a linear combination of a set of orthonormal vectors that form a basis for $\mathbb{R}^d$.

Remark: Suppose we have a vector $\vec{r} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and we want to represent $\vec{r}$ using another set of three orthonormal vectors $\hat{u}$, $\hat{v}$, and $\hat{w}$. We can do this by writing

$$\vec{r} = (\vec{r}\cdot\hat{u})\hat{u} + (\vec{r}\cdot\hat{v})\hat{v} + (\vec{r}\cdot\hat{w})\hat{w}.$$

Example: Suppose we have $\vec{r} = 3\hat{\imath} + 5\hat{\jmath}$. Express $\vec{r}$ in terms of the orthonormal vectors

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \hat{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution:

We know

$$\vec{r} = (\vec{r}\cdot\hat{u})\hat{u} + (\vec{r}\cdot\hat{v})\hat{v}.$$

Computing our dot products, we find

$$\vec{r}\cdot\hat{u} = \frac{3\cdot 1 + 5\cdot 1}{\sqrt{2}} = \frac{8}{\sqrt{2}} \quad \text{and} \quad \vec{r}\cdot\hat{v} = \frac{3\cdot 1 + 5\cdot(-1)}{\sqrt{2}} = \frac{-2}{\sqrt{2}}.$$

Thus,

$$\boxed{\vec{r} = \frac{8}{\sqrt{2}}\hat{u} - \frac{2}{\sqrt{2}}\hat{v}.}$$

To check that we have the right answer, observe that

$$\frac{8}{\sqrt{2}}\hat{u} - \frac{2}{\sqrt{2}}\hat{v} = \frac{8}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \vec{r}.$$

□

2.8 – Parameterization:

Example: Consider $\vec{r}(t)$ and $\vec{q}(s)$ in $\mathbb{R}^3$. Verify that these points intersect and identify this point of intersection using

$$\vec{r}(t) = \langle \pi t, 7 + 3\pi t \rangle \quad \text{and} \quad \vec{q}(t) = \langle 5 - 2s, 4 + 3s \rangle \,.$$

Solution:

Step 1: Set $\vec{r}(t) = \vec{q}(t)$. (Since $\vec{r}(t)$ and $\vec{q}(s)$ intersect, this must be true for some s and t.)
Step 2: Solve for t and/or s.
Step 3: Plug one of the found values back into the respective vector $\vec{r}(t)$ or $\vec{q}(t)$.

First set $\vec{r}(t) = \vec{q}(t)$ so that

$$\langle \pi t, 7 + 3\pi t \rangle = \langle 5 - 2s, 4 + 3s \rangle \,.$$

Equating the $\hat{\imath}$ component, we find

$$\pi t = 5 - 2s \qquad \Rightarrow \qquad t = \frac{5 - 2s}{\pi}.$$

Also, for the $\hat{\jmath}$ component we get

$$4 + 3s = 7 + 3\pi t = 7 + 3\pi \left(\frac{5 - 2s}{\pi} \right) = 7 + 3(5 - 2s) = 22 - 6s \qquad \Rightarrow \qquad 9s = 18 \qquad \Rightarrow \qquad s = 2.$$

So, the point of intersection is at

$$\vec{q}(2) = \langle 5 - 2 \cdot 2, 4 + 3 \cdot 2 \rangle = \boxed{\langle 1, 10 \rangle \,.}$$

□

Example: Let $\vec{q}(t) = \langle t^3 + 5, e^{2t}, \sin(t) \rangle$. Find the corresponding velocity vector (i.e., $\vec{q}\,'(t)$).

Solution:

To find the derivative, we simply differentiate each part respectively. This gives

$$\vec{q}\,'(t) = \langle 3t^2 + 0, 2e^{2t}, \cos(t) \rangle \,.$$

□

Example (Hard): Find the parametric equation $\vec{r}(t)$ of a helix of radius 3 oriented/centered along the z axis for which $\vec{r}(0) = \langle 3, 0, 1\rangle$ and $\vec{r}(t+\pi) = \vec{r}(t) + \pi\hat{k}$.

Solution:

We are looking for a solution of the form $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. The equation for $x(t)$ and $y(t)$ of a helix along the z axis is given by

$$x(t) = A\cos(\omega t + \delta) \quad \text{and} \quad y(t) = A\sin(\omega t + \delta)$$

for some A and δ. Since our radius is 3 we have $A = 3$. Since $3 = x(0) = 3\cos(\delta)$, we also know $\delta = 0$. Because $\vec{r}(t+\pi) = \vec{r}(t) + \pi\hat{k}$, we also know $\omega = 2$ so that our period is π instead of the normal 2π. Now we consider $z(t)$. We know $z(t+\pi) = z(t) + \pi t$. This tells us that $z(t) = \pi t + z(0) = \pi t + 1$. Combining everything together, we get

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \boxed{\langle 3\cos(2t), 3\sin(2t), \pi t + 1.\rangle}$$

□

REMARK: If we were to take a derivative of $\vec{r}(t)$ with respect to t, we would have

$$\vec{r}\,'(t) = \langle -6\sin(2t), 6\cos(2t), \pi\rangle.$$

Example: Suppose the position of a particle is given by $\vec{r}(t) = \langle a\cosh(kt), -a\sinh(kt), e^t\rangle$. Find the acceleration vector $\vec{a}(t)$ of the particle.

Solution:

First the velocity vector is given by

$$\vec{v}(t) = \vec{r}\,'(t) = \langle ak\sinh(kt), -ak\cosh(kt), e^t\rangle .$$

Acceleration is given by the second derivative. So,

$$\vec{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{r}\,'(t)\right] = \boxed{\langle ak^2\cosh(kt), -ak^2\sinh(kt), e^t\rangle .}$$

□

Example: For normal trigonometric functions we have $\cos^2\theta + \sin^2\theta = 1$. However, this is not the case for hyperbolic functions. Verify that instead we have $\cosh^2\theta - \sinh^2\theta = 1$.

Solution:

Use the definition of each term to get

$$\begin{aligned}
\cosh^2\theta - \sinh^2\theta &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{1}{4}\left[e^{2x} + e^x e^{-x} + e^{-x}e^x + e^{-2x}\right] - \frac{1}{4}\left[e^{2x} - e^x e^{-x} - e^{-x}e^x + e^{-2x}\right] \\
&= \frac{1}{4}\left[e^{2x} + 2 + e^{-2x}\right] - \frac{1}{4}\left[e^{2x} - 2 + e^{-2x}\right] \\
&= \frac{1}{4}\left[e^{2x} - e^{2x} + e^{-2x} - e^{-2x} + 2 - (-2)\right] \\
&= \frac{1}{4}\cdot 4 \\
&= 1.
\end{aligned}$$

□

Example: Find and parameterize the intersection of $z = 5x^2$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution:

By inspection, we know for an ellipse that

$$x(t) = a\cos(t) \quad \text{and} \quad y(t) = b\sin(t) \quad \text{for } t \in [0, 2\pi).$$

Then, plugging in our third point, we get $z = 5x^2 = 5a^2\cos^2(t)$. This implies our curve is given by

$$\langle a\cos(t), a\sin(t), 5a^2\cos^2(t)\rangle \quad \text{for } t \in [0, 2\pi).$$

$\square$

Example: Suppose a particle starts at initial position $\langle 3/4, 0\rangle$ and has velocity $\vec{v}(t) = \langle \sin(t)\cos(t), e^t\rangle$. Find the position $\vec{r}(t)$ for the particle for $t \geq 0$.

Solution:

Since the velocity is the derivative of position, we use the fundamental theorem of calculus to say

$$\begin{aligned}
\vec{r}(t) - \vec{r}(0) &= \int_0^t \vec{v}(t^*)\ \mathrm{d}t^* \\
&= \int_0^t \langle \sin(t^*)\cos(t^*), e^{t^*}\rangle\ \mathrm{d}t^* \\
&= \int_0^t \langle \frac{\sin(2t^*)}{2}, e^{t^*}\rangle\ \mathrm{d}t^* && \text{Substitute with trig identity} \\
&= \left[\langle -\frac{\cos(2t^*)}{4}, e^{t^*}\rangle\right]_0^t && \text{Integrate term by term} \\
&= \langle -\frac{\cos(2t)}{4}, e^t\rangle - \langle -\frac{\cos(0)}{4}, e^0\rangle && \text{Evaluate at limits} \\
&= \langle -\frac{\cos(2t)}{4}, e^t\rangle - \langle -\frac{1}{4}, 1\rangle && \text{Simply expression} \\
&= \langle \frac{1}{4} - \frac{\cos(2t)}{4}, e^t - 1\rangle\,. && \text{Add vectors together}
\end{aligned}$$

Thus, adding $\vec{r}(0)$ to each side we obtain

$$\boxed{\vec{r}(t) = \langle \frac{1}{4} - \frac{\cos(2t)}{4}, e^t - 1 \rangle + \langle \frac{3}{4}, 0 \rangle = \langle 1 - \frac{\cos(2t)}{4}, e^t - 1 \rangle\,.}$$

□

Example: Suppose we have $\vec{r}(t) = \langle 3t, 1, t^2 \rangle$ and $\vec{q}(t) = \langle t, 0, -2t \rangle$. If the position of a particle is given by $\vec{r}(t) \times \vec{q}(t)$, find an expression in terms of t for its speed.

Solution:
We must first compute its velocity, which we shall denote by $\vec{v}(t)$. Here[1]

$$\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{r}(t) \times \vec{q}(t)\right] = \vec{r}\,'(t) \times \vec{q}(t) + \vec{r}(t) \times \vec{q}\,'(t).$$

We will compute each of these cross products individually, and then add them together at the end to find $\vec{v}(t)$. Computing these, we find

$$\vec{r}\,'(t) \times \vec{q}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 2t \\ t & 0 & -2t \end{vmatrix} = 0\hat{i} - (-6t - 2t^2)\hat{j} + 0\hat{k} = \langle 0, 6t + 2t^2, 0 \rangle$$

and

$$\vec{r}(t) \times \vec{q}\,'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t & 1 & t^2 \\ 1 & 0 & -2 \end{vmatrix} = -2 \cdot 1\hat{i} - (-2 \cdot 3t - t^2 \cdot 1)\hat{j} - 1\hat{k} = \langle -2, 6t + t^2, -1 \rangle\,.$$

So,

$$\vec{v}(t) = \langle 0, 6t + 2t^2, 0 \rangle + \langle -2, 6t + t^2, -1 \rangle = \langle -2, 12t + 3t^2, -1 \rangle\,.$$

The speed of the particle is, thus, given by

$$\|\vec{v}(t)\| = \|\,\langle -2, 12t + 3t^2, -1 \rangle\,\| = \sqrt{2^2 + (12t + 3t^2)^2 + 1^2} = \boxed{\sqrt{(12t + 3t^2) + 3}.}$$

(Note we probably could simplify farther, but that is not the point of this problem. The goal of

[1]Note: As discussed in class, we could also go about this problem by first computing $\vec{r}(t) \times \vec{q}(t)$, and then taking the derivative. This approach is fine and, in this case, more simple. The purpose of this problem, however, is to recognize the tool of being able to take derivatives of cross products.

this problem is to be able to recognize the product rule for cross products and to find the quantity for speed given position.) ☐

Example: Suppose the position of two particles is given by $\vec{r}(t) = \langle \cosh^2(t), t^2 \rangle$ and $\vec{q}(t) = \langle \sinh^2(t), 1 \rangle$. Find the rate at which the distance between the particles changes.

Solution:

The distance $d(t)$ between the particles is given by

$$\begin{aligned}
d(t) &= \|\vec{r}(t) - \vec{q}(t)\| \\
&= \| \langle \cosh^2(t), t^2 \rangle - \langle \sinh^2(t), 1 \rangle \| && \text{Substitute for } \vec{r} \text{ and } \vec{q} \\
&= \| \langle \cosh^2(t) - \sinh^2(t), t^2 - 1 \rangle \| && \text{Combine vectors} \\
&= \| \langle 1, t^2 - 1 \rangle \| && \text{Use hyperbolic identity} \\
&= \sqrt{1^2 + (t^2 - 1)^2} && \text{Use definition of } \| \cdot \| \\
&= \sqrt{t^4 - t^2 + 2}. && \text{Simplify}
\end{aligned}$$

Thus, our answer is given by

$$d'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{t^4 - t^2 + 2}\right] = \frac{1}{2}\frac{4t^3 - 2t}{\sqrt{t^4 - t^2 + 2}} = \boxed{\frac{2t^3 - t}{\sqrt{t^4 - t^2 + 2}}}.$$

☐

Example: Suppose we have a particle with position given parametrically by $\vec{r}(t) = \langle \cosh(t), \sinh(t) \rangle$. Show that this particle's position and velocity are orthogonal only when $t = 0$.

Solution:

Our velocity $\vec{v}(t) = \vec{r}\,'(t)$ is given by $\vec{v}(t) = \langle \sinh(t), \cosh(t) \rangle$. So, taking the dot product we find

$$\begin{aligned}
\vec{r}(t) \cdot \vec{v}(t) &= \langle \cosh(t), \sinh(t) \rangle \cdot \langle \sinh(t), \cosh(t) \rangle \\
&= 2\sinh(t)\cosh(t) \\
&= 2\left[\frac{e^t - e^{-t}}{2}\right]\left[\frac{e^t + e^{-t}}{2}\right] \\
&= \frac{1}{2}\left[e^{2t} - e^{-2t}\right].
\end{aligned}$$

This is only zero when $e^{2t} = e^{-2t}$, which implies $2t = -2t$, which implies $t = 0$, and we are done. ☐

Example: Let $y = 5 - x$. Show that the shortest distance from the origin to this line is $5/\sqrt{2}$.

Solution:
First we verify that the distance from the origin to the line $y = -x + 5$ is given by $5/2$. To do this, we find a line that is perpendicular to $y = -x+5$ and contains the origin. This is satisfied by $y = x$. Indeed, here our slopes are perpendicular.[2] We can now find where these two lines intersected by setting their y values to be equal; that is,

$$x = y = -x + 5 \quad \Rightarrow \quad 2x = 5 \quad \Rightarrow \quad x = 5/2.$$

This gives us our x component, and the y component is also $y = x = 5/2$. So, we have found a point where on the line $y = 5 - x$ where the distance from the origin to this point is given by $\sqrt{(5/2)^2 + (5/2)^2} = \sqrt{2}(5/2) = 5/\sqrt{2}$.

Now pick any arbitrary point $\langle x_0, y_0 \rangle$ on the line $y = 5 - x$. Then the distance from the origin is $\| \langle x_0, y_0 \rangle \| = \sqrt{x_0^2 + y_0^2}$. We could equivalently write our vector $\langle x_0, y_0 \rangle$ as

$$\langle x_0, y_0 \rangle = \langle 5/2, 5/2 \rangle + \langle x_0 - 5/2, y_0 - 5/2 \rangle .$$

And, we know $\langle 5/2, 5/2 \rangle$ and $\langle x_0 - 5/2, y_0 - 5/2 \rangle$ are orthogonal. Why? (Well, remember that above we chose our line $y = x$ that was perpendicular to $y = 5 - x$). This means we can use the Pythagorean theorem. That is,

$$\| \langle x_0, y_0 \rangle \|^2 = \| \langle 5/2, 5/2 \rangle \|^2 + \| \langle x_0 - 5/2, y_0 - 5/2 \rangle \|^2 \geq \| \langle 5/2, 5/2 \rangle \|^2$$

where we note $\| \langle x_0 - 5/2, y_0 - 5/2 \rangle \|^2 \geq 0$. Taking square roots of each side, we find that $\| \langle x_0, y_0 \rangle \| \geq \| \langle 5/2, 5/2 \rangle \| = 5/\sqrt{2}$, as desired. This completes our solution. $\square$

[2]Remember that if m is the slope of a line, then a line with slop $-1/m$ is perpendicular.

2.9 – Arc Length:

Definition: Suppose we have a vector $\vec{r}(t)$. Then the arc length along the path $\vec{r}(t)$ from $t = a$ to $t = b$ is given by

$$\int_a^b \|\vec{r}\,'(t)\| \; \mathrm{d}t.$$

△

Example: Suppose we the position of a particle is denoted by $\vec{r}(t)$ and we note its velocity is

$$\vec{r}\,'(t) = \left\langle \sqrt{9t^2 + 6t}, 1 \right\rangle.$$

Find the arc $\mathcal{L}$ length along $\vec{r}(t)$ from $t = 0$ to $t = 1$.

Solution:
First we compute $\|\vec{r}\,'(t)\|$ to find

$$\|\vec{r}\,'(t)\| = \sqrt{(\sqrt{9t^2 + 6t})^2 + 1^2} = \sqrt{9t^2 + 6t + 1} = \sqrt{(3t+1)^2} = 3t + 1.$$

So, the arc length $\mathcal{L}$ is

$$\mathcal{L} = \int_0^1 \|\vec{r}\,'(t)\| \; \mathrm{d}t = \int_0^1 3t + 1 \; \mathrm{d}t = \left[\frac{3t^2}{2} + t \right]_{t=0}^1 = \boxed{\frac{5}{2}.}$$

□

Example 726/37: The Bernoulli spiral has the parameterization $\vec{r}(t) = \langle e^t \cos(4t), e^t \sin(4t)\rangle$. Evaluate

$$s(t) = \int_{-\infty}^{t} \|\vec{r}\,'(u)\| \; \mathrm{d}u.$$

Solution:

First we compute $\vec{r}\,'(t)$:

$$\vec{r}\,'(t) = \langle (\cos(4t) - 4\sin(4t)e^t, (\sin(4t) + 4\cos(4t)e^t \rangle \,.$$

So,

$$\begin{aligned}\|\vec{r}\,'(t)\|^2 &= \left[\cos^2(4t) - 8\sin(4t)\cos(4t) + 16\sin^2(4t)\right] e^{2t} \\ &+ \left[\sin^2(4t) + 8\sin(4t)\cos(4t) + 16\cos^2(4t)\right] e^{2t} \\ &= (1+16)e^{2t} \\ &= 17e^{2t}.\end{aligned}$$

Thus, $\|\vec{r}\,'(t)\| = \sqrt{17}e^t$, and so

$$s(t) = \int_{-\infty}^{t} \|\vec{r}\,'(u)\| \; \mathrm{d}u = \int_{-\infty}^{t} \sqrt{17}e^u \; \mathrm{d}u = \sqrt{17}\cdot [e^u]_{-\infty}^{t} = \sqrt{17}\left[e^t - e^{-\infty}\right] = \boxed{\sqrt{17}e^t.}$$

□

2.10 – Differentiating Vectors:

Remark: Recall that our standard basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are fixed. This is not the case for all basis vectors with which we may represent a vector. For instance, if we represent the position of a particle with curvilinear coordinates, we use a tangent vector and a normal vector. This causes a subtle change when differentiating. To see this, recall that a velocity vector points in the tangential direction (represented by $\hat{T}$). So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{v}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\|\vec{v}\| \; \hat{T}\right] = \frac{\mathrm{d}\|\vec{v}\|}{\mathrm{d}t}\hat{T} + \|\vec{v}\|\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}.$$

When we use the standard basis vectors, we didn't have to worry about this subtlety because our basis vector were constant in time (i.e., $\mathrm{d}\hat{\imath}/\mathrm{d}t = \mathrm{d}\hat{\jmath}/\mathrm{d}t = \mathrm{d}\hat{k}/\mathrm{d}t = 0.$)

2.11 – Polar Coordinates:

REMARK: At this point in the course, we are quite familiar we the standard basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ for $\mathbb{R}^3$. However, sometimes it is advantageous to use other coordinate systems. Here we will talk about polar coordinates for representing points in $\mathbb{R}^2$.

Definition: The **polar coordinate system** is described as follows. Each point $\vec{p}$ in the plane can be assigned polar coordinates (r, θ) where

$$r := \text{length of } p, \text{ i.e., } r = \|\vec{p}\|,$$
$$\theta = \text{angle counterclockwise from the origin of } \vec{p}.$$

△

REMARK: The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad \tan(\theta) = y/x, \quad r^2 = x^2 + y^2.$$

Converting to polar coordinates can sometimes make a problem much simpler, as we shall see in later example problems.

REMARK: Note that we have $r = x/\cos\theta$. So, using the chain rule we find

$$\frac{\partial r}{\partial x} = \frac{1}{\cos\theta} - \frac{1}{\cos^2\theta} \cdot \sin\theta \cdot \frac{\partial \theta}{\partial x}.$$

That is, in general, $\partial r/\partial x \neq 1/\cos\theta$. This above way of finding the partial is rather messy. Instead, observe that

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left[\sqrt{x^2 + y^2}\right] = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos\theta}{r} = \cos\theta.$$

Similarly, $\partial r/\partial y = \sin\theta$.

Section 3: Functions of Multiple Variables

3.1 – Level Curves:

Definition: A scalar field is one way to visualize a function of two variables. Here at the point (x, y) the value is $f(x, y)$. A scalar field is characterized by **level curves** (aka contour lines) along which the value $f(x, y)$ is constant. △

Remark: Topographical and weather maps provide great examples of level curves.

Example: Describe the level curves of the function

$$f(x, y) = 4x^2 + y^2.$$

Solution:

Each level curve equation is of the form

$$c = f(x, y) = 4x^2 + y^2.$$

For example, the level surfaces corresponding to the values $c = 0$, $c = 1$, and $c = 4$ are given by

$$\begin{aligned} &\text{For } c = 0: && 0 = 4x^2 + y^2 && \text{(single point)} \\ &\text{For } c = 1: && 1 = \frac{x^2}{(1/2)^2} + \frac{y^2}{1} && \text{(ellipse)} \\ &\text{For } c = 4: && 1 = \frac{x^2}{1} + \frac{y^2}{2^2} && \text{(ellipse)} \end{aligned}$$

□

Example: For $(x, y) \neq (0, 0)$, find the level curves of $f(x, y) = \dfrac{x^2y + y^3}{x^3 + xy^2}$.

Solution:

The given expression for f is ugly. First, let's simplify a little to get

$$f(x, y) = \frac{y(x^2 + y^2)}{x(x^2 + y^2)} = \frac{y}{x}.$$

This makes it look better. So, we could guess that a level curve looks like a line with fixed slope.

Can we be more precise, for instance, on the angle of that slope? Let's convert to polar coordinates. This gives

$$f(r,\theta) = \frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta).$$

Let c be a constant. Then the level curve $c = f(x,y)$ gives $c = \tan(\theta)$, which implies $\theta = \arctan(c)$. What does this tell us? Well, notice that we have found a fixed equation for θ, but there is no r dependence. This implies that along the line going through the origin (but not including the origin itself) at angle $\theta = \arctan(c)$ we have a constant function value c, i.e., $f(r,\arctan(c)) = c$ for all r.

□

3.2 – Limits with Multiple Variables:

Definition: Let f be a function of two variables defined, except possibly at (x_0,y_0), on an open disk centered at (x_0,y_0) and let L be a real number. Then we say

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|(x,y) - (x_0,y_0)\| < \delta \quad \text{implies} \quad |f(x,y) - L| < \varepsilon.$$

△

Example: Define $f(x,y)$ by

$$f(x,y) = \left(\frac{xy}{x^2+y^2}\right)^3$$

whenever $(x,y) \neq 0$. Compute the of $f(x,y)$ as $(x,y) \to (0,0)$ or show that the limit does not exist.

Solution:

We claim the limit does not exist. To show this, it suffices to show that the limit along two different paths to $(0,0)$ yield different limiting values. First, note that $f(x,0) = 0$ for all x. Hence

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{(x,0)\to(0,0)} 0 = 0.$$

Also,

$$f(x,x) = \left(\frac{x^2}{2x^2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}.$$

So,

$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{(x,x)\to(0,0)} \frac{1}{8} = \frac{1}{8}.$$

This shows that the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ is different when approaching along different paths. Hence the limit does not exist. □

Remark: One way to show that a limit does not exist is to identify two paths approaching the limit point (x_0, y_0) which yield different limiting values of f.

Example: Define $f(x,y)$ by

$$f(x,y) = \frac{x^5 + x^4 + 2x^2y^2 + y^4}{(x^2+y^2)^2}$$

whenever $(x,y) \neq 0$. Compute the of $f(x,y)$ as $(x,y) \to (0,0)$ or show that the limit does not exist.

Solution:

This limit looks quite ugly to try and evaluate. First let's simplify the expression for f. Observe that

$$f(x,y) = \frac{x^5 + (x^2+y^2)^2}{(x^2+y^2)^2} = 1 + \frac{x^5}{(x^2+y^2)^2}.$$

This is a lot nicer. Now let's instead change to polar coordinates. Then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Plugging this in, we obtain

$$f(r,\theta) = 1 + \frac{r^5\cos^5(\theta)}{(r^2)^2} = 1 + \frac{r^5\cos^5(\theta)}{r^4} = 1 + r\cos(\theta).$$

Now, taking the limit as $(x,y) \to (0,0)$ is the same as taking the limit as $r \to 0$. Hence

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} 1 + r\cos(\theta) = 1.$$

□

Example: Evaluate

$$\lim_{(x,y)\to(2,0)} \frac{x^2\sin(y)}{e^x y}.$$

Solution:

Here we can use our rule that the limit of a product equals the product of the limits. Namely, since $\lim_{x\to 2} x^2/e^x$ and $\lim_{y\to 0} \frac{\sin(y)}{y}$ exist, we can say

$$\lim_{(x,y)\to(2,0)} \frac{x^2\sin(y)}{e^x y} = \left(\lim_{x\to 2}\frac{x^2}{e^x}\right)\left(\lim_{y\to 0}\frac{\sin(y)}{y}\right) = \frac{4}{e^2}\cdot 1 = 4e^{-2}.$$

□

Example: Evaluate

$$\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{where} \quad f(x,y) = \frac{5x^2y}{x^2+y^2}.$$

Solution:

We evaluate our limit in two ways. First, let's use polar coordinates to say

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = \lim_{r\to 0}\frac{5r^2\cos(\theta)\cdot r\sin(\theta)}{r^2} = \lim_{r\to 0} 5r\sin(\theta)\cos(\theta) = 5\sin(\theta)\cos(\theta)\cdot\lim_{r\to 0} r = 0.$$

But, what if we can't use polar coordinates? Then what do we do?

In this limit, we see that the numerator and the denominator are both 0, and so we cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. However, from above we figure that the limit ought to be 0. So, we can verify this using an $\varepsilon - \delta$ argument with our above definition of a limit.

Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$\|(x,y)-(0,0)\| < \delta \quad \text{implies} \quad |f(x,y)-0| < \varepsilon.$$

(Note that $\|(x,y)-(0,0)\| = \|(x,y)\|$.) Then observe that for $(x,y) \neq (0,0)$ we have

$$\begin{aligned}
|f(x,y) - 0| &= \left|\frac{5x^2y}{x^2+y^2}\right| && \text{Substitute} \\
&= 5|y| \cdot \left(\frac{x^2}{x^2+y^2}\right) && \text{Factor out } 5|y| \\
&\leq 5|y| && \text{Note } x^2/(x^2+y^2) \leq 1 \\
&\leq 5\sqrt{x^2+y^2} && \text{Note } |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} \\
&= 5\|(x,y)\|. && \text{Substitute by definition of the norm}
\end{aligned}$$

Picking $\delta = \varepsilon/5$, it follows that if $\|(x,y)\| < \delta$, then

$$|f(x,y) - 0| \leq 5\|(x,y)-(0,0)\| < 5\delta = 5\left(\frac{\varepsilon}{5}\right) = \varepsilon.$$

Thus, we conclude

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

□

3.3 – Partial Derivatives:

REMARK: Now that we have functions of several variables, we ask ourselves what does it mean to differentiate? As you'll see, it is quite similar to the one dimensional case. If we have a function $f(x,y)$, we can take the *partial* derivative with respect to either x or y. We write the partial of f with respect to x as f_x or $f_x(x,y)$ (and do similarly for y). And, this partial derivative is essentially just the derivative of f if we were to take y as a constant.

Example: Let $f(x,y) = x^2y + x^3y^3$. Compute f_x and f_y.

Solution:

When taking the partial with respect to one variable, we hold all the other ones fixed. This gives

$$f_x = 2xy + 3x^2y^3 \quad \text{and} \quad f_y = x^2 + x^3 \cdot 3y^2.$$

□

Example: Define $f(x,y) = \int_0^x g(s,y)\ dy$ for some function $g(s,y)$. Find expressions for f_x and f_y.

Solution:

Using the fundamental theorem of calculus, $f_x(x,y) = g(s,x)$. Now, for the second part, we shall bring the derivative inside the integral to say[3]

$$f_y(x,y) = \int_0^x g_y(s,y)\ \mathrm{d}y.$$

□

Remark: Now we illustrate how to use the chain rule for two independent variables. This is used if we are differentiating the composition of functions.

Chain Rule for 2 Variables: Suppose $f(x,y)$ is a differentiable function of x and y. If $x = g(s,t)$ and $y = h(s,t)$ and all of our partials $\partial x/\partial s, \partial x/\partial t, \partial y/\partial s$, and $\partial y/\partial t$ exist, then the partial of f with respect to s and with respect to t are given by

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

△

Example: Suppose we have a differentiable function

$$f(x,y) = (g \circ h)(x,y) = g(h(x,y))$$

where $g(x) = x^2$ and $h(x,y) = x^2 + y^2$. Compute f_x and f_y.

Solution:

The notation for a problem like this can be a bit tricky. First note that h maps $\mathbb{R}^2$ to $\mathbb{R}$ and g maps $\mathbb{R}$ to $\mathbb{R}$. (So, we'd usually say $h : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.) Now, using the chain rule in multiple variables,

$$f_x(x,y) = \frac{\partial g}{\partial h}\frac{\partial h}{\partial x}.$$

But, what is $\partial g/\partial h$? Here this is simply the derivative of g and note we are essentially using g as a function of h here. So, this becomes

$$f_x(x,y) = g'(h(x,y)) \cdot \frac{\partial h}{\partial x}(x,y) = 2(h(x,y))^1 \cdot 2x = 2(x^2+y^2)\cdot 2x = 4x(x^2+y^2).$$

For a student writing this up, we could much more briefly write our solution as follows:

[3] For a more general form of differentiation of integrals, see the Leibniz integral rule.

Use the chain rule to say

$$f_x(x,y) = g'(h) \cdot \frac{\partial h}{\partial x} = 2h \cdot 2x = 2(x^2+y^2) \cdot 2x = \boxed{4x(x^2+y^2).}$$

□

Example: Suppose $f(x,y) = h(x)g(y)$ for some differentiable h and g. Show that the mixed partials of f are equal.

Solution:

Observe that $f_x(a,b) = h'(a)g(b)$, which implies $f_{xy}(a,b) = h'(a)g'(b)$. Similarly, $f_y(a,b) = h(a)g'(b)$ and $f_{yx}(a,b) = h'(a)g'(b)$. Thus, $f_{xy}(a,b) = h'(a)g'(b) = f_{yx}(a,b)$, as desired. □

Definition: Assume that $f(x,y)$ is defined in a disk D containing (a,b) and that $f_x(a,b)$ and $f_y(a,b)$ exist. Then $f(x,y)$ is **differentiable** at (a,b) provided that there is some linear $L(x,y)$ such that

$$f(x,y) = L(x,y) + e(x,y)$$

such that

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = \lim_{(x,y)\to(a,b)} \frac{|e(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

△

Definition: If a function f is defined as above and is differentiable, then the **tangent plane** to the graph at $(a,b,f(a,b))$ is given by

$$z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

△

Example: Suppose $x = 5t^2$ and $y = 3t$ and $f(x,y,t) = y^3 + \sin(t)$. Compute $\mathrm{d}f/\mathrm{d}t$.

Solution:

Use the chain rule to find

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \cancelto{0}{\frac{\partial f}{\partial x}}\frac{\mathrm{d}x}{\mathrm{d}t} + \cancelto{3y^2}{\frac{\partial f}{\partial y}}\,\cancelto{3t}{\frac{\mathrm{d}y}{\mathrm{d}t}} + \cancelto{\cos(t)}{\frac{\partial f}{\partial t}}\frac{\mathrm{d}t}{\mathrm{d}t} = 3y^2 \cdot 3t + \cos(t) = \boxed{81t^3 + \cos(t).}$$

□

3.3.1 – Differentiation with Multivariate Polynomials:

REMARK: First let's review polynomials a little bit. A polynomial is given by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0$$

for some nonnegative integer n. When $a_n \neq 0$, we say that n is the degree of the polynomial. As an example, a polynomial of degree 4 is given by

$$p(x) = x^4 + 2x^2 - 3$$

where $a_4 = 1$, $a_3 = 0$, $a_2 = 2$, $a_1 = 0$, and $a_0 = -3$. We can more compactly write polynomials as

$$p(x) = \sum_{i=0}^{n} a_i x^i.$$

This gives us a nice way of differentiating them. Namely,

$$p'(x) = \sum_{i=0}^{n} a_i \cdot i \cdot x^{i-1}$$

and note the degree of p' is $n - 1$.

With this, we now extend our considerations to multiple variables. For a two dimensional polynomial, we can write $p(x, y)$ as

$$p(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{i,j} x^i y^j$$

so that the degree of the polynomial is $(n + m)$, assuming $a_{n,m} \neq 0$.

REMARK: If we are given a two dimensional polynomial $p(x, y)$, then we can compute the partials as follows:

$$p_x(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j}\cdot i\cdot x^{i-1}y^j \quad \text{and} \quad p_y(x,y)= \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j}\cdot j\cdot x^{i}y^{j-1}.$$

3.3.2 – Clairaut's Theorem:

Clairaut's Theorem: If f_{xy} and f_{yx} are both continuous functions on a disk $D \subset \mathbb{R}^2$, then $f_{xy}(a,b) = f_{yx}(a,b)$ for all $(a,b) \in D$. That is,

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}.$$

△

Example: Suppose we have a function f that satisfies $f_x = yx^2$ and f_{yx} is continuous. Express f_y up to some arbitrary constant $c \in \mathbb{R}$.

Solution:
First note $f_{xy} = x^2$, which is continuous since polynomials are continuous. Because f_{xy} and f_{yx} are continuous, it follows that $f_{yx} = f_{xy} = x^2$. This implies

$$f_y = \int_x \tilde{x}^2 \, \mathrm{d}\tilde{x} = \frac{x^3}{3} + c$$

where c is some constant in $\mathbb{R}$. □

Example: Show that there is no function f such that $f_x = x^3 y$ and $f_y = y^2 + \pi x$.
Solution:
We shall proceed by way of contradiction. Suppose such a function f exists. Then

$$f_{xy} = x^3 \quad \text{and} \quad f_{yx} = \pi.$$

Clearly, f_{yx} and f_{xy} are continuous since polynomials are continuous. It follows from Clairut's theorem that $f_{xy} = f_{yx}$. But, this implies $f_{xy}(1,1) = 1^3 = 1 \neq \pi = f_{yx}(1,1)$, which contradicts our claim by Clairut's theorem. Thus, such a function f cannot exist. □

3.4 – Gradients:

Definition: The **gradient** of a function $f(x, y, z)$ is given by

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle .$$

$\triangle$

Remark: The gradient points in the direction of greatest increase.

Remark: The chain rule can now be rewritten in terms of the gradient. Let $\vec{r} = \langle x, y, z \rangle$. Then for a function $f(x, y, z)$ we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \nabla f(\vec{r}) \cdot \vec{r}\,'$$

where $\vec{r}\,' = \langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \rangle$.

Example: Suppose functions $f(x, y, z)$ and $h(x, y)$ are given. Then define $g(t) = f(h(t, t^2), h(e^t, \pi t))$. Compute $g'(t)$ in terms of t, leaving the answer in terms of f, h, and the partials of f and h.

Solution:
Define $\vec{r}(t) = \langle h(t, t^2), h(e^t, \pi t) \rangle$. Then we can write $g(t) = f(\vec{r}(t))$ and so

$$g'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

We know $\nabla f(\vec{r}(t)) = \langle f_x(\vec{r}(t)), f_y(\vec{r}(t)) \rangle$. So now we compute $\vec{r}\,'(t)$ using the chain rule.

$$\begin{aligned} \vec{r}\,'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \langle h(t, t^2), h(e^t, \pi t) \rangle \\ &= \langle h_x(t, t^2) \cdot 1 + h_y(t, t^2) \cdot 2t, \quad h_x(e^t, \pi t) \cdot e^t + h_y(e^t, \pi t) \cdot \pi \rangle . \end{aligned}$$

Thus,

$$\boxed{g'(t) = f_x(\vec{r}(t)) \cdot \left(h_x(t, t^2) + 2t h_y(t, t^2)\right) + f_y(\vec{r}(t)) \cdot \left(e^t h_x(e^t, \pi t) + +\pi h_y(e^t, \pi t)\right).}$$

$\square$

Remark: There are two important remarks to make from the above example.

i) The notation you choose can greatly impact the amount of writing. By using the $\vec{r}(t)$ notation we saved a lot of space above and were able to more easily break our problem into two parts.

ii) It is important to include the arguments of each function. Recall the *arguments* of a function are what are passed into it. For instance, note we have $h_x(t, t^2)$ while elsewhere we have $e^t h_x(e^t, \pi t)$. So, it would not be sufficient to just use compact notation and write h_x in our answer for g'. This would be problematic because it would be unclear what arguments are being passed into h_x in each place it occurs.

Example: Suppose we have a function $f(x, y) = x^2 + 5y^2$ and we want to instead represent this function in polar coordinates. Let $\tilde{f}(r, \theta)$ be this representation in polar coordinates. Compute $\tilde{f}_r(r, \theta)$ and $\tilde{f}_\theta(r, \theta)$.

Solution:
Remember that we can convert between polar and cartesian coordinates using $x = r\cos\theta$ and $y = r\sin\theta$. In other words, we have $\tilde{f}(r, \theta) = f(r\cos\theta, r\sin\theta) = f(x, y)$. To be more specific, we could write $\tilde{f}(r, \theta) = f(x(r, \theta), y(r, \theta))$. In any case, the computation is as follows:

$$\begin{aligned}
\tilde{f}_r(r, \theta) &= \frac{\partial}{\partial r}\left[f(x, y)\right] \\
&= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} + \frac{\partial y}{\partial r} && \text{Use chain rule} \\
&= f_x \cdot \cos\theta + f_y \cdot \sin\theta && \text{Compute partials of } x \text{ and } y \text{ and simplify partials of } f \\
&= 2x\cos\theta + 10y\sin\theta && \text{Compute partials of } f \\
&= \boxed{2r\cos^2\theta + 10r\sin^2\theta.} && \text{Rewrite strictly in terms of } r \text{ and } \theta
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{f}_\theta(r,\theta) &= \frac{\partial}{\partial\theta}\left[f(x,y)\right] \\
&= \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y} + \frac{\partial y}{\partial\theta} && \text{Use chain rule} \\
&= f_x \cdot (-r\sin\theta) + f_y \cdot (r\cos\theta) && \text{Compute partials of } x \text{ and } y \text{ and simplify partials of } f \\
&= -2xr\sin\theta + 10yr\cos\theta && \text{Compute partials of } f \\
&= -2r^2\sin\theta\cos\theta + 10r^2\sin\theta\cos\theta && \text{Rewrite strictly in terms of } r \text{ and } \theta \\
&= \boxed{8r^2\sin\theta\cos\theta.}
\end{aligned}$$

□

Example: Suppose we have a chipmunk that runs in the direction of greatest increase of our obscure function

$$f(x,y) = \cos(x) + x^2 + e^{2y}.$$

If we throw the chipmunk so that it hits the xy-plane at the point $(3,4)$, what direction will it start running in? (Use a unit vector to describe the direction.)

Solution:
Our direction $\hat{d}$ is given by

$$\hat{d} = \frac{\nabla f(3,4)}{\|\nabla f(3,4)\|}.$$

That is,

$$\hat{d} = \left[\frac{\langle -\sin(x) + 2x, 2e^{2y}\rangle}{\|\langle -\sin(x) + 2x, 2e^{2y}\rangle\|}\right]_{(x,y)=(3,4)} = \boxed{\frac{\langle -\sin(3) + 6, 2e^8\rangle}{\|\langle -\sin(3) + 6, 2e^8\rangle\|}.}$$

(I leave the algebra steps to you if you really want to find a numerical answer.) □

Remark: One way to approximate the uncertainty/error in a measurement is to use differentials. For a function $f(x,y,z)$, if we let $(\Delta x, \Delta y, \Delta z)$ have sufficiently small values, then

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta = \nabla f \cdot \langle \Delta x, \Delta y, \Delta z\rangle.$$

This is essentially just the chain rule where instead of using infinitesimally small quantities (i.e., $\mathrm{d}f$), we use the 'slightly larger' Δ's and we aren't dividing by some differential amount (e.g., $\mathrm{d}t$).

Example: Approximate $e^{0.05} + [0.99 \cdot 0.51]^2$ with differentials.

Solution:

Write $f(x, y, z) = e^x + y^2 z^2$. Then we wish to approximate $f(0.05, 0.99, 0.51)$. We can easily see that $f(0, 1, 1/2) = e^0 + (1/2)^2 = 5/4$. This can be used to approximate our point of interest by

$$f(0.05, 0.99, 0.51) \approx f(0, 1, 1/2) + \nabla f(0, 1, 1/2) \cdot \langle 0.05, -0.01, 0.01 \rangle$$

where this last term is the difference between our point that we can easily evaluate and the point of interest. Then

$$\nabla f(x, y, z) = \langle e^x, 2yz^2, 2y^2 z \rangle \quad \Rightarrow \quad \nabla f(0, 1, 1/2) = \langle e^0, 2 \cdot 1 \cdot (1/2)^2, 2 \cdot 1^2 \cdot (1/2) \rangle = \langle 1, 1/2, 1 \rangle .$$

Thus, we find

$$\begin{aligned} e^{0.05} + [0.99 \cdot 0.51]^2 &= f(0.05, 0.99, 0.51) \\ &\approx \frac{5}{4} + \langle 1, 1/2, 1 \rangle \cdot \langle 0.05, -0.01, 0.01 \rangle \\ &= 1.25 + (0.05 - 0.005 + 0.01) \\ &= \boxed{1.305.} \end{aligned}$$

This is a reasonable approximation since the actual solution is ≈ 1.3062. $\square$

Example *(An application)***:** Suppose we have a particle shot vertically with height y given by

$$y = vt - \frac{1}{2}gt^2.$$

We can easily predict the height of the particle given v, t, and g. In addition to this, we would also like to put an estimate on the error of our calculation of this height given that our measurements are given by $v = 15.0 \pm 0.1$, $g = 9.80 \pm 0.01$, and $t = 2.00 \pm 0.01$ where the $\pm$ tells us the uncertainty in the measurement. (This happens in practice due to the limited accuracy and precision of measurements made.) Compute y and identify an associated uncertainty corresponding to that made in our measurements.

Solution:
First we identify that $\Delta v = 0.1$, $\Delta t = 0.01$, and $\Delta g = 0.01$. Then we say

$$\begin{aligned}
\Delta y &\approx \left|\frac{\partial y}{\partial v}\Delta v\right| + \left|\frac{\partial y}{\partial g}\Delta g\right| + \left|\frac{\partial y}{\partial t}\Delta t\right| \\
&= |t \cdot \Delta v| + \left|\frac{1}{2}t^2 \cdot \Delta g\right| + |(v - gt)\Delta t| \\
&= |2 \cdot 0.1| + \left|\frac{1}{2} \cdot 2^2 \cdot 0.01\right| + |(10 - 9.8 \cdot 2) \cdot 0.01| \\
&= 0.124
\end{aligned}$$

where we note the absolute values are used since we are concerned with an error bound, i.e., we want to estimate the greatest amount of uncertainty and, thus, don't want to let the errors cancel each other out. Computing y with the values given we have $y = 10.40$. Accounting for our uncertainties in our quantities for v, t, and g, to two decimal places, we could then estimate that $\boxed{y = 10.40 \pm 0.12.}$ □

Example: Suppose I have a function $f(x, y)$ in Cartesian coordinates. Here we consider how to represent ∇f in a different coordinate system. Let $g(a, b)$ be the representation of f in terms of some coordinate system with orthonormal basis vector $\hat{e}_a$ and $\hat{e}_b$. Find an expression for ∇g.

Solution:

Since f and g are equal, just represented in terms of different coordinate systems, we know $\nabla g(a, b) = \nabla f(x(a, b), y(a, b))$. So, we need only rewrite ∇f in terms of this new coordinate system. For our partial derivatives, we see

$$f_x = \frac{\partial g}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial x} = g_a\frac{\partial a}{\partial x} + g_b\frac{\partial b}{\partial x} \quad \text{and} \quad f_y = \frac{\partial g}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial y} = g_a\frac{\partial a}{\partial y} + g_b\frac{\partial b}{\partial y}.$$

Recall from earlier this quarter that if we have orthonormal vectors we can use dot products to easily find the representation of vectors with respect to this basis. That is,

$$\hat{\imath} = (\hat{\imath} \cdot \hat{e}_b)\hat{e}_b + (\hat{\imath} \cdot \hat{e}_b)\hat{e}_b, \tag{1}$$

and similarly for representing $\hat{\jmath}$ with respect to this basis. This tells us that

$$\begin{aligned}
\nabla g(a, b) &= \nabla f(x(a, b), y(a, b)) \\
&= f_x\hat{\imath} + f_y\hat{\jmath} \\
&= \underbrace{\left(g_a\frac{\partial a}{\partial x} + g_b\frac{\partial b}{\partial x}\right)}_{f_x}\underbrace{\left((\hat{\imath} \cdot \hat{e}_b)\hat{e}_b + (\hat{\imath} \cdot \hat{e}_b)\hat{e}_b\right)}_{\hat{\imath}} \\
&+ \underbrace{\left(g_a\frac{\partial a}{\partial y} + g_b\frac{\partial b}{\partial y}\right)}_{f_y}\underbrace{\left((\hat{\jmath} \cdot \hat{e}_b)\hat{e}_b + (\hat{\jmath} \cdot \hat{e}_b)\hat{e}_b\right)}_{\hat{\jmath}}.
\end{aligned}$$

With some good fortune, if a and b were given in terms of x and y, then one would then be able to simplify this expression to be in terms of a and b and the partials of g. □

3.5 – Implicit Differentiation:

REMARK: Recall that in single-variable calculus we could use implicit differentiation to compute $\mathrm{d}y/\mathrm{d}x$ when y was implicitly defined as a function of x given some equation $f(x, y) = 0$. We give an example of this below:

Example: Consider the curve given by $5x^2 + 5e^y + y + 6 = 0$. Assuming this implicitly defines y as a function of x, compute $\mathrm{d}y/\mathrm{d}x$.

Solution:

Step 1: **Differentiate both sides of our equation.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[5x^2 + 5e^y + y + 6\right] = \frac{\mathrm{d}}{\mathrm{d}x}[0] = 0 \qquad \Rightarrow \qquad 5 \cdot 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}x} + 5e^y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} + 0 = 0.$$

Of course, $\mathrm{d}x/\mathrm{d}x = 1$.

Step 2: **Rearrange for the desired derivative.** Rearranging the above, we find

$$(5e^y + 1)\frac{\mathrm{d}y}{\mathrm{d}x} = -10x \qquad \Rightarrow \qquad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{10x}{5e^y + 1}.}$$

□

Remark: There is actually another way to compute the above, using the chain rule for implicit differentiation, which we know state.

Chain Rule: Implicit Differentiation. If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x, y)}{F_y(x, y)} \quad \text{if } F_y(x, y) \neq 0.$$

And, if the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\partial F_x(x, y, z)}{\partial F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial F_y(x, y, z)}{\partial F_z(x, y, z)} \quad \text{if } F_z(x, y, z) \neq 0.$$

△

Example: We know return to our example above, letting $F(x, y) = 5x^2 + 5e^y + 6 = 0$. Then

$$F_x = 10x + 0 + 0 + 0 \quad \text{and} \quad F_y = 0 + 5e^y + 1 + 0.$$

Then, using our theorem,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{10x}{e^y + 1},$$

which is the same as what we got above.

Example (From Larson Ch. 13 Example 7): Find $\partial z/\partial x$ and $\partial z/\partial y$ given that

$$3x^2z - x^2y^2 + 2z^3 = 5 - 3yz.$$

Solution:
To apply our theorem, we let

$$F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0.$$

Then

$$F_x = 6xz - 2xy^2, \quad F_y = -2x^2y + 3z, \quad F_z = 3x^2 + 6z^2 + 3y.$$

Thus,

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}.}$$

□

3.6 – Tangent Planes & Normal Lines:

Definition: Let f be differentiable at the point $(x_0, y_0, z_0) \in \mathbb{R}^3$ on the surface S given by $f(x, y, z) = 0$ such that $\nabla f(x_0, y_0, z_0) \neq \vec{0}$. Then

i) the plane P that is normal to $\nabla f(x_0, y_0, z_0)$ is the **tangent plane to** S **at** P;

ii) the line through P in the direction of $\nabla f(x_0, y_0, z_0)$ is called the **normal line to** S **at** P**.**

△

Remark: The tangent plane for the surface S described above is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

In vector notation, if $\vec{r}_0 = (x_0, y_0, z_0)$ and $\vec{r} = (x, y, z)$, then this tangent plane is given by

$$\nabla f(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0.$$

Example: Suppose we have $x^2 + 3y^2 = 5$. Find where the normal vector along this curve is proportional to $\langle 2, 1\rangle$.

Solution:
Define $f(x, y) = x^2 + y^2/2 - 5 = 0$. Then ew know at the point (a, b) that

$$k\langle 2, 1\rangle = \nabla f(a, b) = \langle 2a, b\rangle \quad \Rightarrow \quad \begin{cases} 2a = 2k, \\ b = k. \end{cases}$$

We know see that $a = b = k$. To find k, we just plug this back into our given equation to find

$$k^2 + \frac{k^2}{2} - 5 = 0 \quad \Rightarrow \quad k^2 = \frac{2}{3} \cdot 5 = \frac{10}{3} \quad \Rightarrow \quad k = \pm\sqrt{\frac{10}{3}}.$$

Thus, the points (a, b) where the normal vector is proportional to $\langle 2, 1\rangle$ are given by

$$\boxed{\langle a, b\rangle = \pm\sqrt{\frac{10}{3}}\langle 1, 1\rangle.}$$

□

3.7 – Critical Points:

Definition: Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if either $\nabla f(x_0, y_0) = \mathbf{0}$ or $f_x(x_0, y_0)$ does not exist or $f_y(x_0, y_0)$ does not exist. △

Second Derivative Test: Let f have continuous partial derivatives on an open region containing a point (a, b) where $\nabla f(a, b) = \mathbf{0}$. Let

$$d = \det\left(\frac{\partial(\nabla f)}{\partial(x,y)}\right) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2.$$

Then we can classify the critical point (a, b) using the following:

i) If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a *relative minimum* at (a, b).

ii) If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a *relative maximum* at (a, b).

iii) If $d < 0$, then $(a, b, f(a, b))$ is a *saddle point*.

iv) The test is inconclusive if $d = 0$.

△

Example*(From Larson page 958.)***:** Find the relative extrema of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

Solution:

Because $f_x = -3x^2 + 4y$ and $f_y = 4(x - y)$ exists for all x and y, the only critical points occur where $\nabla f = \mathbf{0}$. Setting $\nabla f = \mathbf{0}$, we see from $f_y = 0$ that $x = y$. Substituting this into $0 = -3x^2 + 4y$, we find two solutions: $(0, 0)$ and $(4/3, 4/3)$. Because

$$f_{xx} = -6x, \quad f_{yy} = -4, \quad f_{xy} = 4.$$

for the critical pint $(0, 0)$ we have

$$d = f_{xx}(0,0)f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2 = 0 \cdot (-4) - 4^2 = -16 < 0,$$

it follows from the Second Derivative Test that $\boxed{(0, 0, f(0, 0)) = (0, 0, 1) \text{ is a saddle point of } f.}$ For the critical point $(4/3, 4/3)$ we see

$$d = f_{xx}(4/3,4/3)f_{yy}(4/3,4/3) - (f_{xy}(4/3,4/3)^2 = (-8)\cdot(-4) - 4^2 = 16 > 0.$$

Since $f_{xx}(4/3,4/3) = -8 < 0$, by the Second Derivative test, $\boxed{f \text{ has a relative max at } (4/3,4/3).}$

□

Example*(Failure of the Second Derivative Test)***:** (This problem is based on an example from Larson page 958.) Find the relative extrema of $f(x,y) = x^4y^4$.

Solution:
Since $f_x(x,y) = 4x^3y^4$ and $f_y(x,y) = 4x^4y^3$, we see that $\nabla f(x,y) = 0$ if $x = 0$ or $y = 0$. That is, every point along the x-axis and along the y-axis is a critical point. Now,

$$d = f_{xx}f_{yy} - f_{xy}^2 = (12x^2y^4)(12x^4y^2) - (16x^3y^3)^2 = 144x^6y^6 - 256x^6y^6 = -112x^6y^6.$$

And, if x or y is zero, then $d = 0$, which tells us the second derivative test cannot be applied. However, since $f(x,y) = 0$ along the x and y-axes and is positive everywhere else, we know that at each of the points along these axes there is an absolute minimum. □

Section 4: Differential Equations

Remark: In discussion, we have also talked about some standard differential equations and their solutions. Here we summarize the three types of differential equations we have solved this quarter.

4.1 – Exponentials:

Remark: Here we consider problems of the form

$$y'' - k^2 y = 0.$$

There are multiple ways to describe the solutions of this. It is easily verified that e^{kx} and e^{-kx} are solutions. Since this is a *linear differential equation*, the *general solution* is given by a linear combination of these. That is, we say

$$y = c_1 e^{kx} + c_2 e^{-kx}$$

for some constants $c_1, c_2 \in \mathbb{R}$. To find the solution for a set of conditions, we then must solve for c_1 and c_2. We illustrate this with the following example.

Example: Solve the equation $y'' - 9y = 0$ given that $y(0) = 1$ and $y'(0) = 3$.

Solution:
Our general solutions is given by

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

for some $c_1, c_2 \in \mathbb{R}$. We now solve for c_1 and c_2. Using our initial conditions,

$$1 = y(0) = c_1 e^{3 \cdot 0} + c_2 e^{-3 \cdot 0} = c_1 + c_2$$

and

$$3 = y'(0) = 3c_1 e^{3 \cdot 0} - 3c_2 e^{-3 \cdot 0} = 3(c_1 - c_2) \quad \Rightarrow \quad 1 = c_1 - c_2.$$

Adding our equations together, we see

$$2 = 1 + 1 = (c_1 + c_2) + (c_1 - c_2) = 2c_1 \quad \Rightarrow \quad c_1 = 1.$$

It follows that $c_2 = 0$. Thus, the solution to our differential equation is $\boxed{y(x) = e^{3x}.}$ □

Example: Solve $y' = \alpha y$ where $y(1) = \pi$.

Solution:

If we have something where the derivative is proportional to the original function, then we know it's an exponential, i.e., $y = c_1 e^{\alpha x}$ for some c_1. Then

$$\pi = y(1) = c_1 e^{\alpha} \quad \Rightarrow \quad c_1 = \pi e^{-\alpha} \quad \Rightarrow \quad \boxed{y = \pi e^{-\alpha} e^{\alpha x} = \pi e^{\alpha(x-1)}.}$$

□

4.2 – Simple Harmonic Oscillators:

Remark: The general solution to differential equations of the form

$$y'' + \omega^2 y = 0$$

is

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

for some constants $c_1, c_2 \in \mathbb{R}$.

Example: Find the solution to

$$y'' + \pi^2 y = 0, \quad y(0) = 1, y(1/2) = 3.$$

Solution:

Our solution is of the form $y = c_1 \cos(\pi t) + c_2 \sin(\pi t)$. Then

$$1 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \cdot 1 + c_2 \cdot 0 \quad \Rightarrow \quad c_1 = 1.$$

Also,

$$3 = y(1/2) = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) = c_1 \cdot 0 + c_2 \cdot 1 = c_2 \quad \Rightarrow \quad c_2 = 3.$$

Thus, $\boxed{y(t) = \cos(\pi t) + 3\sin(\pi t).}$ □

4.3 – Cauchy-Euler Equations:

REMARK: Cauchy-Euler equations are of the form

$$r^n y^{(n)} + a_{n-1} r^{n-1} y^{(n-1)} + \cdots + a_1 r y' + a_0 y = 0.$$

Solutions to this are of the form $y = r^k$ for some constant k. To find this k, we plug r^k into the equation. We illustrate further details of this in the following example.

Example: Suppose we are given the Cauchy-Euler equation

$$x^3 y''' - 2x^2 y'' + 4xy' - 4y = 0.$$

Find the general solution for $y(x)$.

Solution:
For this type of equation we assume a solution of the form $y = x^m$. Plugging this in, we find

$$\begin{aligned} 0 &= x^3 m(m-1)(m-2)x^{m-3} - 2x^2 m(m-1)x^{m-2} + x4mx^{m-1} - 4x^m \\ &= x^m \left(m(m-1)(m-2) - 2m(m-1) + 4m - 4\right). \end{aligned}$$

Since this differential equation holds for all x, it must hold whenever $x \neq 0$. Because of this, we obtain the *characteristic equation*

$$0 = m(m-1)(m-2) - 2m(m-1) + 4m - 4 = m^3 - 5m^2 + 8m - 4 = (m-2)^2(m-1).$$

So, x and x^2 are solutions. But, because we have a third-order linear equation, there should be three solutions. It turns out that because there are two roots $m = 2$, we say that $m = 2$ is a root of multiplicity two. This gives the solution x^2 and a second solution $\ln(x)x^2$. (If it was of multiplicity three, then we'd also have a solution $(\ln(x))^2 x^2$, and so on.) The general solution is given as a linear combination of our three solutions we have found. That is,

$$y = c_0 x + c_1 x^2 + c_2 \ln(x) x^2$$

for some constants c_0, c_1, and c_2. If we are given additional conditions on our solution, then we can solve for these constants c_i explicitly. □

Example: Suppose we find that the general solution to a Cauchy-Euler equation is of the form $y = c_1 x + c_2 x \ln(x)$ and we are given that $y(1) = 1$ and $y(e) = 2$. Find y.

Solution:
From the first condition,

$$1 = y(1) = c_1 \cdot 1 + c_2 \cdot 1 \ln(1) = c_1 + 0 \qquad \Rightarrow \qquad c_1 = 1.$$

Then

$$2 = y(e) = c_1 e + c_2 e \ln(e) = e + c_2 e \qquad \Rightarrow \qquad c_2 = \frac{2-e}{e}.$$

Thus, $\boxed{y = x + \frac{2-e}{e} \cdot x \ln(x).}$ □