Discussion Notes for Optimization (MATH 164)

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Purpose: This document is a compilation of notes generated for discussion in MATH 164 with reference credit due to Chong and Zak's *Introduction to Optimization* text. If the reader finds any errors/typos, please feel free to email me at `heaton@math.ucla.edu` and I will address these and post an updated set of notes to my webpage.

Contents

Section 1: Introduction

Remark: In these notes I will have, in places, two separate solutions to an example problem. The first solution will be a more elaborative answer that should build the student's intuition and understanding for how to tackle the problem. The second solution is more compact and along the lines of what a student may be expected to provide as a solution.

Section 2: Linear Programming

2.1 – Introduction to LP:

We consider systems of constraints $Ax = b$, $x \geq 0$ where $A \in \mathbb{R}^{m \times n}$ with $m < n$ and $\text{rank}(A) = m$. The objective function is also linear and can, thus, be written in the form $c^T x$ for some $c \in \mathbb{R}^n$. The standard form of a linear programming problem is

$$\min c^T x \quad \text{subject to} \quad Ax = b, \quad x \geq 0. \tag{1}$$

We will often encounter problems with $Ax \leq b$ constraints. Such problems can be converted to standard form. For example, a constraint $x \leq 5$ can be equivalently rewritten as $x + w = 5$, $w \geq 0$. And, in general,

$$Ax \leq b \quad \Longleftrightarrow \quad b = Ax + y = Ax + Iy = [A,\ I] \begin{bmatrix} x \\ y \end{bmatrix} = b, \quad y \geq 0. \tag{2}$$

Note $y \in \mathbb{R}^m$ and so $[x,\ y]^T \in \mathbb{R}^{m+n}$. Letting $c^* = [c^T,\ 0]^T \in \mathbb{R}^{m+n}$ where here $0 \in \mathbb{R}^m$, we are able to obtain the problem in standard form. Namely,

$$\min\ (c^*)^T \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{subject to} \quad [A,\ I] \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq 0. \tag{3}$$

Example 15.9 in the text provides an example on how to introduce a slack variable. To gain intuition for these problems, we now move to a graphical approach to visualize and solve simple linear programming problems.

Example 2: Graphically solve the problem

$$\min c^T x \quad \text{subject to} \quad Ax \le b, \quad x \ge 0 \tag{4}$$

where

$$c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 5 \\ 20 \end{bmatrix}. \tag{5}$$

Elaborative Solution:

We first visualize the region of feasible points. For the constraints[1] $x \ge 0$ and $A_1 x \le b_1$ we have $x_1 \le 10$. We visualize $x \ge 0$ and $A_1 x \le b_1$ in Figure 1.

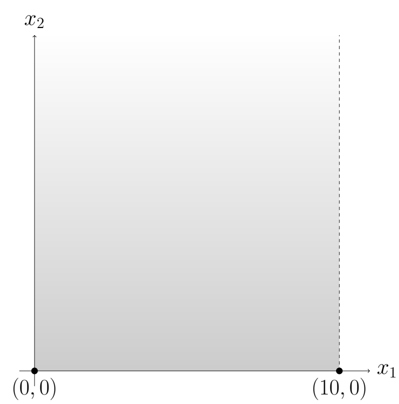

Figure 1: The feasible region for $x \ge 0$, $A_1 x \le b_1$ is shaded.

Adding the constraint $A_2 x \le b_2$, i.e., $-x_1 + x_2 \le 5$, we obtain Figure 2.

[1] Note here I use A_i to denote the i-th row of the matrix A.

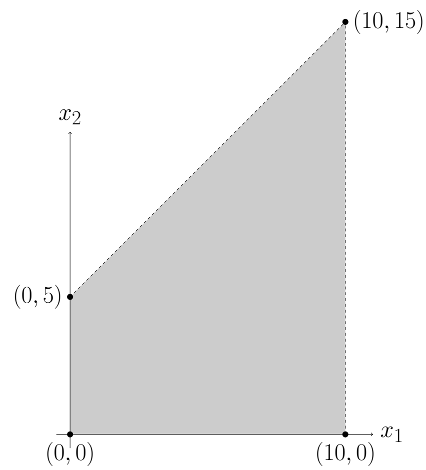

Figure 2: The feasible region for $x \geq 0$, $A_1 x \leq b_1$, $A_2 x \leq b_2$ is shaded.

Adding $A_3 x \leq b_3$, i.e., $x_1 + 2x_2 \leq 20$, we obtain Figure 3. In Figure 3 we also visualize the level curves of $c^T x = -x_2$, which are simply horizontal lines. And, as x_2 increases, $c^T x = -x_2$ decreases. So, a solution will be given by any feasible point with maximal component x_2. From Figure 3, we see where two of the constraint equations intersect gives the feasible point with maximal value x_2, namely $(10/3, 5 + 10/3)$. Thus, the solution to our optimization problem is $(10/3, 5 + 10/3)$.

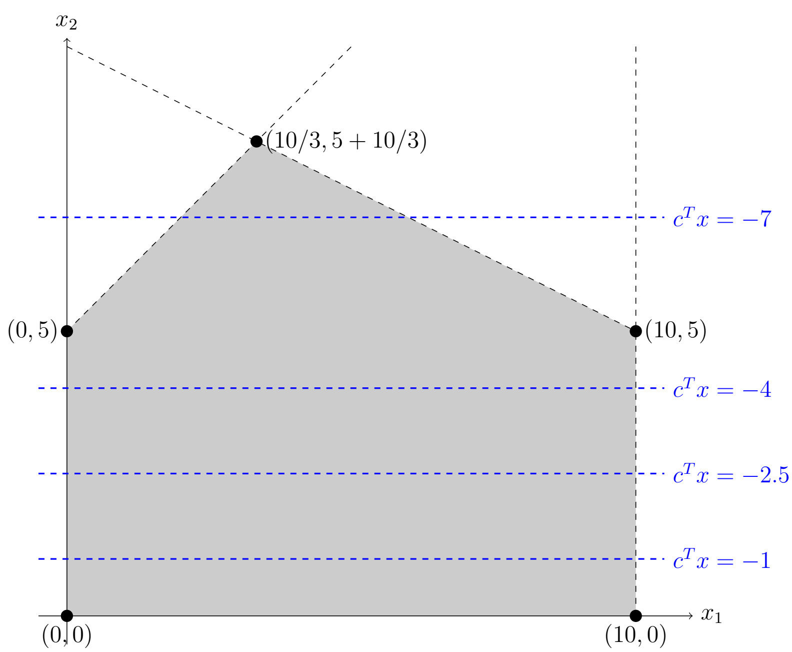

Figure 3: The feasible region for $Ax \leq b$, $x \geq 0$ with objective function level curves drawn in blue.

□

Remark: Now suppose we are able to write the matrix A in the form $A = [B,\ N]$ for some submatries B and N with B invertible. By setting $x_B = B^{-1}b$, we discover

$$[B,\ N]\begin{bmatrix} x_B \\ 0 \end{bmatrix} = Bx_B = BB^{-1}b = Ib = b. \tag{6}$$

This gives rise to the following definition.

Definition: We call $[x_B^T, 0^T]^T$ a *basic solution* to $Ax = b$ with respect to the basis B provided that $x_B = B^{-1}b$. △

Definition: Any vector x that satisfies $Ax = b$ and $x \geq 0$ is said to be *feasible.* △

Remark: We may combine the adjectives to say x is a basic feasible solution if it is both basic and is a feasible solution. The following example illustrates how to find basic solutions and that not all basic solutions are feasible.

Example 3: Find every feasible basis solution to $Ax = b$, $x \geq 0$ given

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}. \tag{7}$$

Solution:

We must find each submatrix[2] $B \in \mathbb{R}^{3\times 3}$ of A such that B is invertible. Write $A = [a_1,\ a_2,\ a_3,\ a_4]$ where a_i denotes the i-th column of A. To construct a submatrix B we must choose to use 3 of the 4 columns of A. This gives $\binom{4}{3} = 4$ possibilities. First take $B_1 = [a_1,\ a_2,\ a_3] = I$. Then set

$$x_{B_1} = I^{-1}b = Ib = b = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}. \tag{8}$$

[2]We now this must be a 3×3 matrix since $b \in \mathbb{R}^3$.

Then $[x_{B_1}^T, 0] = [2,\ 1,\ 5,\ 0]^T \in \mathbb{R}^4$ is a basic solution with respect to the basis B_1. Indeed, we see

$$A \begin{bmatrix} x_{B_1} \\ 0 \end{bmatrix} = b \quad \text{and} \quad \begin{bmatrix} x_{B_1} \\ 0 \end{bmatrix} \geq 0. \tag{9}$$

In similar fashion, let

$$B_2 = [a_2,\ a_3,\ a_4] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \tag{10}$$

Then

$$x_{B_2} = B_2^{-1} b = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}. \tag{11}$$

Note we used the 2nd, 3rd, and 4th columns to form B_2, and so we find $[0,\ 1,\ 3,\ 2]^T$ forms a basic feasible solution.

Next consider

$$B_3 = [a_1,\ a_3,\ a_4] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \tag{12}$$

Since $a_4 = a_1 + a_3$, we see B_3 is singular. Thus, there does not exist a basic solution with respect to B_3 since the columns of B_3 do not form a basis for $\mathbb{R}^3$.

Lastly, consider

$$B_4 = [a_1,\ a_2,\ a_4] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

Then

$$x_{B_4} = B_4^{-1} b = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}. \tag{14}$$

This implies $[-3,\ 1,\ 0,\ 5]^T$ is a basic solution.[3] However, the first entry is negative and so it is *not* a feasible basic solution.

[3]Note we used the 1st, 2nd, and 4th columns and so $x_{B_4} = [x_1,\ x_2,\ x_4]^T$ and we take $x_3 = 0$.

Having worked out each of the possible cases, we conclude the two basic feasible solutions to $Ax = b,\ x \geq 0$ are

$$\begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}. \tag{15}$$

□

Example 4: Consider the optimization problem[4]

$$\max\ c_1|x_1| + \cdots + c_n|x_n| \quad \text{subject to} \quad Ax = b \tag{16}$$

where $c_i \neq 0$ for $i = 1, \ldots, n$. Convert this problem into an equivalent standard form linear programming problem.

Elaborative Solution:

We first tackle the issue of the absolute values. This is accomplished by effectively separating each x_i into a positive part and a negative part. For instance, if $x_i \geq 0$ we can set $u_i := x_i$ and $v_i := 0$ and if $x_i < 0$ set $u_i := 0$ and $v_i := -x_i$. Note this gives $u_i, v_i \geq 0$ and $x_i = u_i - v_i$. Moreover, we also obtain the equality $|x_i| = u_i + v_i$, which is key because now we have rewritten the absolute value as the sum of two nonnegative terms. Having these two equations relating x_i to u_i and v_i, in fact, uniquely determines u_i and v_i. Using u_i and v_i in this fashion for $i = 1, \ldots, n$, our objective function may be rewritten as

$$c_1(u_1 + v_1) + \cdots + c_n(u_n + v_n). \tag{17}$$

In matrix form, this becomes

$$c^T(u + v) = c^T u + c^T v = [c^T c^T] \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix}. \tag{18}$$

The standard form is a minimization problem, but we have a maximization problem. So, we just add a negative sign out front so that our problem equivalently becomes

$$\min\ -\begin{bmatrix} c \\ c \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{or} \quad \min\ \begin{bmatrix} -c \\ -c \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix}. \tag{19}$$

Now we turn to our constraints. By construction of u and v, we can write $x = u - v$ and we know $u, v \geq 0$. Thus,

$$b = Ax = A(u - v) = Au - Av = [A,\ -A] \begin{bmatrix} u \\ v \end{bmatrix}. \tag{20}$$

[4]This is Exercise 15.1 in the text.

From this, we obtain the standard form problem. Namely,

$$\min \underbrace{\begin{bmatrix} -c \\ -c \end{bmatrix}}_{c^*}^T \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{x^*} \quad \text{subject to} \quad \underbrace{[A,\ -A]}_{A^*} \begin{bmatrix} u \\ v \end{bmatrix} = b, \quad \begin{bmatrix} u \\ v \end{bmatrix} \geq 0. \tag{21}$$

If we let c^*, x^*, and A^* denote the underbraced quantities, then we obtain

$$\min\ (c^*)^T x^* \quad \text{subject to} \quad A^* x^* = b, \quad x^* \geq 0. \tag{22}$$

This problem (22) is equivalent to (21) and more clearly shows (21) is in standard form. □

Student Solution:
For each $i = 1, \dots, n$, recall there are unique $u_i, v_i \geq 0$ such that $x_i = u_i - v_i$. If $x_i \geq 0$ set $u_i = x_i$ and $v_i = 0$, and if $x_i < 0$ set $u_i = 0$ and $v_i = -x$. Then $x = u - v$ and $|x_i| = u_i + v_i$ for $i = 1, \dots, n$ and $u, v \geq 0$ where $u, v \in \mathbb{R}^n$. This implies

$$c_1|x_1| + \cdots + c_n|x_n| = c_1(u_1 + v_1) + \cdots + c_n(u_n + v_n) = c^T(u + v) = \begin{bmatrix} c \\ c \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix}. \tag{23}$$

Thus, the objective function portion of our problem can be equivalently rewritten as

$$\min \begin{bmatrix} -c \\ -c \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix}. \tag{24}$$

Substituting $x = u - v$ into the constraint equation, we discover

$$b = Ax = A(u - v) = [A,\ -A] \begin{bmatrix} u \\ v \end{bmatrix}. \tag{25}$$

From this, we obtain the standard form problem. Namely,

$$\min \begin{bmatrix} -c \\ -c \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{subject to} \quad [A,\ -A] \begin{bmatrix} u \\ v \end{bmatrix} = b, \quad \begin{bmatrix} u \\ v \end{bmatrix} \geq 0. \tag{26}$$

□

2.2 – Fundamental Theorem of Linear Programming:

The following is a statement of the Fundamental Theorem of Linear Programming (FTLP).

Theorem: Consider a linear program in standard form.

i) If there is a feasible solution, then there is a basic feasible solution.

ii) If there is an optimal feasible solution, then there is an optimal basic feasible solution.

△

Proof Sketch:
The core ideas of the proof of i) are as follows. Assume $x = [x_1, \ldots, x_p, \ldots, x_n]^T$ is a solution with only the first p entries nonzero.
Case 1: If $a_1, \ldots, a_p$ are linearly independent, then x is basic.
Case 2: Suppose $a_1, \ldots, a_p$ are linearly dependent. Then there is $y = [y_1, \ldots, y_p, 0, \ldots, 0]^T \in \mathbb{R}^n$ with

$$Ay = y_1 a_1 + \cdots + y_p a_p = 0 \quad \Rightarrow \quad 0 = \varepsilon 0 = \varepsilon A y = A(\varepsilon y) \tag{27}$$

for any $\varepsilon \in \mathbb{R}$. This implies

$$b = b - 0 = Ax - A(\varepsilon y) = A(x - \varepsilon y) = (x_1 - \varepsilon y_1)a_1 + \cdots + (x_p - \varepsilon y_p)a_p. \tag{28}$$

We pick ε such that $(x - \varepsilon y) \geq 0$ and at least one of the coefficients $(x_i - \varepsilon y_i)$ is zero. This gives a feasible solution with no more than $p - 1$ positive entries. This process can be repeated to obtain a vector x^* with nonzero entries corresponding to a linearly independent set of columns of A. Then *Case 1* can be applied to say x^* is basic, and we are done.

The proof for ii) follows almost identically. We again have *Case 1* and *Case 2* to consider. *Case 1* is identical to above. The only added step for *Case 2* is to show that $c^T x = c^T x^*$. We claim that if εy is defined as above, then $c^T x = c^T(x - \varepsilon y)$. Then through induction we obtain $c^T x = c^T x^*$. By way of contradiction, suppose $c^T y \neq 0$. Then pick small ε such that $c^T(\varepsilon y) > 0$ and $x - \varepsilon y \geq 0$. This implies

$$c^T(x - \varepsilon y) = c^T x - c^T(\varepsilon y) < c^T x, \tag{29}$$

which contradicts the fact that x is optimal. Hence $c^T x = 0$ and we are done. □

The following is a more thorough/elaborative proof of the FTLP.

Proof:

i) Suppose $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is a feasible solution with p positive components. Regardless of the order of the entries of x, we can equivalently rearrange the columns of A and entries of x so that the first p entries of x are the nonzero ones. Doing so, we write $A = [a_1, \ldots, a_p, \ldots, a_n]$ where the $a_i \in \mathbb{R}^m$ denote the columns of A. This gives

$$b = Ax = [a_1, \ldots, a_p, \ldots, a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x_1 a_1 + \cdots + x_p a_p. \tag{30}$$

We now consider two cases.

Case 1: Suppose $a_1, \ldots, a_p$ are linearly independent. Then $p \leq m$ since the a_i are in $\mathbb{R}^m$. If $p = m$, then $a_1, \ldots, a_p$ form a basis for $\mathbb{R}^m$ so that x is a basic solution and we are done. If $p < m$, then because $\text{rank}(A) = m$, we can pick $m - p$ columns from $a_{p+1}, \ldots, a_n$ so that the collection of p total columns forms a basis for $\mathbb{R}^m$. In this case, x is a (degenerate) basic feasible solution corresponding to these m columns.[5]

Case 2: We now assume $a_1, \ldots, a_p$ are linearly dependent. Then $p > m$. We must show there is a solution feasible solution with only m nonzero entries. We do this in an iterative fashion. Let $x^1 := (x_1, \ldots, x_n)$, which has p nonzero entries. We show how to construct a feasible solution $x^2 = (x_1^2, \ldots, x_n^2)$ with $\leq p - 1$ nonzero entries. Because p is finite, repeating this process finitely many times will eventually give us a solution x^k with $\leq m$ nonzero entries such that $Ax^k = b$ and $x^k \geq 0$. Then the a_i's corresponding to nonzero x_i's will form a linearly independent set and the argument in Case 1 can be applied to conclude x^k is a basic feasible solution.

[5] Recall that a degenerate basic feasible solution is simply one where we have a basis, say, $a_1, \ldots, a_q$, with some of the q entries in x corresponding to these q columns are zero.

All that remains for Case 2 is to show how to construct each of the x^ℓ's. First consider x^1. Since $a_1, \ldots, a_p$ are linearly independent, there are $y_1, \ldots, y_p \in \mathbb{R}$ that are not all zero and such that $y = [y_1, \ldots, y_p, 0, \ldots, 0] \in \mathbb{R}^n$ and

$$0 = Ay = y_1 a_1 + \cdots + y_p a_p = 0. \tag{31}$$

If none of the y_i above are positive, then we can multiply the equation by negative one to get one of them to be positive. So, we can assume at least one of the y_i is positive. (This will be important when we define ε below.) Now, recall we have

$$b = Ax = x_1 a_1 + \cdots + x_p a_p. \tag{32}$$

Multiplying (31) by $\varepsilon > 0$ and subtracting it from (32), we discover

$$b = Ax - \varepsilon Ay = A(x - \varepsilon y) = (x_1 - \varepsilon y_1)a_1 + \cdots + (x_p - \varepsilon y_p). \tag{33}$$

Our goal is to pick ε such that $(x_i - \varepsilon y_i) = 0$ for at least one i. This will enable us to define $x^2 := x^1 - \varepsilon y$ so that x^2 has $\leq p-1$ nonzero entries. Picking a nonzero y_i and letting $\varepsilon = x_i/y_i$ would do the trick. However, we must be careful since we need $x^2 = (x^1 - \varepsilon y) \geq 0$. To make this happen, it is sufficient to choose ε to be the smallest possible value x_i/y_i where $y_i > 0$. (Why don't we have to worry about when $y_i < 0$?) In mathematical terms, we set

$$\varepsilon := \min\{x_i/y_i \ \mid \ y_i > 0\}. \tag{34}$$

And, from above, we know this choice of ε is well-defined since at least one of y_i is positive. So, take $x^2 := x^1 - \varepsilon y$. In general, each $x^{\ell+1}$ can be computed from x^ℓ in this fashion. This completes *Case 2*.

ii) Suppose x is an optimal feasible solution with $x_1, x_2, \ldots, x_p > 0$ for $p \leq n$. If $a_1, \ldots, a_p$ are linearly independent, then identical argument to *Case 1* above can be applied to conclude x is basic. Alternatively, suppose $a_1, \ldots, a_p$ are linearly dependent. Then the argument in *Case 2* can be applied to define an iterative procedure to obtain a basic optimal solution x^k. The one remaining detail to verify in order to apply the argument in *Case 2* is to show that the choice of y gives $c^T y = 0$. By way of contradiction, suppose $c^T y \neq 0$. Provided

$|\varepsilon| \leq \min\{|x_i/y_i| \ \mid y_i \neq 0\}$, we can choose ε so that

$$c^T x > c^T x - \varepsilon c^T y = c^T(x - \varepsilon y). \tag{35}$$

This contradicts the fact x is optimal, from which we conclude $c^T y = 0$. This completes the proof.

□

Example 5: From the optimal feasible solution $x = [4, 1, 0, 2]^T \in \mathbb{R}^4$ to the linear program

$$\max\ c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0, \tag{36}$$

construct an optimal basic feasible solution. Here

$$A = [a_1,\ a_2,\ a_3,\ a_4] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}. \tag{37}$$

Solution:

First observe that

$$b = Ax = 4a_1 + a_2 + 2a_4. \tag{38}$$

But, from the definition of A,

$$a_4 = a_2 - a_1, \tag{39}$$

and so a_1, a_2, a_4 are linearly dependent. To find a basic optimal solution, we must find a new optimal feasible solution x^* expressed as a linear combination of a set of linearly independent a_i. We follow the construction in the proof of the Fundamental Theorem of Linear Programming to do this. We can rewrite (39) as

$$0 = a_1 - a_2 + a_4. \tag{40}$$

Then let $y = (1, -1, 0, 1)$ so that

$$0 = Ay \quad \text{and} \quad c^T y = [1,\ 1,\ 0,\ 0] \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot 1 + 1 \cdot (-1) + 0 \cdot 0 + 0 \cdot 1 = 0. \tag{41}$$

Let $\varepsilon \in \mathbb{R}$. Then $0 = \varepsilon 0 = \varepsilon Ay = A(\varepsilon y)$ and we discover

$$b = b - 0 = Ax - A(\varepsilon y) = A(x - \varepsilon y) = (4 - \varepsilon)a_1 + (1 + \varepsilon)a_2 + (2 - \varepsilon)a_4. \tag{42}$$

To construct a new feasible solution as a linear combination of only two of the a_i, we pick ε such that one of the terms cancels above and all the coefficients remain nonnegative. A choice that satisfies this is $\varepsilon = 2$. (What about $\varepsilon = -1$?) From this we see

$$b = (4-2)a_1 + (1+2)a_2 + (2-2)a_4 = 2a_1 + 3a_2. \tag{43}$$

Thus, $\boxed{x^* := x - 2y = [2,\ 3,\ 0,\ 0]^T}$ forms a basic feasible solution. And, by (41) we see

$$c^T x^* = c^T(x-2y) = c^T x - 2c^T y = c^T x - 2 \cdot 0 = c^T x, \tag{44}$$

from which we conclude x^* is an optimal basic feasible solution. □

Example 6: Let's repeat the above problem by instead choosing $\varepsilon = -1$ and seeing what happens. Doing so gives

$$b = (4-(-1))a_1 + (1+(-1))a_2 + (2-(-1))a_4 = 5a_1 + 3a_4, \tag{45}$$

from which we deduce $\boxed{x^* = x + y = [5, 0, 0, 3]^T}$ forms a basic solution. And, by (44), we again conclude this choice of x^* is also a basic optimal feasible solution. We have discovered that there are two basic optimal feasible solutions to this problem. To discover why this is, see the remark on the next page for an illustration.

Remark: Note that in the proof of the FTLP, we took $\varepsilon = \min\{x_i/y_i \ : \ y_i > 0\}$. This uniquely defines $\varepsilon > 0$. But, we could equally well pick $\varepsilon < 0$ (if such a choice cancels one of the coefficients and keeps $x - \varepsilon y \geq 0$). In other words, there may be two possible values for ε, which is fine. For sake of proofs and algorithms, we will typically take $\varepsilon > 0$.

Remark: Earlier this quarter, we talked about how to introduce slack variables to take a linear system of inequalities and rewrite it as a system of equalities. We can sometimes do the reverse process. Namely, the linear program in (36)–(37) can be rewritten as

$$\max c^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0, \tag{46}$$

where

$$A = [a_1,\ a_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}. \tag{47}$$

What we have done is remove the slack variables and replace them with inequalities so that the problem can be expressed solely in terms of x_1 and x_2. This gives the inequalities $x_2 \leq 5 - x_1$ and $x_2 \leq 8 - x_1/2$ and, as usual, $x_1, x_2 \geq 0$. Plotting this, we obtain Figure 4. Note the starting point in the above example in the (x_1, x_2) plane is $(4, 1)$. From the figure, we know $(2, 3)$ and $(5, 0)$ are basic feasible solutions since they are extreme points of Ω. Note also that the level curves of the objective function (shown in blue) are parallel to the constraint $x_2 = 5 - x_1$, and so the maximal value is given along any point along the line segment between $(2, 3)$ and $(5, 0)$. That is, these are all optimal feasible solutions. And, the two "corners" $(2, 3)$ and $(5, 0)$ form basic optimal feasible solutions. This should give intuition as to why we found two possible answers for an optimal basic feasible solution to the linear program.

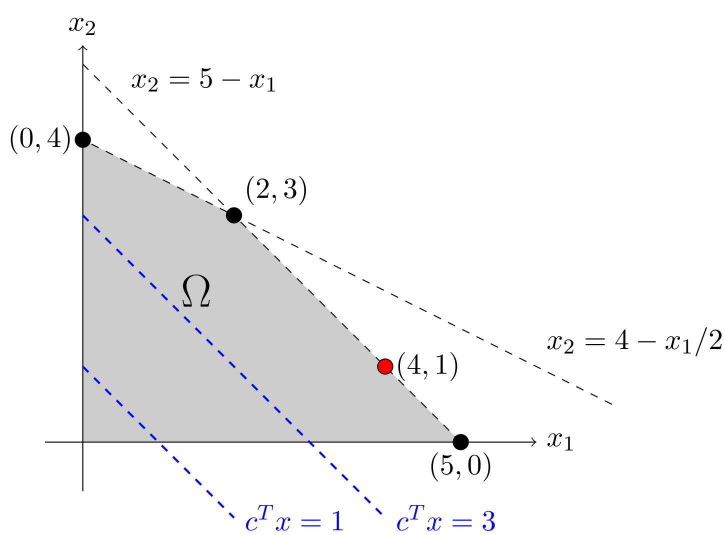

Figure 4: Illustration of the problem in (46)–(47). The level curves of the objective function $c^T x$ are shown in blue and are parallel to the constraint line $x_2 = 5 - x_1$. The red circle denotes initial the optimal feasible point in the above example.

Example 7: Consider the feasible solution $x^1 = [3,\ 3,\ 2,\ 2]^T \in \mathbb{R}^4$ to the linear program

$$\min\ c^T x \qquad \text{subject to} \quad Ax = b,\ x \geq 0 \tag{48}$$

with

$$c = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ 5 \end{bmatrix}. \tag{49}$$

Use x^1 and the ε construction in the proof of the FTLP to find a basic feasible solution.

Solution:

We have

$$b = Ax^1 = 3a_1 + 3a_2 + 2a_3 + 2a_4. \tag{50}$$

In order to be basic, we need to find a feasible solution that has only two nonzero entries (since $\text{rank}(A) = 2$). Note we can write $a_1 = a_2 + a_4$, which implies $0 = -a_1 + a_2 + a_4$. So, setting $y^1 = [-1,\ 1,\ 0,\ 1]^T$, we find $0 = Ay^1$. Then ofr any $\varepsilon^1 \in \mathbb{R}$

$$b = b - \varepsilon^1 0 = Ax^1 - \varepsilon^1 Ay^1 = A(x^1 - \varepsilon^1 y^1) = (3+\varepsilon^1)a_1 + (3-\varepsilon^1)a_2 + 2a_3 + (2-\varepsilon^1)a_4. \tag{51}$$

We need $(x^1 - \varepsilon^1 y^1) \geq 0$ to be feasible and we seek to cancel one of the above coefficients. This can be accomplished by choosing $\varepsilon^1 := 2$. Then set

$$x^2 := x^1 - \varepsilon^1 y^1 = [5,\ 1,\ 2,\ 0]^T \tag{52}$$

and note x^2 is feasible, by construction. However, x^2 is not basic since it has 3 nonzero entries. So, use the fact that $0 = a_2 + a_3$ to pick $y^2 := [0,\ 1,\ 1,\ 0]^T$ so that $Ay^2 = 0$. Then for any $\varepsilon^2 \in \mathbb{R}$

$$b = b - \varepsilon^2 0 = Ax^2 - \varepsilon^2 Ay^2 = A(x^2 - \varepsilon^2 y^2) = 5a_1 + (1-\varepsilon)a_2 + (2-\varepsilon)a_3. \tag{53}$$

Choosing $\varepsilon^2 := 1$, we obtain $(x^2 - \varepsilon^2 y^2) \geq 0$, as desired. Then

$$x^3 := x^2 - \varepsilon^2 y^2 = [5, 0, 1, 0]^T \tag{54}$$

gives a feasible basic solution with respect to $[a_1,\ a_3]$. □

2.3 – Extreme Points & Basic Feasible Solutions:

Definition: Let Ω be a convex set. Then a point x is said to be an *extreme point* of Ω provided that there are no two distinct points in $z_1, z_2 \in \Omega$ and $\alpha \in (0,1)$ such that $x = \alpha z_1 + (1-\alpha)z_2$. In other words, x is a "corner" of the set Ω. △

The second major theorem in Chapter 15 of our text is stated as follows:

Theorem 15.2: Let Ω be the convex set consisting of all feasible solutions x, i.e., all x such that $Ax = b$, $x \geq 0$, where $A \in \mathbb{R}^{m\times n}$ with $m < n$. Then x is an extreme point of Ω if and only if x is a basic feasible solution. △

Proof:

($\Rightarrow$) Assume $x = [x_1, \ldots, x_p, 0, \ldots, 0]^T \in \mathbb{R}^n$ is feasible and an extreme point of Ω with $x_1, \ldots, x_p > 0$. Then let $y_1, \ldots, y_p \in \mathbb{R}$ be such that $0 = y_1 a_1 + \cdots + y_p a_p = Ay$ where $y := [y_1, \ldots, y_p, 0, \ldots, 0]^T$. To show x is basic, it suffices to show $y_1, \ldots, y_p = 0$. Define $z_1 = x - \varepsilon y$ and $z_2 = x + \varepsilon y$, picking $\varepsilon \in \mathbb{R}$ small enough such that $z_1, z_2 \geq 0$ (e.g., pick $\varepsilon = \min\{|x_i/y_i| \ : \ y_i \neq 0\})$). Then

$$Az_1 = Ax - \varepsilon Ay = b - \varepsilon 0 = b \quad \text{and} \quad Az_2 = Ax + \varepsilon Ay = b + \varepsilon 0 = b. \tag{55}$$

This implies z_1 and z_1 are feasible, i.e., $z_1, z_2 \in \Omega$. And, by definition, $x = \frac{1}{2}z_1 + \frac{1}{2}z_2$. Since, by hypothesis, x is an extreme point, it must follow that $z_1 = z_2$, from which we deduce $-y = y$, which implies $y = 0$, as desired.

($\Leftarrow$) Assume $x = [x_1, \ldots, x_p, 0, \ldots, 0]^T \in \mathbb{R}^n$ is a basic feasible solution. Let $y, z \in \Omega$ and $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha)z$. We must show that $y = z$. Since the last $n - p$ components of x are zero, so also must be those for y and z. Thus, we can write $y = [y_1, \ldots, y_p, 0, \ldots, 0]^T$ and $z = [z_1, \ldots, z_p, 0, \ldots, 0]^T$ so that, because y and z are feasible,

$$b = Ay = y_1 a_1 + \ldots y_p a_p \quad \text{and} \quad b = Az = z_1 a_1 + \ldots z_p a_p. \tag{56}$$

Subtracting the equations, we obtain

$$0 = b - b = (y_1 - z_1)a_1 + \ldots + (y_p - z_p)a_p. \tag{57}$$

Since x is basic, $a_1, \ldots, a_p$ are linearly independent. Thence $(y_i - z_i) = 0$ for $i = 1, \ldots, p$, from which we conclude $y = z$, as desired. □

Example 8: Consider the linear program

$$\max c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0 \tag{58}$$

where $x \in \mathbb{R}^4$ and

$$A = [a_1,\ a_2,\ a_3,\ a_4] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}. \tag{59}$$

Starting from the basic feasible solution $x^1 := [0, 4, 1, 0]^T \in \mathbb{R}^4$, use the ε construction procedure to obtain a basic optimal feasible solution to the linear program.

Solution:
The Fundamental Theorem of Linear Programming states that if an optimal feasible solution exists, then so also does a basic optimal feasible solution. So, from our set of basic feasible solutions, we will be able to identify an optimal basic solution by evaluating their objective function values. We proceed using the ε construction to iteratively change the basis of our basic feasible solution.

First observe that $\binom{4}{2} = 6$ and so there are six possible basic solutions since there are four columns of A and any two of them may form a basis for $\mathbb{R}^2$. Using the definition of x^1, we find

$$b = Ax^1 = 4a_2 + a_3. \tag{60}$$

We next obtain a new basic feasible solution by writing another column of A as a linear combination of a_2 and a_3. Indeed, we can write

$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}a_2 + \frac{1}{2}a_3 \quad \Rightarrow \quad 0 = 2a_1 - a_2 - a_3. \tag{61}$$

Let $y^1 := [-2, -1, -1, 0]^T$. Then

$$0 = Ay^1 \quad \Rightarrow \quad 0 = \varepsilon 0 = \varepsilon A y^1 = A(\varepsilon y^1). \tag{62}$$

This implies

$$b = b - 0 = Ax^1 - A(\varepsilon y^1) = A(x^1 - \varepsilon y^1) = -2\varepsilon a_1 + (4+\varepsilon)a_2 + (1+\varepsilon)a_3. \tag{63}$$

We pick ε to cancel out a_2 or a_3 and keep all coefficients nonnegative. A valid choice is $\varepsilon = -1$, which gives

$$b = 2a_1 + 3a_2 + 0a_3 = 2a_1 + 3a_2. \tag{64}$$

Set $x^2 := [2,\ 3,\ 0,\ 0]^T$ and note x^2 is a basic feasible solution with respect to the basis $[a_1,\ a_2]$. We now construct a new basic feasible solution x^3 from x^2. Here note

$$a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_2 - a_1 \quad \Rightarrow \quad 0 = a_1 - a_2 + a_4. \tag{65}$$

Let $y^2 := [1,\ -1,\ 0,\ 1]^T$. Then, as before, $A(\varepsilon y^2) = 0$ and

$$b = b - 0 = Ax - A(\varepsilon y) = A(x - \varepsilon y) = (2-\varepsilon)a_1 + (3+\varepsilon)a_2 - \varepsilon a_4. \tag{66}$$

We pick ε to cancel the coefficient on either a_1 or a_2 and keep all the coefficients nonnegative. This means $\varepsilon = -3$ and so

$$b = 5a_1 + 0a_2 + 3a_4. \tag{67}$$

Thus, setting $x^3 := [5, 0, 0, 3]^T$ gives a basic feasible solution with respect to $[a_2,\ a_4]$. Now we look for a new basic feasible solution x^4. We write a_3 as a linear combination of a_1 and a_4, i.e.,

$$a_1 = a_3 + a_4 \quad \Rightarrow \quad 0 = -a_1 + a_3 + a_4. \tag{68}$$

Set $y^3 := [-1,\ 0,\ 1,\ 1]^T$. Continuing our routine, we see

$$b = A(x^3 - \varepsilon y^3) = (5+\varepsilon)a_1 - \varepsilon a_3 + (3-\varepsilon)a_4 \quad \Rightarrow \quad \text{Pick } \varepsilon = -5. \tag{69}$$

This gives

$$b = 5a_3 + 8a_4 \quad \Rightarrow \quad x^4 := [0,\ 0,\ 5,\ 8]^T. \tag{70}$$

Let us try to construct another feasible solution. Note $a_2 = a_3 + 2a_4$, which implies

$$0 = a_2 - a_3 - 2a_4 \quad \Rightarrow \quad y^4 := [0,\ 1,\ -1,\ -2]^T, \tag{71}$$

and so

$$b = A(x^4 - \varepsilon y^4) = -\varepsilon a_2 + (5+\varepsilon)a_3 + (8+2\varepsilon)a_4. \tag{72}$$

Picking $\varepsilon = -5$ gives $b = 5a_2 - 2a_4$, which corresponds to the basic solution $[0,\ 5,\ 0,\ ,-2]^T$, which is not feasible. And, picking $\varepsilon = -4$ gives us x^1 where we started. So, there is no new option here. Similarly, we can write $a_1 = a_3 + a_4$ to obtain

$$0 = a_1 - a_3 - a_4 \quad \Rightarrow \quad y^4 := [1,\ 0,\ -1,\ -1]^T, \tag{73}$$

which implies

$$b = A(x^4 - \varepsilon y^4) = -\varepsilon a_1 + (5+\varepsilon)a_3 + (8-\varepsilon)a_4. \tag{74}$$

Taking $\varepsilon = -5$ gets us back to x^3 while taking $\varepsilon = 8$ gives $b = 8a_1 - a_3$, which corresponds to the nonfeasible basic solution $[8,\ 0,\ -1,\ 0]^T$. These two above results where we were not able to find a ε yielding a new basic feasible solution show there are two basic nonfeasible solutions.

Now we have our four basic feasible solutions, $x^1,\ x^2,\ x^3,\ x^4$. Observing that

$$c^Tx^1 = 4, \quad c^Tx^2 = 7, \quad c^Tx^3 = 10, \quad c^Tx^4 = 0, \tag{75}$$

we conclude $\boxed{x^3 = [5,0,0,3]^T}$ gives the optimal basic feasible solution.

□

Example 9: Find a solution to

$$\min\ c^T x \quad \text{subject to} \quad Ax \leq b, \quad x \geq 0 \tag{76}$$

where

$$c = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{77}$$

Elaborative Solution:
First we explicitly identify our constraints. We have

$$-x_1 - x_2 \leq -4, \quad x_1 \leq 5, \quad x_1 \geq 0, \quad x_2 \geq 0. \tag{78}$$

Rewriting these, we see $0 \leq x_1 \leq 5$ and the following restriction on x_2:

$$x_2 \geq 4 - x_1 \quad \text{for } 0 \leq x_1 \leq 4 \quad \text{and} \quad x_2 \geq 0 \quad \text{for } 4 \leq x_1 \leq 5. \tag{79}$$

These constraints are visualized in Figure 5.

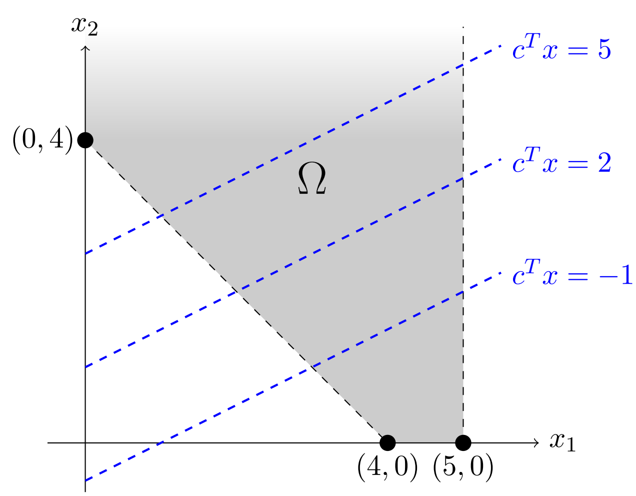

Figure 5: Example 9 illustration. Level curves of the objective are shown in blue. The feasibility region is shaded and labeled here by Ω. Note Ω is unbounded as x_2 gets large.

In Figure 5, $\Omega \subset \mathbb{R}^2$ denotes the feasibility region, which is convex and nonempty. (How would we Ω is convex and nonempty?) Note also that Ω is unbounded since $(0, x_2) \in \Omega$ for all $x_2 \geq 4$. Now Theorem 15.2 tells us that (x_1, x_2) is a basic feasible solution to $Ax = b$, $x \geq 0$ if and only if (x_1, x_2) is an extreme point of Ω. Recall that $(x_1, x_2) \in \Omega$ is an extreme point of Ω if there are no distinct points $(y_1, y_2), (z_1, z_2) \in \Omega$ and $\alpha \in (0,1)$ such that $(x_1, x_2) = \alpha(y_1, y_2) + (1-\alpha)(z_1, z_2)$. Pictorially speaking, these are the "corners" of Ω. That is, in this example, the basis feasible solutions are $(0,4)$, $(4,0)$, and $(5,0)$.

Referring to Figure 5, we see the that as the objective function value decreases, the level curves are shifted downward (i.e., to a lower x_2 value). And, the x_2 components of elements in Ω are bounded below by zero. So, we know there is an optimal solution. The Fundamental Theorem of Linear Programming (Theorem 15.1) states that if an optimal feasible solution exists, then there exists an optimal basic feasible solution. This implies an optimal basic feasible solution is given by one of the extreme points of Ω. Let's evaluate each of these:

$$[-1,\ 2]\begin{bmatrix} 0 \\ 4 \end{bmatrix} = 8, \qquad [-1,\ 2]\begin{bmatrix} 4 \\ 0 \end{bmatrix} = -4, \qquad [-1,\ 2]\begin{bmatrix} 5 \\ 0 \end{bmatrix} = -5. \tag{80}$$

Thence we conclude $(5,0)$ is an optimal basic feasible solution to the problem in (76)–(77). □

Student Solution:
The constraints for this system are given by $0 \leq x_1 \leq 5$ and

$$x_2 \geq 4 - x_1 \quad \text{for } 0 \leq x_1 \leq 4 \quad \text{and} \quad x_2 \geq 0 \quad \text{for } 4 \leq x_1 \leq 5. \tag{81}$$

This feasibility region Ω is visualized in Figure 5. The only extreme points of Ω are $(0,4)$, $(4,0)$, and $(5,0)$. Theorem 15.2 tells us that a point is a feasible solution of $Ax = b$, $x \geq 0$ if and only if it is an extreme point of Ω and so these three listed points give basic feasible solutions. Then the Fundamental Theorem of Linear Programming (Theorem 15.1) states that if an optimal feasible solution exists, then so also does an optimal basic feasible solution. From Figure 5, we see the level curves of the objective function shift downward (i.e., x_2 decreases) as the objective function value decreases. And, the x_2 component of elements in Ω is bounded below. So, an optimal solution subject to the constraints must exist. Thus, by Theorem 15.1, either $(0,4)$, $(4,0)$, or $(5,0)$ gives an optimal basic feasible solution. Evaluating these, we discover

$$[-1,\ 2]\begin{bmatrix} 0 \\ 4 \end{bmatrix} = 8, \qquad [-1,\ 2]\begin{bmatrix} 4 \\ 0 \end{bmatrix} = -4, \qquad [-1,\ 2]\begin{bmatrix} 5 \\ 0 \end{bmatrix} = -5. \tag{82}$$

and, thus, conclude $\boxed{(5,0)}$ is a basic feasible optimal solution to the problem in (76)–(77). □

2.4 – The Simplex Algorithm:

Herein we will describe and provide examples of the simplex algorithm for solving linear programming problems. This method enables us to solve linear programs by essentially moving from one basic solution to another basic solution (with basis differing by one column of A) that has a lesser objective function value. The algorithm terminates when either we arrive at a basic optimal feasible solution or are able to identify the given linear program is unbounded. We begin with a few definitions.

Definition: The *tableau* corresponding to a linear program in standard form (see (1)) is given in these notes by

$$\begin{array}{r|l} A & b \\ \hline c^T & \end{array} \tag{83}$$

△

Definition: The *reduced cost coefficients*, denoted by r_q, are those values in the positions where c^T is above. We shall illustrate more concisely what this means in the following examples. △

Definition: We say a tableau is in *canonical form* with respect to a basis if there are m variables[6] with the property that each appears in precisely one equation, and its coefficient in that equation is unity. △

Simplex Algorithm:

1. Form a canonical tableau for the linear program with respect to some basis.
2. Calculate the reduced cost coefficients r_q corresponding to the nonbasic variables.
3. If $r_j \geq 0$ for each j, stop – the current basic feasible solution is optimal.
4. Pick the first q such that $r_q < 0$.
5. If $a_q \leq 0$, stop – the problem is unbounded. Otherwise, set

$$p = \arg\min_i \left\{ \frac{y_{i0}}{y_{iq}} \;\middle|\; y_{iq} > 0 \right\}. \tag{84}$$

6. Update the tableau by pivoting about the (p, q)-th element.
7. Return to Step 2.

[6]Recall we assume $\text{rank}(A) = m$.

Example 10: Solve the linear program

$$\min\ x_2 - 3x_1 \quad \text{subject to} \quad \begin{array}{rrcr} -x_1 & -2x_2 & \leq & -4, \\ 2x_1 & -x_2 & \leq & 12, \\ x_1 & & \leq & 8, \end{array} \quad \text{and} \quad x_1, x_2 \geq 0. \tag{85}$$

Solution:

First we convert this problem to standard form and write the tableau

$$\begin{array}{c|c} A & b \\ \hline c^T & \end{array} \quad = \quad \begin{array}{rrrrr|r} -1 & -1 & 1 & 0 & 0 & -4 \\ 2 & -1 & 0 & 1 & 0 & 12 \\ 1 & 0 & 0 & 0 & 1 & 8 \\ \hline -3 & 1 & 0 & 0 & 0 & \end{array} \tag{86}$$

Observe this is the canonical tableau with respect to $\{a_3, a_4, a_5\}$. We see only one of the reduced cost coefficients (i.e., the entries in bottom line before the x) is negative. This entry is $r_1 = -3$ and so we pivot about the first column. The p-th row to pivot about is given by

$$p = \underset{i=1,2,3}{\arg\min} \left\{ \frac{y_{i0}}{y_{i1}} \ \mid \ y_{i1} > 0 \right\} = \underset{i=1,2,3}{\arg\min} \left\{ \underbrace{\frac{y_{20}}{y_{21}} = \frac{12}{2} = 6}_{i=2},\ \underbrace{\frac{y_{30}}{y_{31}} = \frac{8}{1} = 8}_{i=3} \right\} = 2. \tag{87}$$

This implies we pivot about $(1,2)$. This gives[7]

$$\begin{array}{rrrrr|r} -1 & -1 & 1 & 0 & 0 & -4 \\ \textcircled{2} & -1 & 0 & 1 & 0 & 12 \\ 1 & 0 & 0 & 0 & 1 & 8 \\ \hline -3 & 1 & 0 & 0 & 0 & \end{array} \quad \begin{array}{r} 2R1 + R2 \to R1 \\ R2/2 \to R2 \\ R3 - R2/2 \to R3 \\ 2R4 + 3R2 \to R4 \end{array} \quad \longrightarrow \quad \begin{array}{rrrrr|r} 0 & -3 & 2 & 1 & 0 & 4 \\ 1 & -1/2 & 0 & 1/2 & 0 & 6 \\ 0 & 1/2 & 0 & -1/2 & 1 & 8 \\ \hline 0 & -1 & 0 & 3 & 0 & \end{array} \tag{88}$$

We now find the reduced cost coefficient $r_2 = -1 < 0$ and so we next pivot about the second column. The p-th row to pivot about is given by

$$p = \underset{i=1,2,3}{\arg\min} \left\{ \frac{y_{i0}}{y_{i2}} \ \mid \ y_{i2} > 0 \right\} = \underset{i=1,2,3}{\arg\min} \underbrace{\left\{ \frac{y_{30}}{y_{32}} = \frac{2}{1/2} = 4 \right\}}_{i=3} = 3. \tag{89}$$

[7]This first time I give the elementary row operations for the pivot. In proceeding computations, these will be omitted their inclusion makes the notation convoluted and their computation should be straightforward.

Pivoting about $(2,3)$ gives

$$\begin{array}{ccccc|c} 0 & -3 & 2 & 1 & 0 & 4 \\ 1 & -1/2 & 0 & 1/2 & 0 & 6 \\ 0 & \textcircled{1/2} & 0 & -1/2 & 1 & 8 \\ \hline 0 & -1 & 0 & 3 & 0 & \end{array} \longrightarrow \begin{array}{ccccc|c} 0 & 0 & 1 & -1 & 0 & 8 \\ 1 & 0 & 0 & 0 & 1 & 8 \\ 0 & 1 & 0 & -1 & 2 & 4 \\ \hline 0 & 0 & 0 & 2 & 2 & \end{array} \tag{90}$$

Note the reduced cost coefficients $[0,\ 0,\ 0,\ 2,\ 2]$ are all nonnegative. Thus, the basic feasible optimal solution is the basic feasible solution with respect to the basis of the current canonical tableau, i.e., $\{a_3, a_1, a_2\}$. This gives $x_{opt} = [8,\ 4,\ 8,\ 0,\ 0]^T$ for the standard form of our linear program. Returning to the initial program form, which only had the variables x_1 and x_2, we conclude the solution is $\boxed{x_{opt} = [8,\ 4]^T.}$ □

Remark: In Figure 6 below, we illustrate the linear program in the above example. The set Ω of feasible solutions is convex and unbounded above. The basic feasible points are labeled. Furthermore, in (90) we are moving from a basis $\{a_3, a_1, a_5\}$ to to the basis $\{a_3, a_1, a_2\}$. These are basic feasible solutions $[6,\ 0,\ 2,\ 0,\ 8]^T$ and $[8,\ 4,\ 8,\ 0,\ 0]^T$, which correspond to the points $[6,\ 0]^T$ and $[8,\ 4]^T$ in the (x_1, x_2) plane. That is, in the last step of the simplex algorithm above we "hopped" from $[6,\ 0]^T$ to $[8,\ 4]^T$ to arrive at the optimal solution.

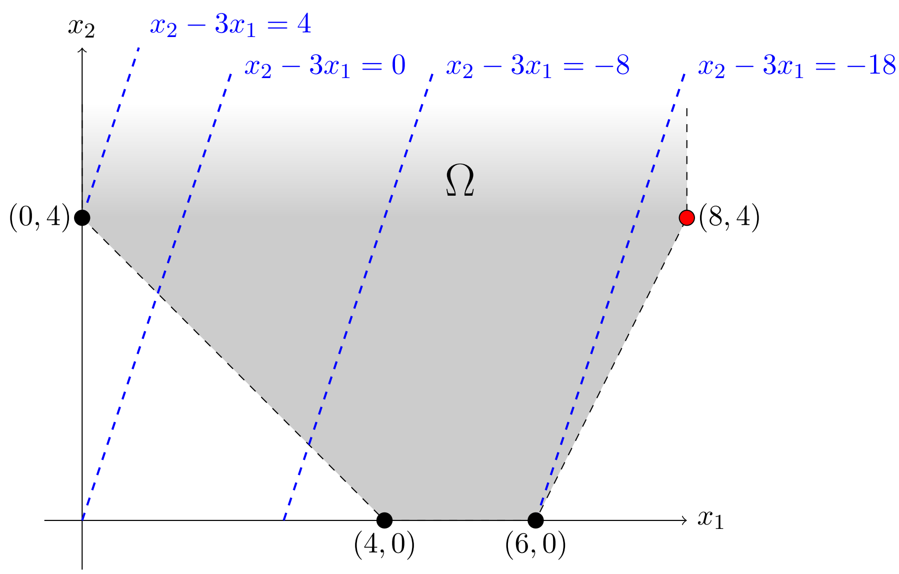

Figure 6: Example 10 illustration. Level curves of the objective are shown in blue. The feasibility region is shaded and labeled here by Ω. Note Ω is unbounded as x_2 gets large.

Example 11: Show the following linear program is unbounded:

$$\max\ -x_1 + 2x_2 \quad \text{subject to} \quad \begin{array}{rrcr} -x_1 & -x_2 & \leq & -4, \\ x_1 & & \leq & 5, \end{array} \quad \text{and} \quad x_1, x_2 \geq 0. \tag{91}$$

Solution:

We first convert the linear program to its standard form (i.e., flipping things to make it a minimization problem and adding slack variables) and write the tableau:

$$\begin{array}{c|c} A & b \\ \hline c^T & \end{array} \quad = \quad \begin{array}{rrrr|r} -1 & -1 & 1 & 0 & -4 \\ 1 & 0 & 0 & 1 & 5 \\ \hline 1 & -2 & 0 & 0 & \end{array} \tag{92}$$

This tableau is in canonical form with respect to the basis $\{a_3, a_4\}$. To apply the simplex algorithm, we pivot about the second column since $r_2 = -2 < 0$. But, $a_2 \leq 0$ and so the problem is unbounded. We show this as follows.

Let the nonbasic variable x_1 be zero and set $x_2 = k$. Then we use the linear system to solve for x_3 and x_4, i.e.,

$$\left.\begin{array}{rcll} -4 & = & -x_1 - x_2 + x_3 + 0 & = 0 - k + x_3 + 0 \\ 5 & = & x_1 + 0 + 0 + x_4 & = 0 + 0 + 0 + 0 \end{array}\right\} \quad \Rightarrow \quad \begin{cases} x_3 = k - 4, \\ x_4 = 5. \end{cases} \tag{93}$$

Then define $x^k := [0,\ k,\ k-4,\ 5]^T$. By construction, we have $Ax^k = b$. And, for $k \geq 4$ we have $x^k \geq 0$. We can therefore define a sequence $\{x^k\}_{k=4}^{\infty}$ of feasible solutions. But, using the fact $c^T = [1,\ -2,\ 0,\ 0]$, we discover

$$\lim_{k\to\infty} c^T x^k = \lim_{k\to\infty} 1 \cdot 0 - 2 \cdot k = \lim_{k\to\infty} -2k = -\infty. \tag{94}$$

Thus the objective values are unbounded, from which we conclude the linear program is unbounded.

□

Remark: Again, let us ask what is geometrically happening in the above example? We illustrate the linear program below in Figure 7. The points x^4, x^5, and x^6 are plotted along the x_2 axis. The objective function value for x^k is $2k$ and as k increases, we see the objective function values go to infinity. Thus, the problem is unbounded.

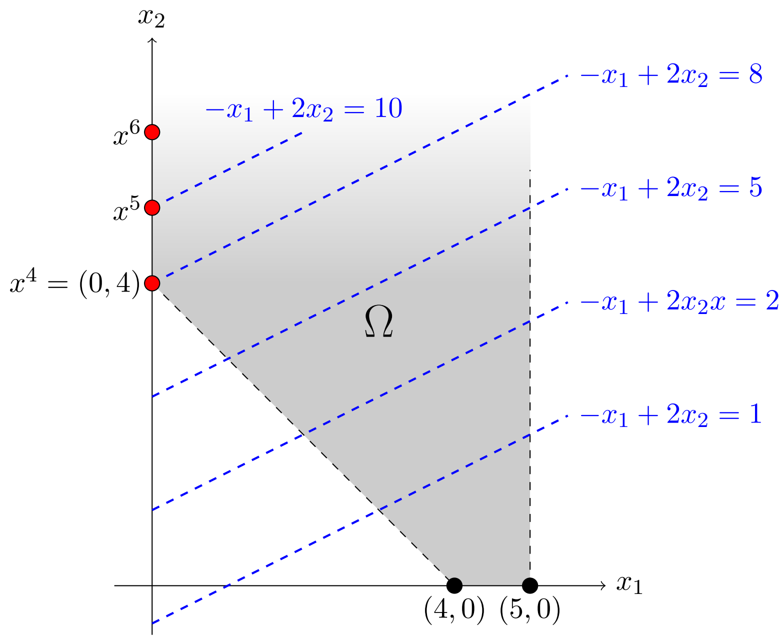

Figure 7: Example 11 illustration. Level curves of the objective are shown in blue. The feasibility region is shaded and labeled here by Ω. Note Ω is unbounded as x_2 gets large.

Remark: If we are asked to show a linear program is unbounded, we proceed as follows. First write a canonical tableau for the linear program. Then apply the simplex algorithm until one of the columns, say column q, in the tableau has a negative reduced cost coefficient (i.e., $r_q < 0$) and the entries in a_q are nonpositive. We then construct a sequence of x^k. Let all the nonbasic variables of x^k be zero, except set $(x^k)_q = k$. Then solve for the remaining basic variables in terms of k. Then take the limit of the objective function values for x^k as $k \longrightarrow \infty$. If the limit is infinite, then we have shown there is a sequence of feasible x^k with unbounded function values, as desired.

Example 12: Consider the tableau

$$\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 3 \\ 1 & -1 & 2 & 0 & 1 & 3 \\ \hline 2 & -1 & 0 & 0 & 0 & \end{array} \tag{95}$$

Use the simplex algorithm to solve the corresponding linear program.

Solution:

We first note the given tableau is not in canonical form, which is needed to use the simplex algorithm. To put it in canonical form with respect to $\{a_1, a_2, a_3\}$, we pivot about $(1, 1)$ to get the tableau

$$\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ \hline 0 & 1 & -2 & 0 & 0 & \end{array} \tag{96}$$

We now pivot about a position in the third column since $r_3 = -2 < 0$ is the only negative reduced cost coefficient. The row p to pivot about is

$$p = \underset{i=1,2,3}{\arg\min} \left\{ \frac{y_{i0}}{y_{i3}} \mid y_{i3} > 0 \right\} = \underset{i=1,2,3}{\arg\min} \left\{ \underbrace{\frac{y_{20}}{y_{23}} = \frac{1}{1} = 1}_{i=1}, \underbrace{\frac{y_{30}}{y_{33}} = \frac{2}{1} = 2}_{i=3} \right\} = 1. \tag{97}$$

Pivoting around $(1, 3)$ gives the tableau

$$\begin{array}{ccccc|c} 1 & -1 & \textcircled{1} & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ \hline 0 & 1 & -2 & 0 & 0 & \end{array} \quad \longrightarrow \quad \begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 3 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ \hline 2 & -1 & 0 & 0 & 0 & \end{array} \tag{98}$$

We see $r_2 = -1 < 0$ and so we pivot now about a position in the second column. As before, the row p is given by

$$p = \underset{i=1,2,3}{\arg\min} \left\{ \frac{y_{i0}}{y_{i2}} \mid y_{i2} > 0 \right\} = \underset{i=1,2,3}{\arg\min} \left\{ \underbrace{\frac{y_{20}}{y_{22}} = \frac{3}{1} = 3}_{i=2}, \underbrace{\frac{y_{30}}{y_{32}} = \frac{1}{1} = 1}_{i=3} \right\} = 3. \tag{99}$$

Pivoting about $(3, 2)$ yields

$$\begin{array}{rrrrr|r} 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 3 \\ -1 & \textcircled{1} & 0 & 0 & 1 & 1 \\ \hline 2 & -1 & 0 & 0 & 0 & \end{array} \longrightarrow \begin{array}{rrrrr|r} 0 & 0 & 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 & -1 & 2 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & \end{array} \tag{100}$$

Since all the reduced cost coefficients are nonnegative, we conclude the optimal feasible solution is the basic solution with respect to $\{a_3, a_4, a_2\}$. Namely, $\boxed{x_{opt} = [0,\ 1,\ 2,\ 2,\ 0]^T.}$ □

REMARK: Once more, let's investigate this example geometrically. A linear program corresponding to the tableau on the left hand side in (100) is

$$\min\ 2x_1 - x_2 \quad \text{subject to} \quad \begin{aligned} x_1 - x_2 &\leq 1, \\ x_1 + x_2 &\leq 3, \\ -x_1 + x_2 &\leq 1, \end{aligned} \quad \text{and} \quad x_1, x_2 \geq 0. \tag{101}$$

We illustrate the linear program below in Figure 8. As we may deduce geometrically, the optimal solution is $[x_1,\ x_2] = [0,\ 1]^T$, which is consistent with our result in the above example.

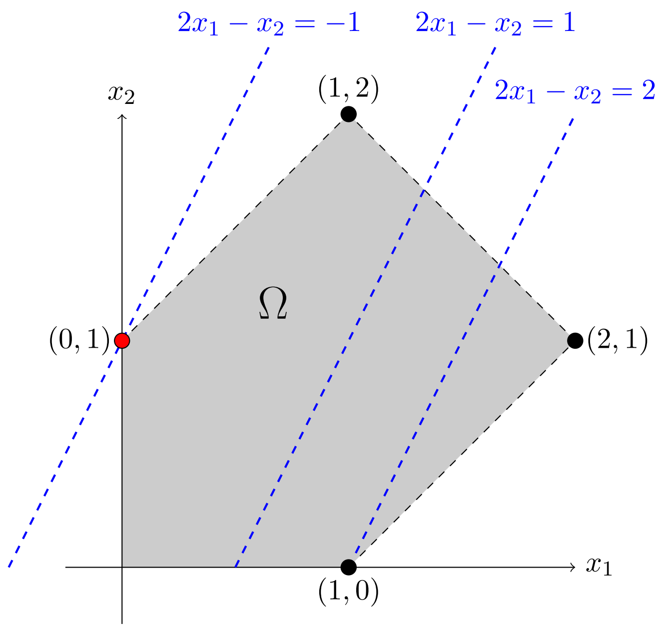

Figure 8: Example 12 illustration. Level curves of the objective are shown in blue. The feasibility region is shaded and labeled here by Ω. Note Ω is unbounded as x_2 gets large.

Example 13: Use the simplex algorithm to solve the linear program

$$\min -2x_1 - 5x_2 \quad \text{subject to} \quad 0 \le x_1 \le 4, \quad 0 \le x_2 \le 6, \quad x_1 + x_2 \le 8. \tag{102}$$

Solution:

First convert this program to standard form and write the corresponding tableau

$$\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 1 & 0 & 0 & 1 & 8 \\ \hline -2 & -5 & 0 & 0 & 0 & \end{array} \tag{103}$$

Note this tableau is in canonical form with respect to the basis $\{a_3, a_4, a_5\}$. Since $r_1 = -2 < 0$ and $r_2 = -5 < 0$, we can pivot around either the first column or the second. Let's pick to pivot around the second column to bring a_2 into the basis. The row p of the second column about which to pivot is given by

$$p = \underset{i=1,2,3}{\arg\min} \left\{ \frac{y_{i0}}{y_{i2}} \;\middle|\; y_{i2} > 0 \right\} = \underset{i=1,2,3}{\arg\min} \left\{ \underbrace{\frac{y_{20}}{y_{22}} = \frac{6}{1} = 6}_{i=2}, \underbrace{\frac{y_{30}}{y_{32}} = \frac{8}{1} = 8}_{i=3} \right\} = 2. \tag{104}$$

Pivoting about $(2,2)$ yields

$$\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & \textcircled{1} & 0 & 1 & 0 & 6 \\ 1 & 1 & 0 & 0 & 1 & 8 \\ \hline -2 & -5 & 0 & 0 & 0 & \end{array} \longrightarrow \begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \\ \hline -2 & 0 & 0 & 5 & 0 & \end{array} \tag{105}$$

Since $r_1 = -2 < 0$, we now pivot about the a position in the first column. The row p is given by

$$p = \underset{i=1,2,3}{\arg\min} \left\{ \frac{y_{i0}}{y_{i1}} \;\middle|\; y_{i1} > 0 \right\} = \underset{i=1,2,3}{\arg\min} \left\{ \underbrace{\frac{y_{10}}{y_{11}} = \frac{4}{1} = 4}_{i=1}, \underbrace{\frac{y_{30}}{y_{31}} = \frac{2}{1} = 2}_{i=3} \right\} = 3. \tag{106}$$

Pivoting about $(3, 1)$ yields

$$\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ \textcircled{1} & 0 & 0 & -1 & 1 & 2 \\ \hline -2 & 0 & 0 & 5 & 0 & \end{array} \quad \longrightarrow \quad \begin{array}{rrrrr|r} 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \\ \hline 0 & 0 & 0 & 3 & 2 & \end{array} \tag{107}$$

Since all the reduced cost coefficients are nonnegative, the optimal feasible solution is the basic solution with respect to the basis of this tableau, i.e., $\{a_3, a_2, a_1\}$. Thus, $[2,\ 6,\ 2,\ 0,\ 0]^T$ gives the optimal solution to the standard form of the linear program. Returning to the original problem, we conclude $\boxed{x_{opt} = [2,\ 6]^T.}$ $\square$

2.5 – Duality in Linear Programming:

In this section, we introduce and discuss duality in linear programming. Say we are given a linear programming problem. We shall call this original problem the 'primal' problem. Then we can construct a corresponding 'dual' linear programming problem. This is noteworthy because the solution to the primal solution can be obtained from the solution to the dual, and vice versa. And, sometimes one of the problems may be much easier to solve than the other. Suppose we have the primal problem

$$\min\ c^T x \qquad \text{subject to} \quad Ax \geq b, \quad x \geq 0. \tag{P}$$

The corresponding dual problem is

$$\max\ b^T \lambda \qquad \text{subject to} \quad A^T \lambda \leq c, \quad \lambda \geq 0. \tag{D}$$

Note $x \in \mathbb{R}^n$ while $\lambda \in \mathbb{R}^m$. We may use the above definition of duality to also derive a dual corresponding to the primal problem

$$\min\ c^T x \qquad \text{subject to} \quad Ax = b, \quad x \geq 0. \tag{P}$$

Namely,

$$\max\ b^T \lambda \qquad \text{subject to} \quad A^T \lambda \leq c^T. \tag{D}$$

Note here there is *no restriction* on the sign of the terms in λ. From the above, we notice a trend in a certain type of symmetry between the primal and dual problems. We consider the problems to have a symmetric form if they have corresponding types of inequalities. An asymmetric form arises when one of the problems has a strict equality and its dual does not have a restriction on the parameter. Following Tables 17.1 and 17.2 in our text, we provide the following table as summary.

Primal	Dual
$\min\ c^T x$	$\max\ \lambda^T b$
subject to $Ax \geq b$	subject to $\lambda^T A \leq c^T$
$x \geq 0$	$\lambda \geq 0$

Table 1: Symmetric Form of Duality

Primal	Dual
$\min\ c^T x$	$\max\ \lambda^T b$
subject to $Ax = b$	subject to $\lambda^T A \leq c^T$
$x \geq 0$	

Table 2: Asymmetric Form of Duality

In these notes, we present two methods for finding the dual to a corresponding problem. The first is to note every problem we are interested in can be written in the form of one of the above problems. Then we can rearrange our problem at hand to match the above form, and substitute in the result. We illustrate this with the following example.

Example 14: Find the dual to the problem

$$\min\ c^T x \qquad \text{subject to} \quad Ax \le b, \quad x \ge 0. \tag{P}$$

Solution:
This problem can be rewritten as

$$\min\ c^T x \quad A^* x \ge b^*, \quad x \ge 0 \tag{108}$$

where $A^* = -A$ and $b^* = -b$. Then using Table 1 we know the dual is

$$\max\ (b^*)^T \lambda \qquad \text{subject to} \quad (A^*)^T \lambda \le c, \quad \lambda \ge 0, \tag{109}$$

or, equivalently,

$$\boxed{\max\ -b^T \lambda \qquad \text{subject to} \quad -A^T \lambda \le c, \quad \lambda \ge 0.} \tag{110}$$

□

Remark: As near as I can tell, this is how the book expects these problems to be solved. So, you can memorize the tables for the symmetric and asymmetric forms and use the above type substitution. Or, you can use the latter method (below), whichever is easier.

Weak Duality: Suppose x and λ are feasible solutions to corresponding primal and dual problems. Then $c^T x \geq \lambda^T b$. $\triangle$

Strong Duality: If the primal problem has an optimal solution, say x_{opt}, then so does the dual, say λ_{opt}, and $c^T x_{opt} = b^T \lambda_{opt}$. $\triangle$

REMARK: From these two theorems, the following theorem may be intuitive.

A condition showing optimality: If x and λ are feasible solutions to corresponding primal and dual problems and $c^T x = b^T \lambda$, then x and λ are optimal solutions to their corresponding problems. $\triangle$

Example 15: Consider the problem of Example 9, which is

$$\min\ -x_1+2x_2 \qquad \text{subject to} \quad -x_1-x_2\le -4,\ \ x_1\le 5,\ \ x_1,x_2\ge 0. \tag{P}$$

Write down and solve the dual of the linear program. Then, using the fact $x_{opt}=[5,\ 0]^T$ solves the primal problem, verify the strong duality theorem holds in this case.

Solution:

We can write this problem in matrix form as

$$\min\ \underbrace{[-1,\ 2]}_{c^T}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{x} \quad \text{subject to} \quad \underbrace{\begin{bmatrix}1&1\\-1&0\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix}\ge\underbrace{\begin{bmatrix}4\\-5\end{bmatrix}}_{b},\quad \begin{bmatrix}x_1\\x_2\end{bmatrix}\ge\begin{bmatrix}0\\0\end{bmatrix}. \tag{111}$$

The dual problem is then

$$\max\ b^T\lambda \qquad \text{subject to} \quad A^T\lambda\le c,\ \ \lambda\ge 0 \tag{D}$$

where A, b, and c are defined to be the underbraced quantities above. Then we introduce slack variables and write our problem in standard form, i.e.,

$$\min -4\lambda_1+5\lambda_2 \qquad \text{subject to} \quad \lambda_1-\lambda_2+\lambda_3=-1,\ \ \lambda_1+\lambda_4=2,\ \ \lambda_1,\lambda_2,\lambda_3,\lambda_4\ge 0. \tag{112}$$

Writing the corresponding tableau and pivoting about the (1,2) entry, we obtain[8]

$$\begin{array}{cccc|c} 1 & \textcircled{-1} & 1 & 0 & -1\\ 1 & 0 & 0 & 1 & 2\\ \hline -4 & 5 & 0 & 0 & \end{array} \quad\longrightarrow\quad \begin{array}{cccc|c} -1 & 1 & -1 & 0 & 1\\ 1 & 0 & 0 & 1 & 2\\ \hline 1 & 0 & 5 & 0 & \end{array} \tag{113}$$

Because the reduced cost coefficients are nonnegative, this shows[9] an optimal basic feasible solution to (112) is $[0,\ 1,\ 0,\ 2]^T$. This implies an optimal basic feasible solution to the dual problem (D) is $\boxed{\lambda_{opt}=[0,\ 1]^T.}$ Lastly, observe that

$$c^Tx_{opt}=-1\cdot 5+2\cdot 0=-5=4\cdot 0-5\cdot 1=b^T\lambda_{opt}, \tag{114}$$

which is precisely as claimed by the strong duality theorem. □

[8] We pivot about this entry since the current tableau corresponds to a basic solution that is not feasible.
[9] Note the tableau is in the canonical form with respect to the basis $[a_2\ a_4]$.

Remark: **You do not have to know following method using $\mathcal{L}$ to convert between the primal and dual problems.** For some, this approach may be more intuitive and give insight into the relation between these two problems and what the duality theorems tell us. This approach may also be considered easier because there isn't much to memorize.

To give insight as to where these primal and dual problems come from, we shall briefly introduce the notion of a Lagrangian. In this context, the Lagrangian, denoted by $\mathcal{L}(x, \lambda)$, is a linear function with two arguments, namely, $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. The first argument x is the *primal vector* and the second argument λ is the *dual vector*. The *primal problem* is given by

$$\min_x \max_\lambda \mathcal{L}(x, \lambda) \tag{P}$$

and the *dual problem* by

$$\max_\lambda \min_x \mathcal{L}(x, \lambda). \tag{D}$$

Note well that we may include restrictions on the ranges of x and λ used in the minimization and maximization, respectively (e.g., we often have "$\min_{x \geq 0}$"). And, these same restrictions must be kept when converting between the two problems. The examples on the following pages illustrate both how to find the Lagrangian corresponding to a linear program and also how to then find the unknown primal or dual problem. In these examples, the underlying procedure for converting from a primal to a dual problem is as follows.

1. Write the primal problem as an unconstrained problem.[10]
2. Rewrite the objective function as the maximization over $\lambda \geq 0$.
3. Write the Lagrangian, collecting the λ terms together.
4. Write dual by swapping the order of maximization over λ and minimization over x
5. Simplify.

The process for converting to the dual from the primal is analogous.

Remark: It is important to pay careful attention to **keep the consistent restrictions on x and λ** when using this approach. For example, if we are given a dual problem with no restriction on λ, then we should *not* introduce a restriction on λ when we go to compute the corresponding primal problem.

[10] That is, let the objective function goes to $\pm\infty$ if the constraints are not satisfied. This can be done by breaking the objective function into two cases.

Example 16: Find the dual to the linear program

$$\min\ c^T x \quad \text{subject to} \quad Ax \geq b,\ x \geq 0. \tag{P}$$

Elaborative Solution:
We begin by trying to find the Lagrangian associated with this problem. Observe we can equivalently write the primal problem (P) as

$$\min_{x\geq 0} \begin{cases} c^T x & \text{if } Ax - b \geq 0, \\ \infty & \text{otherwise.} \end{cases} \tag{115}$$

We may wonder why such a formulation is useful, but this will become more apparent if we try to now write this separated expression as the solution to a maximization problem. Namely, (115) can be equivalently written as

$$\min_{x\geq 0} \max_{\lambda \geq 0} c^T x - \lambda^T (Ax - b). \tag{116}$$

Indeed, if $Ax - b < 0$, then we can make this expression blow up. But, if $Ax - b \geq 0$, then the expression is maximized when $\lambda = 0$. This shows the Lagrangian $\mathcal{L}$ for this problem is given by

$$\mathcal{L}(x, \lambda) = c^T x - \lambda^T (Ax - b). \tag{117}$$

The dual problem is given by swapping the order of the minimization over x and the maximization over λ. To make the results more clear, it is best to first rewrite the Lagrangian combining the terms with x first, i.e.,

$$\mathcal{L}(x, \lambda) = \lambda^T b + (c^T - \lambda^T A)x. \tag{118}$$

Then the dual problem is given by

$$\begin{aligned} \max_{\lambda\geq 0} \min_{x \geq 0} L(x, \lambda) &= \max_{\lambda\geq 0} \min_{x \geq 0} \lambda^T b - (\lambda^T A + c^T)x \\ &= \max_{\lambda \geq 0} \begin{cases} \lambda^T b & \text{if } c^T - \lambda^T A \geq 0, \\ -\infty & \text{otherwise,} \end{cases} \\ &= \max_{\lambda\geq 0}\ \lambda^T b \qquad \text{subject to} \quad c^T - \lambda^T A \geq 0 \\ &= \max_{\lambda \geq 0} \lambda^T b \quad \text{subject to} \quad \lambda^T A \leq c^T \\ &= \boxed{\max \lambda^T b \quad \text{subject to} \quad \lambda^T A \leq c^T, \quad \lambda \geq 0.} \end{aligned} \tag{D}$$

□

Student Solution:

The primal problem may be rewritten as

$$\min_{x\geq 0} \begin{cases} c^T x & \text{if } Ax - b \geq 0, \\ \infty & \text{otherwise.} \end{cases} = \min_{x\geq 0} \max_{\lambda \geq 0} c^T x - \lambda^T (Ax - b). \tag{119}$$

This implies the Lagrangian $\mathcal{L}(x, \lambda)$ is given by

$$\mathcal{L}(x, \lambda) = c^T x - \lambda^T (Ax - b) = \lambda^T b + (c^T - \lambda^T A)x, \tag{120}$$

and so, reordering the max and min, we see the dual problem is

$$\begin{aligned} \max_{\lambda\geq 0} \min_{x\geq 0} \mathcal{L}(x,\lambda) &= \max_{\lambda\geq 0} \min_{x\geq 0} \lambda^T b + (c^T - \lambda^T A)x \\ &= \max_{\lambda \geq 0} \begin{cases} \lambda^T b & \text{if } c^T - \lambda^T A \geq 0, \\ -\infty & \text{otherwise,} \end{cases} \\ &= \boxed{\max \ \lambda^T b \qquad \text{subject to} \quad \lambda^T A \leq c^T, \quad \lambda \geq 0.} \end{aligned} \tag{D}$$

□

Example 17: Find the dual of the linear program

$$\min c^T x \quad \text{subject to} \quad Ax \geq b, \quad x \geq 0. \tag{P}$$

Student Solution:

We may rewrite the primal problem as

$$\min_{x\geq 0} \begin{cases} c^T x & \text{if } Ax - b \geq 0, \\ \infty & \text{otherwise,} \end{cases} = \min_{x\geq 0} \max_{\lambda \geq 0} c^T x - \lambda^T (Ax - b), \tag{121}$$

from which we deduce the Lagrangian $\mathcal{L}(x, \lambda)$ is given by

$$\mathcal{L}(x, \lambda) = c^T x - \lambda^T (Ax - b) = \lambda^T b + (c^T - \lambda^T A)x. \tag{122}$$

Thus the dual problem is

$$\begin{aligned} \max_{\lambda\geq 0} \min_{x\geq 0} \mathcal{L}(x, \lambda) &= \max_{\lambda\geq 0} \min_{x\geq 0} \lambda^T b + (c^T - \lambda^T A)x \\ &= \max_{\lambda\geq 0} \begin{cases} \lambda^T b & \text{if } c^T - \lambda^T A \geq 0, \\ -\infty & \text{otherwise.} \end{cases} \\ &= \boxed{\max \lambda^T b \quad \text{subject to} \quad \lambda^T A \leq c^T, \quad \lambda \geq 0.} \end{aligned} \tag{D}$$

□

Remark: The above procedure applies equally well for going from the dual problem to the primal problem, as illustrated in the next example. Carefully note that in the following problem we do not place a nonnegativity restriction (i.e., $\lambda \geq 0$) on the maximization to take place over λ.

Example 18: Find the primal problem corresponding to the dual problem

$$\max \lambda^T b \quad \text{subject to} \quad \lambda^T A \leq c^T. \tag{D}$$

Solution:

The dual problem may be rewritten as

$$\max_{\lambda} \begin{cases} \lambda^T b & \text{if } \lambda^T A - c^T \leq 0, \\ -\infty & \text{otherwise} \end{cases} = \max_{\lambda} \min_{x \geq 0} \lambda^T b - (\lambda^T A - c^T)x. \tag{123}$$

This implies the Lagrangian $\mathcal{L}(x, \lambda)$ is

$$\mathcal{L}(x, \lambda) = \lambda^T b - (\lambda^T A - c^T)x = c^T x - \lambda^T (Ax - b) \tag{124}$$

where the second equality is a rearrangement of the first one, collecting the λ terms together. The primal problem is given by reversing the order of the minimization and maximization, i.e.,

$$\begin{aligned} \min_{x \geq 0} \max_{\lambda} L(x, \lambda) &= \min_{x \geq 0} \max_{\lambda} c^T x - \lambda^T (Ax - b) \\ &= \min_{x \geq 0} \begin{cases} c^T x & \text{if } Ax - b = 0, \\ \infty & \text{otherwise,} \end{cases} \\ &= \boxed{\min c^T x \quad \text{subject to} \quad Ax = b, \quad x \geq 0.} \end{aligned} \tag{P}$$

□

Remark: We now discuss the duality theorems. To give intuition into the first of these, recall the primal and dual problems are, respectively,

$$\min_x \max_\lambda \ \mathcal{L}(x,\lambda) \quad \text{and} \quad \max_\lambda \min_x \ \mathcal{L}(x,\lambda). \tag{125}$$

In terms of $\mathcal{L}(x,\lambda)$, the **weak duality theorem** states

$$\max_\lambda \ \mathcal{L}(x,\lambda) \geq \min_x \ \mathcal{L}(x,\lambda). \tag{126}$$

In linear programming, we also have **strong duality.** This is given Theorem 17.2 in our text, and it states

$$c^T x_{opt} = \min_x \max_\lambda \ \mathcal{L}(x,\lambda) = \max_\lambda \min_x \ \mathcal{L}(x,\lambda) = \lambda_{opt}^T b. \tag{127}$$

Example 19: In Example 13, we had the primal problem

$$\min\ -2x_1 - 5x_2 \quad \text{subject to} \quad 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ x_1 + x_2 \le 8. \tag{P}$$

Write out the dual to this problem and find λ_{opt}.

Hint: Write the tableau with respect to the basis λ_2, λ_3.

Solution:

Note this problem can be written in the form

$$\min c^T x \qquad \text{subject to} \quad Ax \le b,\ x \ge 0 \tag{128}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}. \tag{129}$$

Letting $A^* = -A$ and $b^* = -b$, the problem becomes

$$\min c^T x \qquad \text{subject to} \quad A^* x \ge b^*,\ x \ge 0. \tag{130}$$

The dual of this is given in Table 1 above (and also in Table 17.1 of our text) to be

$$\max \lambda^T b^* \qquad \text{subject to} \quad \lambda^T A^* \le c^T,\ \lambda \ge 0. \tag{131}$$

Substituting for A^* and b^*, we can rearrange our expression to write

$$\max\ -b^T \lambda \qquad \text{subject to} \quad A^T \lambda \ge -c,\ \lambda \ge 0. \tag{D}$$

We can introduce slack variables and write this in standard form to get

$$\min 4\lambda_1 + 6\lambda_2 + 8\lambda_3 \qquad \text{subject to} \quad \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}. \tag{132}$$

The corresponding tableau is

$$\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & \textcircled{1} & 1 & 0 & -1 & 5 \\ \hline 4 & 6 & 8 & 0 & 0 & \end{array}. \tag{133}$$

Making use of the hint, let's pivot about the $(2, 2)$ entry and then the $(1, 3)$ entry:

$$\begin{array}{ccccc|c} 1 & 0 & \textcircled{1} & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 5 \\ \hline 4 & 0 & 2 & 0 & 1 & \end{array} \longrightarrow \begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ \hline 2 & 0 & 0 & 2 & 2 & \end{array} \tag{134}$$

This tableau is now in the canonical form with respect to the basis $[\lambda_3, \lambda_2]$. Because each of the reduced cost coefficients are nonnegative, we can immediately write off the basic optimal solution to be $[\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5]^T = [0,\ 3,\ 2,\ 0,\ 0]^T$. From this, we deduce the solution to the original dual problem (D) is $\boxed{\lambda_{opt} = [0,\ 3,\ 2]^T.}$ □

Remark: In Example 13, we found $x_{opt} = [2,\ 6]^T$. Note that

$$\lambda_{opt}^T b^* = -4 \cdot 0 - 6 \cdot 3 - 8 \cdot 2 = -34 \quad \text{and} \quad c^T x_{opt} = -2 \cdot 2 - 5 \cdot 6 = -34. \tag{135}$$

This is consistent with the strong duality theorem.

Remark: The work required to solve the primal or dual problems can vary. One may be much easier than the other. And, the form you write them in also has an impact. For instance, introducing slack variables in Example 19, we can write the primal problem as

$$\min\ c^T x \qquad \text{subject to} \quad Ax = b,\ x \geq 0 \tag{136}$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{137}$$

From[11] Table 2, we can write immediately write the form of the dual to be

$$\max\ \lambda^T b \qquad \text{subject to} \quad A^T \lambda \leq c. \tag{138}$$

Plugging in our values, we get the monstrous problem

$$\max\ 4\lambda_1 + 6\lambda_2 + 8\lambda_3 \quad \text{subject to} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \leq \begin{bmatrix} -2 \\ -5 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{139}$$

And, to use the simplex algorithm, we would still need to introduce additional slack variables so that we have strict equality constraints. This should reveal the original approach was *much* easier.

[11] Note this is also found in Table 17.2 of our text. Or, if you forget it, you can derive it.

Section 3: Calculus Based Methods

We now transition to calculus based optimization. This means most of our arguments will be about smooth functions and we will be playing close attention to derivatives.

3.1 – Methods of Finding Roots:

A first-order necessary condition for a point x^* to be a minimizer of a smooth function f is that $f'(x^*) = 0$. So, we call discuss three methods for finding zeros of functions: the bisection method, Newton's method, and the secant method.

3.1.1 – Bisection Method:

The *bisection method* is a simple algorithm used to find zeros of continuous functions. It does this by repeated application of the intermediate value theorem (IVT). It works as follows. Suppose we have an interval $[a, b]$ and a function f for which $f(a) \cdot f(b) < 0$ (i.e., $f(a)$ and $f(b)$ have different signs). Then the IVT asserts there is $c \in (a, b)$ such that $f(c) = 0$. If $f((a + b)/2) = 0$, then we take $c = (a + b)/2$. Now suppose this is not the case. If the sign of $f((a + b)/2)$ matches the sign of $f(b)$, then we know $c \in [a, (a + b)/2]$. Otherwise, $c \in [(a + b)/2, b]$. We then repeat this process on our new interval.

We now list the algorithm for the bisection explicitly. Assume an interval $[a, b]$ is given and f is continuous with $f(a) \cdot f(b) < 0$. Let $\varepsilon > 0$ be the tolerance for our approximation. We perform the following steps:

1. Set $a_1 \leftarrow a$ and $b_1 \leftarrow b$ and $k \leftarrow 1$.
2. Set $x^k \leftarrow (a_k + b_k)/2$.
3. If $\text{sgn}\left(f(x^k)\right) = \text{sgn}\left(f(b_k)\right)$, set $a_{k+1} \leftarrow a_k$ and $b_{k+1} \leftarrow x^k$.
   If $\text{sgn}\left(f(x^k)\right) = \text{sgn}\left(f(a_k)\right)$, set $a_{k+1} \leftarrow x^k$ and $b_{k+1} \leftarrow b_k$.
   If $f(x^k) = 0$, stop. Here x^k is a root of f.
4. If $b_k - a_k < 2\varepsilon$, stop because there is $x^* \in [a_k, b_k]$ with $|x^k - x^*| < \varepsilon$ and $f(x^*) = 0$.
   Otherwise, set $k \leftarrow k + 1$ and return to Step 2.

Convergence of the Bisection Method: Suppose $f \in C[a,b]$ and $f(a)\cdot f(b) < 0$. The bisection method generates a sequence $\{x^k\}_{k=1}^{\infty}$ such that $x^k \longrightarrow x^* \in [a,b]$ with $f(x^*) = 0$. Moreover, we have the error estimate

$$|x^k - x^*| \leq \frac{b-a}{2^k} \quad \forall\ k \geq 1. \tag{140}$$

△

Proof:

For each $k \geq 1$, w have

$$b_k - a_k = \frac{(b-a)}{2^{k-1}} \tag{141}$$

and there is $x^* \in (a_k, b_k)$ such that $f(x^*) = 0$. Since $x^k = (a_k + b_k)/2$ for $k \geq 1$, it follows that

$$|x^k - x^*| \leq \frac{b_k - a_k}{2} = \frac{b-a}{2^k}, \tag{142}$$

from which we deduce $x^k \longrightarrow x^*$ as $k \longrightarrow \infty$. □

Example 20: Show $f(x) = x^2 - x + 4$ has a minimizer in $[0, 3/4]$ and then use the bisection method to approximate this minimizer to within $1/4$.

Solution:

Because $f''(x) = 2 > 0$, we know f is strictly convex. This implies that any point x^* for which $f'(x^*) = 0$ is a unique minimizer of f. And we compute $f'(x) = 2x - 1$ to find $f'(0) = -1 < 0$ and $f'(3/4) = 1/2 > 0$. Since f is continuous, the intermediate value theorem (IVT) therefore implies there exists $x^* \in [0, 3/4]$ such that $f'(x^*) = 0$. Now let $[a_1, b_1] = [0, 3/4]$ and $x^1 = (0+3/4)/2 = 3/8$. Then

$$f'\left(x^1\right) = f'\left(\frac{3}{8}\right) = \frac{3}{4} - 1 = -\frac{1}{4} < 0. \tag{143}$$

So, we obtain $[a_2, b_2] = [3/8, 3/4]$ and $x^2 = (3/8 + 3/4)/2 = 9/16$. Then observe

$$f'\left(x^2\right) = f'\left(\frac{9}{16}\right) = \frac{18}{16} - 1 > 0. \tag{144}$$

This implies $x^* \in [a_3, b_3] = [3/8, 9/16]$ and $x^3 = (3/8 + 9/16)/2 = 15/32$. Since

$$b_3 - a_3 = 9/16 - 3/8 = 3/16 < 1/4, \tag{145}$$

we conclude $\boxed{x^3 = 15/32}$ satisfies $|x^3 - x^*| < 1/4$, and we are done. □

Remark: By direct computation, for f as in the above example, we see $f'(1/2) = 2(1/2) - 1 = 0$ and so $x^* = 1/2$. This means $|x^3 - x^*| = 1/32$, which is certainly smaller than $1/4$.

3.1.2 – Newton's Method:

For the bisection method, we needed only the value of a function to approximate its root. Suppose, in addition to this, we have its derivative at hand. That is, at each measurement x^k, we can determine $f(x^k)$ and $f'(x^k)$. Then We can fit a linear equation through x^k that matches the first derivative of f at x^k. This linear function has the form

$$h(x) = f(x^k) + f'(x^k) \cdot (x - x^k). \tag{146}$$

At a root $\bar{x}$ of h, we have

$$\bar{x} = x^k - \frac{f(x^k)}{f'(x^k)}. \tag{147}$$

This gives us an approximation $\bar{x}$ of a root of f, and iteratively repeating this process by defining $x^{k+1} = \bar{x}$, we obtain the sequence

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}. \tag{148}$$

This is Newton's method. We illustrate this procedure in Figure 9 below. Example 21 shows how to apply this method to find the root of a function. And, in the context of minimizing a function, we can use this process with the derivative of a function, as shown in Example 22.

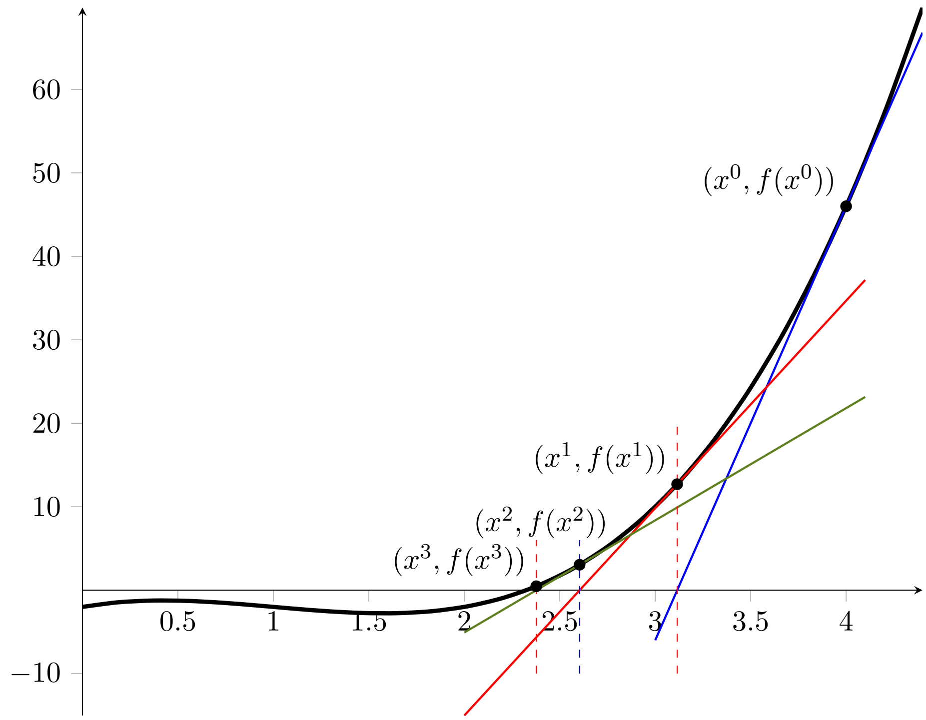

Figure 9: Illustration of Newton's method to find zeros of $f(x) = 2(x-1)^3 - 2x$, starting from $x^0 = 4$.

Example 21: Perform two iterations, using Newton's method, to approximate the zero of the function $f(x) = \cos(x) + 3x$, starting from $x^1 = 0$.

Solution:

Because we are interested in simply finding a zero of f, using Newton's method for this problem gives the sequence

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)} = x^k - \frac{\cos(x^k) - 3x^k}{3 - \sin(x^k)}. \tag{149}$$

Then

$$x^2 = 0 - \frac{\cos(0) - 3 \cdot 0}{3 - \sin(0)} = -\frac{1}{3}, \tag{150}$$

and so

$$x^3 = \frac{1}{3} - \frac{\cos(1/3) - 3(1/3)}{3 - \sin(1/3)} \approx -0.31679. \tag{151}$$

Thus we conclude $\boxed{x^* \approx -0.31679.}$ □

REMARK: To six decimal places of accuracy, the actual solution to the above problem is $x^* = -0.316751$. This means the error was roughly $|x^3 - x^*| \approx 0.000039$.

Example 22: Use two iterations of Newton's method to find approximate a minimizer of $f(x) = e^x + x^2$. Start from $x^1 = 0$.

Solution:
First note $f'(x) = e^x + 2x$ and $f''(x) = e^x + 2$. Since $f'' > 2$, we know f is convex. Thus f has a unique minimizer x^*, for which $f'(x^*) = 0$. Newton's method to find a zero of f', we obtain

$$x^{k+1} := x^k - \frac{f'(x^k)}{f''(x^k)} = x^k - \frac{e^{x^k} + 2x^k}{e^{x^k} + 2}. \tag{152}$$

Thus

$$x^2 = 0 - \frac{e^0 + 2\cdot 0}{e^0 + 2} = 0 - \frac{1+0}{1+2} = -\frac{1}{3}, \tag{153}$$

and

$$\boxed{x^3 = \frac{1}{3} - \frac{e^{1/3} + 2(1/3)}{e^{1/3} + 2} \approx -0.351689.} \tag{154}$$

□

Remark: The true solution is roughly $x^* \approx -0.351734$ and so $|x^3 - x^*| \approx 0.000045$, not a bad estimate for only two iterations. We illustrate Example 22 below in Figure 10.

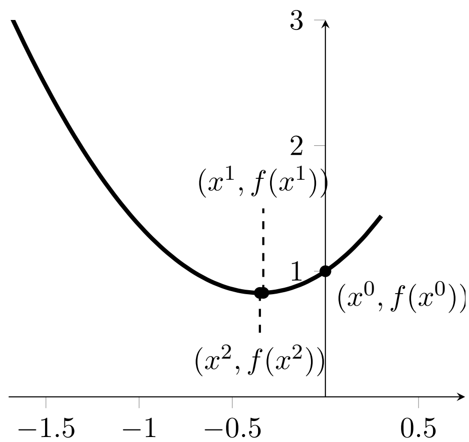

Figure 10: Illustration of Newton's method to find zeros of $f(x) = 2(x-1)^3 - 2x$, starting from $x^0 = 4$. Note the points $(x^1, f(x^1))$ and $(x^2, f(x^2))$ are nearly indistinguishable.

3.1.3 – Secant Method:

If, in applying Newton's method, we use the approximation

$$f'(x^k) \approx \frac{f(x^k) - f^{(}x^{k-1})}{x^k - x^{k-1}} \tag{155}$$

then we obtain a sequence for finding roots that does not compute f'. Namely, (148) becomes

$$x^{k+1} := x^k - \frac{x^k - x^{k-1}}{f(x^k) - f(x^{k-1})} \cdot f(x^k) = \frac{f(x^k)x^{k-1} - f(x^{k-1})x^k}{f(x^k) - f(x^{k-1})}. \tag{156}$$

This is called the *secant method.*

3.2 – Gradient Methods for Unconstrained Optimization:

A common approach to minimizing functions makes use of the gradient. Suppose we have a point x^k and are able to evaluate the gradient ∇f of an objective function. Then we can find a descent direction of f and generate a new iterate x^{k+1} that is a move in the direction. So, gradient descent algorithms produce a sequence of iterates $\{x^k\}_{k=1}^{\infty}$ using a relation of the form

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k), \tag{157}$$

where $\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{R}$ is a sequence of scalars.

3.2.1 – Steepest Descent:

We may construct a particular method of gradient descent, called *steepest descent*, as follows. Here we define each α_k to reduce f as much as possible along the direction of $-\nabla f(x^k)$. That is, we set

$$\alpha_k := \arg\min_{\alpha \geq 0} \underbrace{f\left(x^k - \alpha \nabla f(x^k)\right)}_{\phi_k(\alpha)} = \arg\min_{\alpha \geq 0} \phi_k(\alpha), \tag{158}$$

where we define ϕ_k to be the underbraced quantity above. This means $\phi_k'(\alpha_k) = 0$. Two important properties should be noted of this algorithm. First, the vectors $(x^{k+1} - x^k)$ and $(x^k - x^{k-1})$ are orthogonal from each other. Second, each step reduces the objective function value, i.e., $f(x^k) < f(x^{k-1})$.

Proposition: If $\{x^k\}_{k=0}^{\infty}$ is a steepest descent sequence for an objective function $f : \mathbb{R}^n \to \mathbb{R}$, then, for each k, $(x^{k+1} - x^k)$ is orthogonal to $(x^{k+2} - x^{k+1})$.

Proof:

First we compute the derivative of ϕ_k at α_k to discover

$$\begin{aligned} 0 = \phi_k'(\alpha_k) &= \left[\frac{\partial f}{\partial x}\left(x^k - \alpha \nabla f(x^k)\right) \frac{\partial}{\partial \alpha}\left(x^k - \alpha \nabla f(x^k)\right)\right]_{\alpha=\alpha_k} \\ &= -\left[\frac{\partial f}{\partial x}\left(x^k - \alpha \nabla f(x^k)\right) \nabla f(x^k)\right]_{\alpha=\alpha_k} \\ &= -\frac{\partial f}{\partial x}(x^{k+1}) \nabla f(x^k) \\ &= -\left(\nabla f(x^{k+1})\right)^T \nabla f(x^k). \end{aligned} \tag{159}$$

The first equality follows directly from our choice of α_k. The second equality holds by using the chain rule and definition of ϕ_k. The third equality holds by evaluating the partial derivative with respect to α. The following equality holds by evaluating our expression at $\alpha = \alpha_k$. The final equality is a direct result of the definition[12] of the gradient of f. Then, using the definition of each x^k, we see

$$(x^{k+2} - x^{k+1})^T (x^{k+1} - x^k) = \left(\nabla f(x^{k+1})\right)^T \nabla f(x^k) = 0, \tag{160}$$

where the final equality holds by (159). This completes the proof. □

Proposition: Assume $f : \mathbb{R}^n \to \mathbb{R}$ and ∇f is continuous. If $\{x^k\}_{k=0}^{\infty}$ is the steepest descent sequence for $f : \mathbb{R}^n \to \mathbb{R}$ and $\nabla f(x^k) \neq 0$, then $f(x^{k+1}) < f(x^k)$. △

Proof:

We claim $\phi_k'(0) < 0$. Because ϕ_k' is continuous, this implies[13] there is $\varepsilon > 0$ such that $\phi_k'(\alpha) < 0$ for $\alpha \in [0, \varepsilon)$. From this, we deduce $\phi_k(\alpha) < \phi_k(0)$ for $\alpha \in (0, \varepsilon)$. Hence, by definition of α_k, this shows $\alpha_k \geq \varepsilon$ and so

$$f(x^{k+1}) = \phi_k(\alpha_k) \leq \phi_k(\bar{\varepsilon}) < \phi_k(0) = f(x^k). \tag{161}$$

All that remains is to show $\phi_k'(0) < 0$. Indeed,

$$\begin{aligned}
\phi_k'(0) &= \left[\frac{\partial f}{\partial x}\left(x^k - \alpha \nabla f(x^k)\right) \frac{\partial}{\partial \alpha}\left(x^k - \alpha \nabla f(x^k)\right)\right]_{\alpha=0} \\
&= -\left[\frac{\partial f}{\partial x}\left(x^k - \alpha \nabla f(x^k)\right) \nabla f(x^k)\right]_{\alpha=0} \\
&= -\frac{\partial f}{\partial x}\left(x^k\right) \nabla f(x^k) \\
&= -\left(\nabla f(x^k)\right)^T \nabla f(x^k) \\
&= -\|\nabla f(x^k)\|^2 \\
&< 0.
\end{aligned} \tag{162}$$

□

[12] For a review on how to take derivatives of functions with respect to vectors or matrices, see §3.3.

[13] To be rigorous about this, we should actually make a more formal argument using the definition of continuity and make an $\varepsilon - \delta$ type argument. However, this is outside the scope of this course.

Now suppose we have a quadratic function. That is, there is symmetric $Q \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$f(x) = \frac{1}{2}x^T Qx - b^T x + c. \tag{163}$$

This implies

$$\nabla f(x) = Qx - b. \tag{164}$$

For notational convenience, we set $g^k = \nabla f(x^k)$. We next derive an explicit expression for α_k. Recall that

$$\phi_k(\alpha) := f(x^k - \alpha\nabla f(x^k)). \tag{165}$$

Differentiating this expression gives

$$\begin{aligned} \phi_k'(\alpha) &= \frac{\partial f}{\partial x}\left(x^k - \alpha g^k\right)\frac{\partial}{\partial \alpha}\left[x^k - \alpha\nabla g^k\right] \\ &= \left(Q(x^k - \alpha g^k) - b\right)^T g^k \\ &= (Qx^k - b)^T g^k - \alpha(Qg^k)^T g^k \\ &= (g^k)^T g^k - \alpha(g^k)^T Qg^k. \end{aligned} \tag{166}$$

Since $\phi_k'(\alpha_k) = 0$, we can equate the right hand side of the above to zero and rearrange to get

$$\alpha_k = \frac{(g^k)^T g^k}{(g^k)^T Qg^k}. \tag{167}$$

With this in mind, we give the following example.

Example 23: Using the method of steepest descent for the function

$$f(x_1, x_2, x_3) = \frac{x_1^2}{2} + \frac{x_2^2}{8} + \frac{x_3^2}{25} + 5x_1 + x_3 - 5, \tag{168}$$

starting from $x^1 = [0,\ 0,\ 0]^T$, compute x^2.

Solution:
We first rewrite f as

$$f(x) = \frac{1}{2} x^T Q x + b^T x - 5 \tag{169}$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/25 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}. \tag{170}$$

Then we have

$$g^1 = \nabla f(x^0) = Qx^0 - b = -b. \tag{171}$$

Using our result for α_k when f is quadratic, we write

$$\alpha_1 = \frac{(g^1)^T g^1}{(g^1)^T Q g^1} = \frac{5^2 + 0^2 + 1^2}{5^2 + 0^2/4 + 1^2/25} = \frac{26}{25 + 1/25} = \frac{325}{313}. \tag{172}$$

Then

$$x^2 = x^1 - \alpha_1 g^1 = 0 - \alpha_1 (-b) = \alpha_1 b = \boxed{\frac{325}{313} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}}. \tag{173}$$

□

Example 24: For the same function f as above, what is the largest fixed step size α we can use to guarantee $x^k \longrightarrow x^*$, where x^* is the minimizer of f?

Solution:
We make use of Theorem 8.3 in our text. It states we will have convergence as long as

$$0 < \alpha < \frac{2}{\lambda_{max}(Q)}, \tag{174}$$

where $\lambda_{max}(Q)$ denotes the value of the largest eigenvalue of Q. Since Q is diagonal, we can immediately read of its eigenvalues. Thus,

$$\lambda_{max}(Q) = \max\{1, 1/4, 1/25\} = 1, \tag{175}$$

and so we need

$$0 < \alpha < \frac{2}{1} = 2. \tag{176}$$

□

3.3 – Vector Calculus:

Before diving into constrained optimization, it is important to recall how to perform derivative computations with vectors. In what follows, we will **not** follow the books notation of using a capital D when taking derivatives. The reason for this is that in other situations (not in our text) we use Df to denote the gradient ∇f of the function f. So, in our text, $Df \neq \nabla f$ when working in $\mathbb{R}^n$ with $n > 1$. To avoid any ambiguity, we use $\partial f/\partial x$ where the book uses Df. Also, here we write $\partial^2 f/\partial x^2$ instead of writing $\nabla^2 f$ since in other contexts ∇^2 is commonly understood to be the Laplacian operator.

Definition: Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and $x = [x_1, \ \ldots \ , \ x_n]^T \in \mathbb{R}^n$. Then we say

$$\nabla f(x) := \left(\frac{\partial f}{\partial x}\right)^T \tag{177}$$

where

$$\frac{\partial f}{\partial x} := \left[\frac{\partial f}{\partial x_1}, \ \ldots, \ \frac{\partial f}{\partial x_n}\right]. \tag{178}$$

We take the gradient of a function to be the column vector where the i-th entry is the partial derivative of f with respect to its i-th component x_i. And, $\partial f/\partial x$ is a row vector, the transpose of ∇f. △

Remark: Below we provide a few example computations using vector calculus.

Example 25: Compute $\partial f/\partial x$ when $f(x) := \|x\|$ for $x \in \mathbb{R}^n$.

Solution:

Through direct computation we discover

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= \frac{\partial}{\partial x_i}\left[\|x\|\right] \\ &= \frac{\partial}{\partial x_i}\left[\sum_{j=1}^n x_j^2\right]^{1/2} \\ &= \frac{1}{2}\left[\sum_{j=1}^n x_j^2\right]^{-1/2} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n x_j^2\right) \\ &= \frac{1}{2\sqrt{\sum_{j=1}^n x_j^2}}\left(\sum_{j=1}^n 2x_j \cdot \frac{\partial x_j}{\partial x_i}\right) \\ &= \frac{1}{2\|x\|}\left(\sum_{j=1}^n 2x_j \cdot \delta_{ij}\right) \\ &= \frac{1}{2\|x\|} \cdot 2x_i \\ &= \frac{x_i}{\|x\|} \qquad \text{for } i = 1, \dots, n.\end{aligned} \tag{179}$$

The first equality holds by definition of f. The second equality holds by definition of the Euclidean norm. The third line follows by applying the chain rule. The fourth line follows by again applying the chain rule and rewriting the first term. The fifth line follows by substituting in the definition of the Euclidean norm and note δ_{ij} here denotes the Kronecker δ, i.e., $\delta_{ij} = 1$ when $i = j$ and 0 otherwise. Since this holds for each index i, we write

$$\frac{\partial f}{\partial x} = \left[\frac{x_1}{\|x\|}, \ \dots, \ \frac{x_n}{\|x\|}\right] = \frac{1}{\|x\|}\left[x_1, \ \dots, \ x_n\right] = \frac{x^T}{\|x\|}. \tag{180}$$

□

Example 26: Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) := \|Ax - b\|^2$ and compute $\partial f / \partial x$.

Solution:
Using the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\|Ax-b\|^2\right] = 2\|Ax-b\|\frac{\partial}{\partial x}\left[\|Ax-b\|\right]. \tag{181}$$

The term inside the square root in the definition of $\|Ax - b\|$ is the sum of squared terms. Differentiating this, we find

$$\begin{aligned}
\frac{\partial}{\partial x_k}\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}x_j - b_i\right)^2 &= 2\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}x_j - b_i\right)\cdot\frac{\partial}{\partial x_k}\left(\sum_{j=1}^n a_{ij}x_j - b_i\right)\\
&= 2\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}x_j - b_i\right)\cdot\left(\sum_{j=1}^n a_{ij}\frac{\partial x_j}{\partial x_k} - 0\right)\\
&= 2\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}x_j - b_i\right)\cdot\sum_{j=1}^n a_{ij}\delta_{jk}\\
&= 2\sum_{i=1}^m (Ax-b)_i\cdot\ a_{ik}\\
&= 2(Ax-b)^T a^k.
\end{aligned} \tag{182}$$

where, as usual, a^k denotes the k-th column of A. Then

$$\frac{\partial}{\partial x_k}\|Ax-b\| = \frac{1}{2\|Ax-b\|}\frac{\partial}{\partial x_k}\left[\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}x_j - b_i\right)^2\right] = \frac{1}{\|Ax-b\|}\cdot(Ax-b)^T a^k, \tag{183}$$

and so

$$\frac{\partial f}{\partial x_k} = 2\|Ax-b\|\cdot\frac{1}{\|Ax-b\|}(Ax-b)^T a^k \quad \text{for } k = 1,\dots,n. \tag{184}$$

From this, we conclude

$$\boxed{\frac{\partial f}{\partial x} = 2(Ax-b)^T[a^1,\ \dots,\ a^n] = 2(Ax-b)^T A.} \tag{185}$$

□

Remark: If $n = 1$, then the computation in the above becomes

$$f'(x) = \frac{\partial}{\partial x}|Ax - b|^2 = 2(Ax - b) \cdot A. \tag{186}$$

So, our final result in Example 26 shows an analogous result in higher dimensions.

Example 27: Compute $\partial f/\partial x$ when $f(x) = \frac{1}{2}x^T Qx$ for symmetric $Q \in \mathbb{R}^{n\times n}$.

Solution:

For $k = 1, \ldots, n$, we see

$$\begin{aligned}
\frac{\partial}{\partial x_k}\left[\frac{1}{2}x^T Qx\right] &= \frac{\partial}{\partial x_k}\frac{1}{2}\sum_{i,j=1}^{n} x_i q_{ij} x_j \\
&= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial x_i}{\partial x_k} q_{ij} x_j + x_i q_{ij}\frac{\partial x_j}{\partial x_k} \\
&= \frac{1}{2}\sum_{i,j=1}^{n} \delta_{ik} q_{ij} x_j + x_i q_{ij} \delta_{jk} \\
&= \frac{1}{2}\left(\sum_{j=1}^{n} q_{kj} x_j + \sum_{i=1}^{n} x_i q_{ik}\right) \\
&= \sum_{j=1}^{n} q_{kj} x_j \\
&= (Qx)_k.
\end{aligned} \tag{187}$$

The first equality holds by using the definition of matrix multiplication twice. The second equality follows from the product rule. The fourth equality holds because all terms cancel in the sums except for when the Kronecker δ evaluates to one. The following equality holds by the symmetry of A. The final equality then follows from the definition of matrix multiplication. This shows that

$$\boxed{\frac{\partial f}{\partial x} = (Qx)^T = (x^T Q^T) = x^T Q.} \tag{188}$$

□

3.4 – Constrained Optimization:

We now turn our attention to the minimization of a function with respect to constraints. From our unconstrained material, we know a function is minimized when its gradient is zero. However, how does this change when subject to certain constraints? Do we still seek a constraints-compatible point for which the gradient is zero? It turns out, this is not the case. However, we shall see our primary result for equality constrained optimization, in fact, yields the previous result of the gradient equaling zero in the special case where we set our constraint set to be all of $\mathbb{R}^n$. We first show this in the case of equality constraints.

3.4.1 – Equality Constraints:

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ be smooth functions. Then we can write the constrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad h(x) = 0. \tag{189}$$

The central result here of Lagrange is that a solution x^* to (189) satisfies

$$\frac{\partial f}{\partial x}(x^*) = \lambda^T \frac{\partial h}{\partial x}(x^*) \tag{190}$$

for some $\lambda \in \mathbb{R}^m$. In the case where h is the zero function (i.e., $h(x) = 0$ for each $x \in \mathbb{R}^n$), we see (190) implies the gradient of f will be zero for any solution x^* to (189). We also may use second order information of f to determine sufficient conditions for a point x to be a solution to (189). Before moving onward to stating these results formally, we review some vector calculus results. These will be key for tackling the examples in this subsection.

REMARK: We now state the key results of this chapter.

Lagrange's Theorem: Let x^* be a local minimizer of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $h(x^*) = 0$ where $h : \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$. Assume $\partial h/\partial x(x^*)$ has linearly independent rows. Then there is $\lambda^* \in \mathbb{R}^m$ such that

$$\frac{\partial f}{\partial x}(x^*) + (\lambda^*)^T \frac{\partial h}{\partial x}(x^*) = 0. \tag{191}$$

△

Definition: We define the *tangent space* $T(x^*)$ at a point x^* on the surface $S := \{x \in \mathbb{R}^n \mid h(x) = 0\}$ by

$$T(x^*) := \left\{ y \in \mathbb{R}^n \mid \frac{\partial h}{\partial x}(x^*)\ y = 0 \right\}. \tag{192}$$

This is the set of all points in $\mathbb{R}^n$ in the null space of $\partial h/\partial x(x^*)$. △

Definition: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth objective function and $h : \mathbb{R}^n \to \mathbb{R}^m$ be smooth with $m \leq n$. Then we define the Lagrangian $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$\mathcal{L}(x, \lambda) := f(x) + \lambda^T h(x). \tag{193}$$

△

Remark: Note the similarity between this definition and the Lagrangian used in our linear programming work earlier this quarter. Lagrange's condition (191) also becomes

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda^*) = 0. \tag{194}$$

Second-Order Necessary Conditions: Let x^* be a local minimizer of the smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with $h(x^*) = 0$ where $h : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $m \leq n$. Suppose also that $\partial h/\partial x(x^*)$ has linearly independent rows. Then there is $\lambda^* \in \mathbb{R}^m$ such that

$$\frac{\partial f}{\partial x}(x^*) + (\lambda^*)^T \frac{\partial h}{\partial x}(x^*) = 0, \tag{195}$$

and, for all $y \in T(x^*)$,

$$y^T \left(\frac{\partial^2 \mathcal{L}}{\partial x^2}(x^*) \right) y \geq 0. \tag{196}$$

△

Remark: The above result holds for any minimizer. But, this result does not tell us that if a point satisfies the equations in the above theorem that it will be a minimizer. However, our next theorem does give sufficient conditions.

Second-Order Sufficient Conditions: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ be smooth functions with $m \leq n$. Suppose there is $x^* \in \mathbb{R}^n$ and $\lambda^* \in \mathbb{R}^m$ such that

$$0 = \frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda^*) = \frac{\partial f}{\partial x}(x^*) + (\lambda^*)^T \frac{\partial h}{\partial x}(x^*) \tag{197}$$

and, for each nonzero $y \in T(x^*)$,

$$y^T \left(\frac{\partial^2 \mathcal{L}}{\partial x^2}(x^*, \lambda^*) \right) y > 0. \tag{198}$$

Then x^* is a strict local minimizer of f subject to the constraint $h = 0$. $\triangle$

Remark: The above theorem tells us that if the Hessian matrix of the Lagrangian $\mathcal{L}$ is positive definite for inputs x^* and λ^*, then x^* is a local minimizer. Sometimes we say the Hessian of $\mathcal{L}$ is positive definite by writing $\partial^2\mathcal{L}/\partial x^2(x^*) > 0$. With this notation, we see that in one dimension the above results reduces to the classic second derivative test.

Remark: Not every unconstrained problem can be rewritten as a constrained problem with equality constraints by letting h be the zero function.

Remark: The below examples are given to illustrate the difference between the necessary and sufficient conditions. The first example shows a point may satisfy a first-order necessary condition to be a minimizer, but still not be a minimizer. The second example shows a point may fail to satisfy a second-order sufficient condition, but still be a minimizer.

Example 28: Give a function f and a point x^* that satisfies the first-order necessary condition $\nabla f(x^*) = 0$ to be a minimizer, but for which x^* is not at a local extrema of f.

Solution:
Set $f(x) := x^3$. Then $f'(0) = 3 \cdot 0^2 = 0$. However, 0 is not a local extrema of f. $\square$

Example 29: If a point x^* satisfies the first-order necessary condition

$$0 = \frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda^*), \tag{199}$$

but fails to holds the second order sufficient condition

$$\frac{\partial^2 f}{\partial x^2}(x^*) > 0, \tag{200}$$

can it be a minimizer?

Solution:

Yes, it can. Suppose $f(x) := x^4$ and $h(x) := 0$. Then

$$\frac{\partial \mathcal{L}}{\partial x}(0, \lambda) = 4 \cdot 0^3 + \lambda \cdot 0 = 0. \tag{201}$$

And,

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(0, \lambda) = 12 \cdot 0^2 + \lambda \cdot 0 = 0. \tag{202}$$

However, $f(0) = 0$ and $f(x) = x^4 > 0$ when $x \neq 0$, and so 0 is a minimizer of f. □

Remark: We now provide examples using the Lagrange condition.

Example 30: Consider the problem

$$\min \ \|Ax-b\|^2 \quad \text{subject to} \quad \|x\|^2 = 5 \quad \text{and} \quad c^T x = 7 \tag{203}$$

where $A \in \mathbb{R}^{m\times n}$, $c, x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Write the Lagrange condition satisfied by any to (203.)

Solution:

First we define

$$f(x) := \|Ax-b\|^2 \quad \text{and} \quad h(x) := \begin{bmatrix} \|x\|^2 - 5 \\ c^T x - 7 \end{bmatrix} \tag{204}$$

so that any solution x^* to (203) will satisfy $h(x^*) = 0$. Then we compute

$$\frac{\partial f}{\partial x} = 2(Ax-b)^T A \quad \text{and} \quad \frac{\partial h}{\partial x} = \begin{bmatrix} 2x^T \\ c^T \end{bmatrix}. \tag{205}$$

The Lagrangian $\mathcal{L}$ is given by $\mathcal{L}(x,\lambda) := f(x) + \lambda^T h(x)$ for $\lambda = [\lambda_1 \ \lambda_2]^T \in \mathbb{R}^2$. Then Lagrange's theorem implies that, for a solution x^* to (203),

$$\begin{aligned} 0^T = \frac{\partial \mathcal{L}}{\partial x}(x^*) &= \frac{\partial}{\partial x}\left[f(x^*) + \lambda^T h(x^*)\right] \\ &= 2(Ax^*-b)^T A + \lambda^T \begin{bmatrix} 2(x^*)^T \\ c^T \end{bmatrix} \\ &= 2(Ax^*-b)^T A + 2\lambda_1 (x^*)^T + \lambda_2 c^T. \end{aligned} \tag{206}$$

With the linear system of equations and the fact $h(x^*) = 0$, we could then solve for x^*, if the terms A, c, and b were given explicitly. □

Example 31: Consider the constrained optimization problem

$$\min \ \ 4x_1 + 3x_2 - 10 \quad \text{subject to} \quad x_1 x_2 = 12. \tag{207}$$

a) Use Lagrange's theorem to find all possible local minimizers and maximizers.

b) Use the second-order sufficient conditions to determine which points are strict local maximizers/minimizers. From this, identify the solution to the problem above.

Solution:

a) First we set

$$f(x) := 4x_1 + 3x_2 - 10 \quad \text{and} \quad h(x) := x_1 x_2 = 12 \tag{208}$$

so that our Lagrangian $\mathcal{L}$ becomes $\mathcal{L}(x, \lambda) = f(x) + \lambda h(x)$. Then we compute

$$(\nabla f(x))^T = \frac{\partial f}{\partial x}(x) = [4 \ 3] \quad \text{and} \quad (\nabla h(x))^T = \frac{\partial h}{\partial x}(x) = [x_2 \ x_1] = x^T Q = (Qx)^T \tag{209}$$

where

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{210}$$

and we Q is symmetric. For a point x^* satisfying the Lagrange condition, we see

$$0 = \nabla \mathcal{L}(x^*) = \nabla f(x^*) + \lambda \nabla h(x^*) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} x_2^* \\ x_1^* \end{bmatrix}, \tag{211}$$

which implies

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = -\frac{1}{\lambda} \begin{bmatrix} 3 \\ 4 \end{bmatrix}. \tag{212}$$

Using our constraint, we then discover

$$0 = h(x^*) = \left(-\frac{3}{\lambda}\right)\left(-\frac{4}{\lambda}\right) - 12 = 12\left(\frac{1}{\lambda^2} - 1\right) \quad \Rightarrow \quad \lambda \in \{-1, 1\}. \tag{213}$$

Thus the possible local extrema corresponding to $\lambda = -1$ and $\lambda = 1$, respectively, are

$$\boxed{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad -\begin{bmatrix} 3 \\ 4 \end{bmatrix}.} \tag{214}$$

b) To apply the second-order sufficient conditions, we must compute the tangent space $T(x^*)$. This is defined so that

$$T(x^*) := \left\{ y \;\mid\; \frac{\partial h}{\partial x}(x^*)y = 0 \right\}. \tag{215}$$

Observe

$$0 = \frac{\partial h}{\partial x}(x^*)y = (x^*)^T Q y = [3\ 4] \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} = 3y_2 + 4y_1. \tag{216}$$

This implies

$$T(x^*) = \left\{ \alpha \begin{bmatrix} -3 \\ 4 \end{bmatrix} \;\mid\; \alpha \in \mathbb{R} \right\}. \tag{217}$$

And

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x^*, \lambda) = \frac{\partial^2}{\partial x^2}\left[f(x^*) + \lambda h(x^*)\right] = \frac{\partial}{\partial x}\left([4\ 3] + \lambda Q x\right) = 0 + \lambda Q I = \lambda Q. \tag{218}$$

Thus, for nonzero $y \in T(x^*)$,

$$y^T \mathcal{L}(x^*, \lambda) y = \lambda\alpha^2 [-3\ 4] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \lambda\alpha^2[-3\ 4] \begin{bmatrix} 4 \\ -3 \end{bmatrix} = -24\lambda\alpha^2. \tag{219}$$

For $\lambda = 1$ the above expression is negative and for $\lambda = -1$ the above expression is positive. This implies

$$\boxed{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ is a local minimizer} \quad \text{and} \quad -\begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ is a local maximizer.}} \tag{220}$$

Because there is only one local minimizer, we conclude the solution to (207) is $\boxed{[3\ 4].}$

□

Example 32: Let[14] $A \in \mathbb{R}^{n\times n}$ be symmetric and $S := \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Let $x \in S$ be such that

$$x^T Ax = \sup_{y\in S} y^T Ay \tag{221}$$

(By compactness such x exists.) Prove $\langle x, y\rangle = 0 \;\Rightarrow\; \langle Ax, y\rangle = 0$.

Solution:

Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(y) = \langle Ay, y\rangle$ and define $h : \mathbb{R}^n \to \mathbb{R}$ by $h(y) = \langle y, y\rangle - 1$. Then $\langle Ax, x\rangle$ is the max of f subject to the constraint $h = 0$. From this, we make the computations

$$\nabla f(y) = 2Ay \quad \text{and} \quad \nabla h(y) = 2y. \tag{222}$$

These computations follow directly from Examples 25 and 27. Lagrange's theorem for multipliers asserts that if the first partials of f and h are continuous and f attains an extremum at a point x subject to $h(x) = 0$, then there is $\lambda \in \mathbb{R}$ such that $\nabla f(x) = \lambda\nabla h(x)$. Indeed, ∇f and ∇h are linear and so the partials of f and h are continuous. Hence

$$2Ax = \nabla f(x) = \lambda h(x) = \lambda 2x \quad \Rightarrow \quad Ax = \lambda x, \tag{223}$$

which implies x is an eigenvector of A. Note $x \neq 0$ since $\|x\| = 1$. Moreover,

$$0 = \langle x, y\rangle \quad \Rightarrow \quad 0 = \lambda \cdot 0 = \lambda \langle x, y\rangle = \langle \lambda x, y\rangle = \langle Ax, y\rangle, \tag{224}$$

as desired. □

[14] This problem is derived from Spring 2008, Problem 12 on UCLA's Basic Exam.

Example 33: Consider the problem

$$\min \ 2x_2^2 + 3x_2 + 5x_1 + 4 \quad \text{subject to} \quad x_1 x_2 = 1 \quad \text{and} \quad x_1^2 + x_2^2 = 1. \tag{225}$$

Use Lagrange's theorem to write the linear system (in terms of λ) satisfied by a solution to (225), but do not proceed any further in finding local extrema. Then compute the Hessian of the associated Lagrangian for this problem.

Solution:
First define

$$f(x) := 2x_1^2 + 3x_2 + 5x_1 \quad \text{and} \quad h(x) := \begin{bmatrix} x_1 x_2 - 1 \\ x_1^2 + x_2^2 - 1 \end{bmatrix}. \tag{226}$$

Then

$$\frac{\partial f}{\partial x} = [4x_1 + 5 \ \ 3] \quad \text{and} \quad \frac{\partial h}{\partial x} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}. \tag{227}$$

The Lagrange condition is that the Lagrangian $\mathcal{L}(x, \lambda)$ defined by $\mathcal{L}(x, \lambda) = f(x) + \lambda^T h(x)$ satisfies

$$\begin{aligned} 0^T = \frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda^*) &= [(4x_1^* + 5) \ \ 3] + [\lambda_1 \ \lambda_2] \begin{bmatrix} x_2^* & x_1^* \\ 2x_1^* & 2x_2^* \end{bmatrix} \\ &= [(4x_1^* + 5 + \lambda_1 x_2^* + 2\lambda_2 x_1^*) \ \ (3 + \lambda_1 x_1^* + 2\lambda_2 x_2^*)]. \end{aligned} \tag{228}$$

This can be rewritten as

$$0 = [x_1^* \ x_2^*] \begin{bmatrix} 4 + 2\lambda_2 & 1 \\ \lambda_1 & 2\lambda_2 \end{bmatrix} + [5 \ 3], \tag{229}$$

or

$$\boxed{-\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 + 2\lambda_2 & 1 \\ \lambda_1 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}.} \tag{230}$$

Now

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x) = \frac{\partial}{\partial x} \left(\begin{bmatrix} 4 + 2\lambda_2 & 1 \\ \lambda_1 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right) = \boxed{\begin{bmatrix} 4 + 2\lambda_2 & 1 \\ \lambda_1 & 2\lambda_2 \end{bmatrix}.} \tag{231}$$

□

3.4.2 – Inequality Constraints:

We now extend our consideration to optimization problem with inequality constraints and equality constraints. The methods in this subsection are more general than that above. That is, the key result, the Karush-Kuhn-Tucker (KKT) theorem can be applied to problems with only equality constraints as well as problem that also have inequality constraints. We state it as follows.

Karush-Kuhn-Tucker (KKT) Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^m \to \mathbb{R}$ with $m \leq n$, $g : \mathbb{R}^p \to \mathbb{R}$ and let f, h, and g be smooth. Let $x^* \in \mathbb{R}^n$ be a local minimizer of the problem

$$\min f(x) \qquad \text{subject to} \quad h(x) = 0,\ g(x) \leq 0. \tag{232}$$

Also assume

$$\nabla h_i(x^*),\ \ \nabla g_j(x^*),\quad 1 \leq i \leq m,\ j \in \{j \mid g_j(x^*) = 0\} \tag{233}$$

forms a linearly independent set of vectors. Then there is $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that

$$\mu^* \geq 0, \qquad 0 = \frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda^*, \mu^*), \qquad (\mu^*)^T g(x^*) = 0. \tag{234}$$

△

Definition: In the above theorem, we have

$$0 = \mu_1^* g_1(x^*) + \cdots + \mu_p^* g_p(x^*) \quad \text{and} \quad \mu^* \geq 0. \tag{235}$$

If $g_i(x^*) < 0$ for $i \in \{1, \ldots, p\}$, then we say this constraint is *inactive.* Otherwise, we say the constraint $g_i(x^*)$ is *active.* △

Remark: Note the KKT theorem implies that if $g_i(x^*)$ is an inactive constraint, then $\mu_i^* = 0$ for $i = 1, \ldots, p$.

Remark: The KKT conditions can be used to identify local minimizers to (232), as illustrated by the following example.

Example 34: Find local minimizers of

$$x_1^2 + x_2^2 \qquad \text{subject to} \qquad x_1^2 + 2x_1x_2 + x_2^2 = 1, \quad x_1^2 - x_2 \leq 0. \tag{236}$$

Solution:
Define the objective and constraint functions

$$f(x) := x_1^2 + x_2^2, \quad h(x) := (x_1 + x_2)^2 - 1, \quad \text{and} \quad g(x) := x_1^2 - x_2. \tag{237}$$

Then the Lagrangian $\mathcal{L}$ is given by $\mathcal{L}(x, \lambda, \mu) := f + \lambda h + \mu g$, and so

$$\begin{aligned} \nabla \mathcal{L} = \nabla f + \lambda \nabla h + \mu \nabla g &= 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2\lambda \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} + \mu \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} \\ &= 2 \begin{bmatrix} 1 + \lambda + \mu & \lambda \\ \lambda & 1 + \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\mu \end{bmatrix}. \end{aligned} \tag{238}$$

We must consider two cases: when the constraint g is active and when it is inactive.

Inactive Constraint: First assume the inequality constraint g is inactive. That is, assume for a feasible local minimizer x^* that $g(x^*) < 0$. The KKT conditions then imply $\mu^* = 0$, and so

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla \mathcal{L}(x^*, \mu^*, \lambda^*) = 2 \begin{bmatrix} 1 + \lambda^* + 0 & \lambda^* \\ \lambda^* & 1 + \lambda^* \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{239}$$

Note the determinant of the above matrix is $(1 + \lambda^*)^2 - (\lambda^*)^2 = 1 + 2\lambda^*$. So, if $\lambda^* \neq -1/2$, then it is invertible and we discover

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + \lambda^* + \mu^* & \lambda^* \\ \lambda^* & 1 + \lambda^* \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{240}$$

Thus x^* is the zero vector. But, this implies $h(x^*) = -1$, contradicting the fact x^* is feasible.

Alternatively, suppose $\lambda = -1/2$. Then we obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \qquad \Rightarrow \qquad x_1^* = x_2^*. \tag{241}$$

From our equality constraint, we see

$$0 = h(x^*) = (x_1^* + x_2^*)^2 - 1 = 4(x_1^*)^2 - 1 \quad \Rightarrow \quad x_1^* = \pm\frac{1}{2}. \tag{242}$$

If $x_1^* = -1/2$, then

$$g(x^*) = \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} > 0, \tag{243}$$

yielding x^* to be infeasible. If instead $x_1^* = 1/2$, then

$$g(x^*) = \left(-\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4} < 0, \tag{244}$$

and so x^* is feasible. In summary, we have shown that when the constraint g is inactive, the single candidate minimizer is

$$\mu^* = 0, \quad \lambda^* = -1/2, \qquad \text{and} \qquad x^* = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{245}$$

To show this choice x^* is, in fact, a local minimizer, we apply the second order sufficient conditions (see Theorem 21.3 in our text). We first compute the tangent space $T(x^*)$. By definition,

$$T(x^*) := \left\{ y \in \mathbb{R}^2 \ \mid \ \frac{\partial h}{\partial x}(x*)y = 0 \right\}. \tag{246}$$

This implies, for each $y \in T(x^*)$¡

$$0 = (\nabla h(x^*))^T y = [2 \ 2]\begin{bmatrix} y_1 \\ y_1 \end{bmatrix} = 2(y_1 + y_2) \quad \Rightarrow \quad y_1 = -y_2. \tag{247}$$

Hence we may write the tangent space $T(x^*)$ explicitly as

$$T(x^*) = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ \mid \ \alpha \in \mathbb{R} \right\}. \tag{248}$$

Next we compute the Hessian of our Lagrangian $\mathcal{L}$. Observe

$$\nabla \mathcal{L}(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x^* \quad \Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial x^2}(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{249}$$

Finally, for any nonzero $y \in T(x^*)$, there is $\alpha \neq 0$ such that

$$y^T \left(\frac{\partial^2 \mathcal{L}}{\partial x^2}\right) y = [\alpha \ \ -\alpha] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \alpha^2 [1 \ \ -1] \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \alpha^2(2+2) = 4\alpha^2 > 0. \quad (250)$$

By the second order sufficient conditions (Theorem 21.3), we conclude

$$\boxed{x^* = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad (251)$$

is a local minimizer of (236).

Active Constraint: Now assume the constraint g is active so that $g(x^*) = 0$. This implies $x_2* = (x_1^*)^2$ and so

$$0 = h(x^*) = (x_1^* + x_2^*)^2 - 1 = (x_1^* + (x_1^*)^2)^2 - 1 = (x_1^*)^4 + 2(x_1^*)^3 + (x_1^*)^2 - 1. \quad (252)$$

The only real solutions to (252) are given by

$$x_1^* = \frac{-1 \pm \sqrt{5}}{2}, \quad (253)$$

which implies

$$x_2^* = (x_1)^2 = \left(-\frac{1}{2}\right)^2 \mp \frac{\sqrt{5}}{2} + \left(\frac{\sqrt{5}}{2}\right)^2 = \frac{3 \mp \sqrt{5}}{2}. \quad (254)$$

Our KKT conditions yield

$$0 = \nabla \mathcal{L} = 2 \begin{bmatrix} 1 + \lambda^* + \mu^* & \lambda^* \\ \lambda^* & 1 + \lambda^* \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} - \begin{bmatrix} 0 \\ \mu \end{bmatrix}. \quad (255)$$

The above linear system consists of two equations. The first gives

$$0 = (1 + \lambda^* + \mu^*)x_1^* + \lambda^* x_2^* = (1 + \mu^*)x_1^* + \lambda^* \quad \Rightarrow \quad \lambda^* = -(1 + \mu^*)x_1^*. \quad (256)$$

The second equality follows from the fact $x_1^* + x_2^* = 1$. Substituting for λ^* in the other equation of our linear system, we obtain

$$0 = \lambda^* x_1^* + (1 + \lambda^*)x_2^* - \frac{\mu^*}{2} = x_2^* + \lambda^* - \frac{\mu^*}{2} = x_2^* - (1 + \mu^*)x_1^* - \frac{\mu^*}{2} \quad \Rightarrow \quad \mu^* = \frac{x_2^* - x_1^*}{1/2 + x_1^*}. \quad (257)$$

Now plugging in x_1^* and x_2^*, we see

$$\mu^* = \frac{2 \mp \sqrt{5}}{\pm\sqrt{5}/2} = \pm\left(\frac{4}{\sqrt{5}} \mp 2\right) < 0. \tag{258}$$

The final inequality holds since in both cases the value on the right hand side is negative, a contradiction to the KKT conditions. Thus the only local minimizer of (236) is given in (251). □

We next give a definition for the projection of a point $x \in \mathbb{R}^n$ onto a set $\Omega \subseteq \mathbb{R}^n$.

Definition: Let $\Omega \subseteq \mathbb{R}^n$. Then we define the operator $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Pi[x] := \arg\min_{z\in\Omega} \|z - x\|. \tag{259}$$

△

Remark: This definition means $\Pi[x]$ is point in Ω that is closest to x, in the norm sense. Note this projection is not always well-defined, as illustrated by the next example. However, $\Pi[x]$ is well-defined whenever Ω is closed and convex.

Example 35: Define the set Ω by

$$\Omega := \left\{x \in \mathbb{R}^2 \ \mid \ x_1 \in [-1, 1],\ x_2 \le x_1^2\right\}. \tag{260}$$

This set is illustrated in Figure 11. Let $y = [0,\ 2]^T$. Then

$$\Pi[y] := \arg\min_{z\in\Omega} \|z - y\| = \{[1,\ 1]^T, [-1,\ 1]^T\}. \tag{261}$$

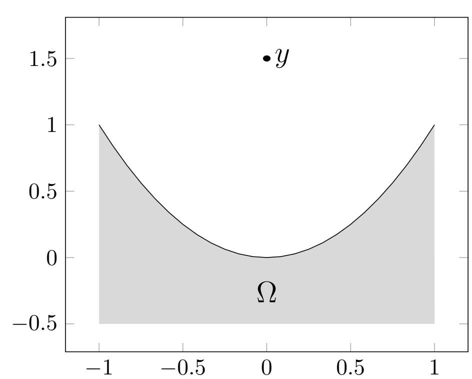

Figure 11: Illustration of the set Ω, which is the shaded region and is unbounded below.

Definition: Consider the problem of minimizing $f(x)$ such that $x \in \Omega$ for some closed, convex set $\Omega \subseteq \mathbb{R}^n$. Let $x^0 \in \Omega$. Then the sequence $\{x^k\}_{k=0}^{\infty}$ defined by the iteration

$$x^{k+1} = \Pi\left[x^k + \alpha_k d^k\right] \tag{262}$$

for sequences $\{\alpha_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ and $\{d_k\}_{k=0}^{\infty} \subseteq \mathbb{R}^n$ is called a projected algorithm. △

REMARK: If, for each k, $d_k = \nabla f(x^k)$, then we refer to the iteration for the x^k's a projected gradient algorithm.

Example 36: Let $A \in \mathbb{R}^{m\times n}$, $m < n$, rank $A = m$, and $b \in \mathbb{R}^m$. Define $\Omega := \{x \mid Ax = b\}$ and let $x_0 \in \Omega$. Show that for each $y \in \mathbb{R}^n$,

$$\Pi[x_0 + y] = x_0 + Py. \tag{263}$$

Solution:

First note, for $y \in \mathbb{R}^n$,

$$y = (I - A^T(AA^T)^{-1}A)y \iff 0 = A^T(AA^T)^{-1}Ay \iff 0 = A0 = AA^T(AA^T)^{-1}Ay = Ay. \tag{264}$$

Thus, $y = Py$ if and only if $y \in N(A)$.[15] So, $(x_0 + y) \in \Omega$ if and only if $y = Py$. Indeed,

$$A(x_0 + y) = Ax_0 + Ay = Ax_0 = b \iff (x_0 + y) \in \Omega. \tag{265}$$

This shows that if $(x_0 + y) \in \Omega$, then $\Pi[x_0 + y] = x_0 + y = x_0 + Py$.

Now assume $(x_0 + y) \notin \Omega$. By definition, $\Pi[x_0 + y]$ is the solution z^* to the problem

$$\min_z \|z - (x_0 + y)\| \qquad \text{subject to} \qquad Az = b. \tag{266}$$

Define $f(z) := \|z - (x_0 + y)\|$ and $h(z) := Az - b$. Then the Lagrange condition implies a minimizer z^* will satisfy

$$0 = \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial f}{\partial z} + \lambda^T \left(\frac{\partial h}{\partial z}\right) = \frac{(z^* - (x_0 + y))^T}{\|z^* - (x_0 + y)\|} + \lambda^T A. \tag{267}$$

Let $\beta(z) := \|z - (x_0 + y)\|$ and note $\beta(z) \neq 0$ for $z \in \Omega$. We can rewrite (267) as

$$0 = (z^*)^T - (x_0 + y)^T + \beta(z^*)\lambda^T A \quad \Rightarrow \quad z^* = x_0 + y - \beta(z^*)A^T\lambda. \tag{268}$$

Using the fact $Az^* = b$, we find

$$b = Az^* = A\left(x_0 + y - \beta(z^*)A^T\lambda\right) = Ax_0 + Ay - \beta(z^*)AA^T\lambda = b + Ay - \beta(z^*)AA^T\lambda. \tag{269}$$

Because A has rank m, AA^T is invertible and so we can rearrange for λ to find

$$\lambda = \frac{1}{\beta(z^*)}(AA^T)^{-1}Ay. \tag{270}$$

[15] Here $N(A)$ denotes the null space of A.

Thus (268) and (270) together imply

$$z^* = x_0 + y - \beta(z^*)A^T\left(\frac{1}{\beta(z^*)}(AA^T)^{-1}Ay\right) = x_0 + y - A^T(AA^T)^{-1}Ay = x_0 + Py. \tag{271}$$

We now verify z^* is a local minimizer of f subject to $h = 0$. Observe

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial z^2}(z^*) &= \frac{\partial}{\partial z}\left[\frac{1}{\beta(z)}(z-(x_0+y)^T + \lambda^T A\right]_{z=z^*}\\ &= -\frac{1}{\beta(z^*)^2}\left(z^*-(x_0+y)\right)\left(z^*-(x_0+y)\right)^T + \frac{1}{\beta(z^*)}I\\ &= -\frac{1}{\beta(z^*)^2}(Py-y)(Py)^T + \frac{1}{\beta(z^*)}I.\end{aligned} \tag{272}$$

The second equality follows by using the product and chain rule, and the third holds by substituting $x_0 + Py$ for z^*. Then for any nonzero $w \in \mathbb{R}^n$ we discover

$$\begin{aligned}w^T\left(\frac{\partial^2\mathcal{L}}{\partial z^2}(z^*)\right)w &= \frac{1}{\beta(z^*)^2}w^T(Py-y)(Py-y)^Tw + \frac{1}{\beta(z^*)}w^TIw\\ &= \underbrace{\left(\frac{w^T(Py-y)}{\beta(z^*)}\right)^2}_{\geq 0} + \underbrace{\frac{1}{\beta(z^*)}\|w\|^2}_{>0}\\ &> 0,\end{aligned} \tag{273}$$

i.e., $\partial^2\mathcal{L}/\partial z^2 > 0$. So, z^* is a local minimizer of f subject to $h = 0$. Therefore we conclude $\Pi[x_0 + y] = z^* = x_0 + Py$, as desired. □

Remark: In particular, if we take $x_0 = x^k \in \Omega$ and $y = \alpha_k \nabla f(x_0)$, then

$$\Pi\left[x^k - \alpha\nabla f(x_0)\right] = x^k - \alpha_k P\nabla f(x^k). \tag{274}$$

Example 37: Consider the problem

$$\min \ \frac{1}{2}\|x\|^2 \qquad \text{subject to} \qquad Ax = b \tag{275}$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$, and rank $A = m$. Show that if $Ax^0 = b$, then the projected steepest descent algorithm converges to the solution in one step.

Solution:

In order to show the algorithm converges to the solution in one step, we must compute the solution, which we do by using Lagrange multipliers. Define $f(x) := \frac{1}{2}\|x\|^2$, $h(x) := Ax - b$, and the Lagrangian $\mathcal{L}(x, \lambda) = f(x) + \lambda^T h(x)$. Then a minimizer x^* to the problem at hand satisfies

$$0 = \frac{\partial \mathcal{L}}{\partial x}(x^*) = \frac{\partial f}{\partial x}(x^*) + \lambda^T \frac{\partial h}{\partial x}(x^*) = (x^*)^T + \lambda^T A \qquad \Rightarrow \qquad x^* = -A^T\lambda. \tag{276}$$

Using our equality constraint, we discover

$$b = Ax^* = -AA^T\lambda. \tag{277}$$

Since A has full rank, AA^T is invertible, from which we deduce

$$\lambda = -(AA^T)^{-1}b. \tag{278}$$

Thus

$$x^* = -A^T\left(-(AA^T)^{-1}b\right) = A^T(AA^T)^{-1}b. \tag{279}$$

Since

$$\frac{\partial^2 \mathcal{L}}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}(x^*) + \lambda^T \frac{\partial h}{\partial x}(x^*)\right] = \frac{\partial}{\partial x}\left[(x)^T + \lambda^T A\right]_{x=x^*} = I + 0 > 0, \tag{280}$$

i.e., $\partial^2\mathcal{L}/\partial x^2$ is positive definite, the second-order sufficient conditions imply x^* is a local minimizer.

We now proceed to compute x^1. The projected steepest descent algorithm is defined by

$$x^{k+1} = \Pi\left[x^k - \alpha_k \nabla f(x^k)\right] = x^k - \alpha_k P \nabla f(x^k) \tag{281}$$

where the second equality holds by our result in Example 36. By definition,

$$\begin{aligned}\alpha_k := \arg\min_{\alpha\geq 0} f\left(x^k - \alpha\nabla f(x^k)\right) &= \arg\min_{\alpha\geq 0} f(x^k - \alpha x^k)\\ &= \arg\min_{\alpha\geq 0} \frac{1}{2}\|(1-\alpha)x^k\|^2\\ &= \arg\min_{\alpha\geq 0} \frac{|1-\alpha|}{2}\|x^k\|^2\\ &= 1.\end{aligned} \tag{282}$$

Then through substitution for P and x^*, we obtain

$$\begin{aligned}x^1 &= x^0 - \alpha_0 P\nabla f(x^0)\\ &= x^0 - Px^0\\ &= x^0 - (I - A^T(AA^T)^{-1}A)x^0\\ &= A^T(AA^T)^{-1}Ax^0\\ &= A^T(AA^T)^{-1}b\\ &= x^*.\end{aligned} \tag{283}$$

All that remains is to show $x^k = x^1 = x^*$ for $k \geq 1$, which we do by induction. The above verifies the base case where $k = 1$. By Theorem 23.1, if $P\nabla f(x^k) \neq 0$, then $f(x^{k+1}) < f(x^k)$. Suppose $x^k = x^*$. Since $f(x^k) = f(x^*)$ is optimal, it follows that $P\nabla f(x^k) = 0$. This implies

$$x^{k+1} = \Pi\left[x^k - \alpha_k P\nabla f(x^k)\right] = \Pi\left[x^k - \alpha_k 0\right] = \Pi\left[x^k\right] = x^k = x^*, \tag{284}$$

which closes the induction. Thus we conclude the steepest descent algorithm converges in one step.

$\square$

Example 38: Consider the optimization problem

$$\min_{x\in\Omega} \frac{1}{2}x^T Qx \tag{285}$$

where $\Omega = [-2,3]\times[0,1]$. Compute x^1 using the projected steepest descent algorithm, using

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad x^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{286}$$

Solution:
Let $f(x) := \frac{1}{2}x^T Qx$. Iterates of the projected steepest descent algorithm are then given by

$$x^{k+1} = \Pi\left[x^k - \alpha_k \nabla f(x^k)\right] \tag{287}$$

where

$$\alpha_k := \arg\min_{\alpha\geq 0} f\left(x^k - \alpha\nabla f(x^k)\right). \tag{288}$$

From Chapter 8 of our text, we know

$$\alpha_k = \frac{\nabla f(x^k)^T \nabla f(x^k)}{\nabla f(x^k)^T Q \nabla f(x^k)} = \frac{\left(Qx^k\right)^T Qx^k}{\left(Qx^k\right)^T Q\left(Qx^k\right)}. \tag{289}$$

Observe that

$$Qx^0 = Q = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad Q^2x^0 = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}. \tag{290}$$

Whence

$$\alpha_0 = \frac{\left(Qx^0\right)^T Qx^0}{\left(Qx^0\right)^T Q\left(Qx^0\right)} = \frac{(-1)^2 + 3^2}{-1\cdot 1 + 3\cdot 9} = \frac{10}{26} = \frac{5}{13}. \tag{291}$$

So,

$$x^0 - \alpha_0\nabla f(x^0) = x^0 - \alpha_0 Qx^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{13}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 18/13 \\ -2/13 \end{bmatrix}. \tag{292}$$

Note that our projection operator is here given by

$$\Pi[x] = \begin{bmatrix} \max\{-3, \min\{x_1, 2\}\} \\ \max\{0, \min\{x_1, 1\}\} \end{bmatrix}. \tag{293}$$

Therefore,

$$\begin{aligned}
x^1 = \Pi\left[x^0 - \alpha_0 \nabla f(x^0)\right] &= \Pi\left[\begin{bmatrix} 18/13 \\ -2/13 \end{bmatrix}\right] \\
&= \begin{bmatrix} \max\{-3, \min\{18/13, 2\}\} \\ \max\{0, \min\{-2/13, 1\}\} \end{bmatrix} \\
&= \begin{bmatrix} \max\{-3, 18/13\} \\ \max\{0, -2/13\} \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 18/13 \\ 0 \end{bmatrix}.}
\end{aligned} \tag{294}$$

□

3.5 – Convexity:

We now introduce the notion of convexity, which enables us to extend results about local behavior to assertions about global behavior. In other words, if a function f is convex, then any local minimizer x^* is also a global minimizer of f. We follow with the definition of a convex set and convex function, a lemma relating the two, and then a couple examples.

Definition: A set Ω is called *convex* if $u, v \in \Omega$ and $\lambda \in [0,1]$ together imply $(\lambda u + (1-\lambda)v) \in \Omega$. △

Definition: Let $f : \Omega \to \mathbb{R}$ for some $\Omega \subseteq \mathbb{R}^n$. The epigraph of f, denoted epi(f) is a subset of $\Omega \times \mathbb{R}$ defined by

$$\text{epi}(f) := \left\{ \begin{bmatrix} x \\ \beta \end{bmatrix} \;\middle|\; x \in \Omega,\ \beta \geq f(x) \right\}. \tag{295}$$

△

Definition: A function $f : \Omega \to \mathbb{R}$ is *convex* if and only if its epigraph is a convex set. △

Theorem: A function $f : \Omega \to \mathbb{R}$ defined on a convex set $\Omega \subseteq \mathbb{R}^n$ is convex if and only if for all $x, y \in \Omega$ and all $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{296}$$

△

Remark: If the inequality in (296) is strict, then we say f is *strictly convex.*

Remark: **How to show convexity:** Suppose we seek to show $\Omega \subset \mathbb{R}^n$ is convex. Let u and v be arbitrary points in Ω and let $\lambda \in (0,1)$. Then, using the properties of u and v inherited from Ω, it suffices to show $(\lambda u + (1-\lambda)v) \in \Omega$.

Suppose we seek to show $f : \Omega \to \mathbb{R}$ is convex for some $\Omega \subseteq \mathbb{R}^n$. Let x and y be arbitrary points in Ω and let $\lambda \in (0,1)$. Then, using the properties of Ω, it suffices to show $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

Example 39: Let $a > 0$ and define $\mathcal{T}$ to be the triangle in the plane with vertices $(0,0)$, $(a,0)$. Show that $\mathcal{T}$ is convex.

Solution:

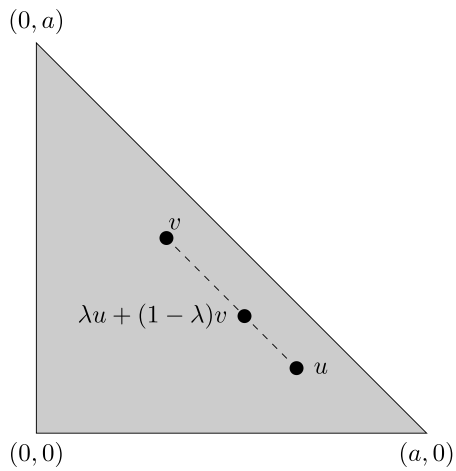

Figure 12: Triangle $\mathcal{T}$ in the plane with vertices $(0,0)$, $(a,0)$ and $(0,a)$

We first illustrate our problem above in Figure 12. Let $u, v \in \mathcal{T}$ and $\lambda \in (0,1)$. We will be done if we show $\lambda u + (1-\lambda)v \in \mathcal{T}$. To do this, we explicitly identify $\mathcal{T}$ as

$$\mathcal{T} = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq a \text{ and } 0 \leq x_2 \leq a - r_1\}.$$

Since u and v are in $\mathcal{T}$, we know $0 \leq u_1, v_1 \leq a$, which implies

$$\lambda \cdot 0 \leq \lambda u_1 \leq \lambda a \quad \text{and} \quad (1-\lambda) \cdot 0 \leq (1-\lambda)v_1 \leq (1-\lambda)a.$$

Adding these inequalities, we obtain

$$0 = \lambda \cdot 0 + (1-\lambda) \cdot 0 \leq \lambda u_1 + (1-\lambda)v_1 \leq \lambda a + (1-\lambda)a = a.$$

This verifies x_1 satisfies the restriction to be in $\mathcal{T}$. For the second component, note that, by hypothesis,

$$0 \leq u_2 \leq a - u_1 \quad \text{and} \quad 0 \leq v_2 \leq a - v_1,$$

which implies

$$\lambda \cdot 0 \leq \lambda u_2 \leq \lambda(a - u_1) \quad \text{and} \quad (1-\lambda) \cdot 0 \leq (1-\lambda)v_2 \leq (1-\lambda)(a - v_1).$$

Again adding these equations together, we obtain

$$0 = \lambda \cdot 0 + (1-\lambda) \cdot 0 \leq \lambda u_2 + (1-\lambda) v_2 \leq \lambda (a - u_1) + (1-\lambda)(a - v_1) = a - (\lambda u_1 + (1-\lambda) v_2) \,.$$

Hence $(\lambda u + (1-\lambda) v) \in \mathcal{T}$, and we are done. $\square$

Example 40: Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{2}x^T Qx - b^T x \tag{297}$$

where $Q = Q^T > 0$ with $Q \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n$. Define the function $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\alpha) := f(x + \alpha d)$ where $x, d \in \mathbb{R}^n$ are fixed vectors and $d \neq 0$. Show $\phi(\alpha)$ is a strictly convex function of α.

Solution:
We first show the second derivative of ϕ, denoted ϕ'', is strictly positive. Then we show ϕ'' is strictly positive if and only if ϕ is strictly convex. By direct computation, we see

$$\phi(\alpha) := f(x + \alpha d) = \frac{1}{2}(x + \alpha d)^T Q(x + \alpha d) - b^T(x + \alpha d) = \alpha^2\left(\frac{1}{2}d^T Qd\right) - \alpha b^T d + f(x) \tag{298}$$

where the third equality follows using linearity and the definition of $f(x)$. This implies ϕ is a quadratic function and

$$\phi'(\alpha) = 2\alpha\left(\frac{1}{2}d^T Qd\right) - b^T d \quad \text{and} \quad \phi''(\alpha) = 2\left(\frac{1}{2}d^T Qd\right) - 0 = \frac{1}{2}d^T Qd > 0. \tag{299}$$

The final inequality for $\phi''(\alpha)$ holds since, by hypothesis, Q is positive definite (written $Q > 0$) and $d \neq 0$.

We now show ϕ is strictly convex if and only if $\phi'' > 0$. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$ and $\lambda \in (0, 1)$. Let $z := \lambda\alpha + (1 - \lambda)\beta$. To verify ϕ is strictly convex, it suffices to show

$$\phi(z) < \lambda\phi(\alpha) + (1 - \lambda)\phi(\beta). \tag{300}$$

By Taylor's theorem, there exists ξ between α and z such that

$$\phi(\alpha) = \phi(z)+\phi'(z)(z-\alpha)+\frac{\phi''(\xi)}{2}(z-\alpha)^2 = \phi(z)+\phi'(z)\cdot(1-\lambda)(\alpha-\beta)+\frac{\phi''}{2}(1-\lambda)^2(\alpha-\beta)^2 \tag{301}$$

where $z - \alpha = (1 - \lambda)(\alpha - \beta)$. Since $\phi''(\alpha)$ is a positive constant for all $\alpha \in \mathbb{R}$, we simply write ϕ'' after the second equality sign. In similar fashion, we obtain

$$\phi(\beta) = \phi(z) + \phi'(z) \cdot (-\lambda)(\alpha - \beta) + \frac{\phi''}{2}\lambda^2(\alpha - \beta)^2. \tag{302}$$

Multiplying (301) and (302) by λ and $(1-\lambda)$ respectively and then adding gives

$$\begin{aligned}\lambda\phi(\alpha)+(1-\lambda)\phi(\beta) &= \phi(z)+\lambda\cdot\phi'(z)\cdot(1-\lambda)(\alpha-\beta)+(1-\lambda)\cdot\phi'(z)\cdot(-\lambda)(\alpha-\beta)\\ &\quad+\frac{\phi''}{2}\left[\lambda(1-\lambda)^2+\lambda^2(1-\lambda)\right](\alpha-\beta)^2\\ &= \phi(z)+\underbrace{\frac{\phi''}{2}\cdot\lambda(1-\lambda)(\alpha-\beta)^2}_{>0}.\end{aligned} \tag{303}$$

Note the second term on the right hand size is positive since ϕ'', λ, $(1-\lambda)$, and $(\alpha-\beta)^2$ are all positive. Then rearranging, we obtain

$$\phi(z)=\lambda\phi(\alpha)+(1-\lambda)\phi(\beta)-\frac{\phi''}{2}\cdot\lambda(1-\lambda)(\alpha-\beta)^2<\lambda\phi(\alpha)+(1-\lambda)\phi(\beta), \tag{304}$$

as desired. The final inequality holds if and only if $\phi''>0$. This shows ϕ is strictly convex and completes the proof. □