Discussion Notes for Undergraduate Analysis (MATH 131B)

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Purpose: This document is a compilation of notes generated for discussion in MATH 131B with reference credit due to Terrence Tao's text *Analysis II*. If the reader finds any errors/typos, please feel free to email me at `heaton@math.ucla.edu` and I will address these and post an updated set of notes to my webpage.

Contents

Introduction

These notes are provided to compliment the TA discussion sessions on Tuesdays for MATH 131B. Typically, more detail is provided here than on the board during discussions since portions of solutions are given orally in class. The examples provided here are meant to be a constructive reference for students. These illustrate how to use certain logical quantifiers, how to guide the reader through your proofs, and the level of rigor desired from students this quarter. Before reading each solution, I highly encourage students to first seriously attempt the problems on their own. I cannot overstate the value of struggling through these problems before comparing your attempts to the example solutions.

These notes will be updated weekly (if not more often) on Mondays or Tuesdays, reflecting the current discussion material.

Metric Spaces

Example 1: Let X be a set.

a) State the four axioms of a metric $d : X \times X \to \mathbb{R}$.

b) Let $X = \{1, 2, 3\}$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) := \begin{cases} 0 & \text{if } x = y, \\ 1/3 & \text{if } x > y, \\ 2/3 & \text{if } x < y. \end{cases} \tag{1}$$

Is (X, d) a metric space? Prove your answer.

Solution:

a) The four axioms are as follows.

   i) For any $x \in X$, $d(x, x) = 0$.

   ii) (positivity) For any distinct $x, y \in X$, $d(x, y) > 0$.

   iii) (triangle inequality) For any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

   iv) (symmetry) For any $x, y \in X$, $d(x, y) = d(y, x)$.

b) We claim (X, d) does *not* form a metric space. Indeed, observe

$$d(1, 2) = \frac{2}{3} \neq \frac{1}{3} = d(2, 1), \tag{2}$$

and so the symmetry axiom iv) for a metric d does not hold.

□

Remark 1: Note in the above example the other three axioms hold. However, to prove (X, d) is not a metric space one only needs to provide a single counter example. ◇

Remark 2: For $p \in [1, \infty)$, we can define the p-norm (also called the ℓ^p norm) of vectors in $\mathbb{R}^n$ by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}. \tag{3}$$

This can in turn be used to define a metric $d_{\ell^p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_{\ell^p}(x, y) := \|x - y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}. \tag{4}$$

We also define the sup norm (or called "max" norm) on vectors in $\mathbb{R}^n$ by

$$\|x\|_\infty := \max_{1 \le i \le n} |x_i|, \tag{5}$$

which leads to the sup metric d_{ℓ^∞} defined by

$$d_{\ell^\infty}(x, y) := \|x - y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|. \tag{6}$$

Consequently, $(\mathbb{R}^n, d_{\ell^p})$ is a metric space for $p \in [1, \infty]$. If we simply say the space $\mathbb{R}^n$, then we implicitly mean the space $(\mathbb{R}^n, d_{\ell^2})$, with which we are well-familiar. $\diamond$

Example 2: Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$ converging to z with respect to the d_{ℓ^∞} metric. Prove $x^{(n)} \longrightarrow z$ with respect to the d_{ℓ^1} metric.

Proof:

Let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$d_{\ell^1}(x^{(n)}, z) < \varepsilon \quad \forall\ n > N. \tag{7}$$

First observe

$$d_{\ell^1}(x^{(n)}, z) = \sum_{i=1}^{m} |x_i^{(n)} - z| \le \sum_{i=1}^{m} \max_{1 \le i \le m} |x_i^{(n)} - z| = m \cdot \max_{1 \le i \le m} |x_i^{(n)} - z| = m \cdot d_{\ell^\infty}(x^{(n)}, z). \tag{8}$$

By the convergence given in our hypothesis, there is $N > 0$ such that

$$d_{\ell^\infty}(x^{(n)}, z) < \frac{\varepsilon}{m} \quad \forall\ n > N, \tag{9}$$

and so

$$d_{\ell^1}(x^{(n)}, z) \le m \cdot d_{\ell^\infty}(x^{(n)}, z) < m \cdot \frac{\varepsilon}{m} = \varepsilon \quad \text{for all } n > N. \tag{10}$$

This verifies (7) and completes the proof. $\blacksquare$

Remark 3: We can also talk about metric spaces in similar fashion to $(\mathbb{R}^n, d_{\ell^p})$, but that are of infinite dimension. Indeed, let $p \in [1,\infty)$ Then for each sequence $x = (x_i)_{i=1}^\infty$ of numbers in $\mathbb{R}$ we can take[1]

$$\|x\|_p := \left(\sum_{i=1}^\infty |x_i|^p\right)^{1/p} \quad \text{and} \quad d_{\ell^p}(x,y) := \|x-y\|_p = \left(\sum_{i=1}^\infty |x_i - y_i|^p\right)^{1/p}. \tag{11}$$

Note the only difference here is that our sums from before now turn into infinite series. Then let X be the space consisting of all sequences with finite p-norm, i.e.,

$$X := \left\{ x = (x_i)_{i=1}^\infty \ : \ \|x\|_{\ell^p} := \left(\sum_{i=1}^\infty |x_i|^p\right)^{1/p} < \infty \right\}. \tag{12}$$

Then (X, d_{ℓ^p}) forms a metric space. This metric space is commonly written simply as ℓ^p.

In similar fashion to above, now consider $p = \infty$. For a sequence $x = (x_i)_{i=1}^\infty)$ we set

$$\|x\|_\infty := \sup_{i\in\mathbb{N}} |x_i| \quad \text{and} \quad d_{\ell^\infty}(x,y) := \sup_{i\in\mathbb{N}} |x_i - y_i|. \tag{13}$$

When

$$X := \left\{ x = (x_i)_{i=1}^\infty \ : \ \|x\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |x_i| < \infty \right\}, \tag{14}$$

the combination (X, d_{ℓ^∞}) forms a metric space, denoted ℓ^∞. $\diamond$

Remark 4: The sequence space ℓ^2 is complete. That is, if a sequence $(x^{(n)})_{n=1}^\infty$ of sequences is in ℓ^2 and is Cauchy, then it converges to some limit in ℓ^2. $\diamond$

[1] Note here we use different notation to denote the sequences in order to avoid confusion in the example below.

Example 3: For each $n \in \mathbb{N}$, let $e^{(n)} = (e_i^{(n)})_{i=1}^{\infty}$ be the sequence of real numbers satisfying

$$e_i^{(n)} = \begin{cases} 1 & \text{if } n = i, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Note $e^{(n)}$ is itself a sequence for fixed $n \in \mathbb{N}$.

a) Show $e^{(n)} \in \ell^2$ for each $n \in \mathbb{N}$.

b) Show the sequence $(e^{(n)})_{n=1}^{\infty}$ is not Cauchy in ℓ^2.

c) Define the sequence $(x^{(n)})_{n=1}^{\infty}$ by $x^{(n)} := n \cdot e^{(n)}$ for each $n \in \mathbb{N}$. Is this sequence $(x^{(n)})_{n=1}^{\infty}$ in ℓ^2?

d) Is $(x^{(n)})_{n=1}^{\infty}$ bounded?

Solution:

a) Observe for each $n \in \mathbb{N}$ we have

$$\|e^{(n)}\|_{\ell^2} = \left(\sum_{i=1}^{\infty} |e_i^{(n)}|^2 \right)^{1/2} = \left(|e_n^{(n)}|^2 \right)^{1/2} = (1^2)^{1/2} = 1 < \infty. \tag{16}$$

Consequently, $e^{(n)} \in \ell^2$ for each $n \in \mathbb{N}$.

b) It suffices to provide a single counter example. Let $\varepsilon = 1$. Then for $m \neq n$ we have

$$\begin{aligned} d_{\ell^2}(e^{(n)}, e^{(m)}) &= \|e^{(n)} - e^{(m)}\|_{\ell^2} \\ &= \left(\sum_{i=1}^{\infty} |e_i^{(n)} - e_i^{(m)}|^2 \right)^{1/2} \\ &= \left(|e_m^{(n)} - e_m^{(m)}|^2 + |e_n^{(n)} - e_n^{(m)}|^2 \right)^{1/2} \\ &= \left(|0-1|^2 + |1-0|^2 \right)^{1/2} \\ &= \sqrt{2}. \end{aligned} \tag{17}$$

Consequently,

$$d_{\ell^2}\left(e^{(n)}, e^{(m)}\right) = \sqrt{2} > \varepsilon \quad \text{whenever } n \neq m. \tag{18}$$

This shows there does *not* exist $N > 0$ such that

$$d_{\ell^2}(e^{(n)}, e^{(m)}) < \varepsilon \quad \forall\ m, n > N, \tag{19}$$

and so the sequence $(e^{(n)})_{n=1}^{\infty}$ is not Cauchy.

c) We claim the sequence $(x^{(n)})_{n=1}^{\infty}$ is in ℓ^2. Indeed, for $n \in \mathbb{N}$ we discover

$$\|x^{(n)}\|_{\ell^2} = \left(\sum_{i=1}^{\infty} |x_i^{(n)}|^2\right)^{1/2} = \left(|x_n^{(n)}|^2\right)^{1/2} = \left(n^2\right)^{1/2} = n < \infty. \tag{20}$$

d) We claim the sequence $(x^{(n)})_{n=1}^{\infty}$ is unbounded. Indeed, by way of contradiction, suppose this sequence is bounded by some $M > 0$. Then by the Archimedean property of $\mathbb{R}$ there is $k \in \mathbb{N}$ such that $k > M$. Consequently, (20) shows

$$\|x^{(k)}\|_{\ell^2} = k > M, \tag{21}$$

contradicting the fact we assumed M was an upper bound. Thus the initial assumption was false and we conclude $(x^{(n)})_{n=1}^{\infty}$ is unbounded.

$\square$

Remark 5: The above example contains a sequence in a sequence space. In other words, $(e^{(n)})_{i=1}^{\infty}$ is a sequence of sequences. $\diamond$

Remark 6: The above example shows results we know in $\mathbb{R}^n$ do not necessarily hold in infinite dimensional spaces. Indeed, recall the Bolzano-Weierstrass theorem discussed in MATH 131A last quarter, which tells us every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence, which in turn is Cauchy. However, in the above example, the sequence $(e^{(n)})_{n=1}^{\infty}$ is bounded, but it does *not* have a Cauchy subsequence. $\diamond$

Lemma: Let $\alpha \in (0,1)$. Then $\alpha^n \longrightarrow 0$ as $n \longrightarrow \infty$. $\triangle$

Proof:

We claim this sequence is decreasing. Indeed, for each $n \in \mathbb{N}$ we see $\alpha^{n+1} = \alpha\alpha^n < \alpha^n$. And the sequence is nonnegative. Consequently, the monotone convergence theorem implies this sequence converges to a limit z. Moreover,

$$z = \lim_{n\to\infty} \alpha^n = \lim_{n\to\infty} \alpha^{n+1} = \alpha \lim_{n\to\infty} \alpha^n = \alpha z, \tag{22}$$

which holds precisely when $z = 0$. This completes the proof. $\blacksquare$

Example 4: Let (X, d) be a metric space. Suppose the sequence $(x^{(n)})_{n=1}^{\infty}$ satisfies the inequality

$$d(x^{(n+1)}, x^{(n)}) \leq \alpha^{n+1} \quad \text{for } n \in \mathbb{N}, \tag{23}$$

where $\alpha \in (0,1)$. Is this sequence Cauchy? Prove your answer.

Proof:

We claim this sequence is Cauchy. Let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$d(x^{(m)}, x^{(n)}) < \varepsilon \quad \forall\ m, n > N. \tag{24}$$

Observe for $m, n \in \mathbb{N}$ with $n \geq m$ the triangle inequality yields

$$d(x^{(m)}, x^{(n)}) \leq \sum_{j=1}^{n-m} d(x^{(m-1+j)}, x^{(m+j)}) \leq \sum_{j=1}^{n-m} \alpha^{m+j}. \tag{25}$$

Evaluating this geometric sum gives

$$d(x^{(m)}, x^{(n)}) \leq \alpha^m \sum_{j=1}^{n-m} \alpha^j = \alpha^m \left(\frac{1-\alpha^{n-m+1}}{1-\alpha}\right) \leq \frac{\alpha^m}{1-\alpha}. \tag{26}$$

From the above lemma, we know $\alpha^m/(1-\alpha) \longrightarrow 0$ as $m \longrightarrow \infty$. This implies there exists $N > 0$ such that

$$\left|\frac{\alpha^m}{1-\alpha}\right| < \varepsilon \quad \forall\ m > N. \tag{27}$$

Equations (26) and (27) together imply (24) holds, and we are done. $\blacksquare$

Example 5: Let (X, d) be a metric space and fix $z \in X$. The define the function $f : X \to \mathbb{R}$ by $f(x) := d(x, z)$. Prove f is continuous.

Proof:

This is currently left to the reader. A solution will be posted at a later date. ■

Example 6: Let (X, d) be a metric space and $x \in X$. Prove the set $S := \{x\} \subset X$ is closed.

Proof:

We use the fact a set is closed if and only if its complement is open. It thus suffices to show $S^c = X - \{x\}$ is open. Pick $y \in S^c$. Then $y \neq x$, and so $d(x, y) > 0$. Let $r = d(x, z)/2$ and $z \in B(y, r)$. Then the triangle inequality yields

$$d(x, y) \leq d(x, z) + d(z, y) \quad \Longrightarrow \quad d(x, z) \geq d(x, y) - d(y, z) = s - d(y, z) > s - \frac{s}{2} = \frac{s}{2} > 0, \tag{28}$$

and so $z \neq x$, which implies $z \in S^c$. This reveals $B(y, r) \subset S^c$, and so $y \in \text{int}(S^c)$. Because y was chosen arbitrarily in S^c, it follows that S^c is open, from which we conclude $S = \{x\}$ is closed. ■

Remark 7: For the following example, it may be helpful to draw several pictures, visualizing E and the multiple open balls inside it, each centered at $x \in E$. $\diamond$

Example 7: Is every point of every open set $E \subset \mathbb{R}^n$ an adherent point of E? (Use the Euclidean norm.)

Proof:

We claim the answer is yes. Let $x = (x_1, x_2, \dots, x_n) \in E$. To verify x is a limit point of E, we must show for each neighborhood of x there exists $y \in E$ distinct from x (i.e., $y \neq x$) contained in the neighborhood. Let $r > 0$ be given. Then it suffices to find $y \in E \cap B(x, r)$ with $y \neq x$. We do this as follows. Since E is open and $x \in E$, there is $s > 0$ such that $B(x, s) \subset E$. Let $d = \frac{1}{2}\min\{s, r\}$ so that $B(x, d) \subset B(x, s) \cap B(x, r) \subset E \cap B(x, r)$. Then choose

$$y = (y_1, y_2, \dots, y_n) = (x_1 + d/2, x_2, \dots, x_n), \tag{29}$$

noting $y \neq x$, and observe

$$\|x - y\|_2 = \left(\sum_{i=1}^{n} (y_i - x_i)^2\right)^{1/2} = \left((d/2)^2\right)^{1/2} = d/2 < d. \tag{30}$$

Consequently, $y \in B(x, d) \subset E \cap B(x, r)$, and we are done. ■

Example 8: Is every point of every closed set $E \subset \mathbb{R}^n$ an adherent point of E?

Proof:

We claim the answer is no. Let $x \in \mathbb{R}$. Then the singleton set $E = \{x\}$ is closed. And, there are no elements in E distinct from x contained in the open ball $B(x, r)$ for any $r > 0$, and we are done. ■

Relatively Open/Closed Sets:

Example 9: Define $X = \mathbb{R}$. Let $Y = [0, 2]$ and $E = [0, 1/2)$. Prove

a) E is not open with respect to X;

b) E is not closed with respect to X;

c) E is open with respect to Y.

Proof:

a) We now show E is not open in X. It suffices consider $y = -1$. For each $r > 0$, the open ball $B(y, r) \subset X$ contains the point $y - r/2 = -1 - r/2 < -1 \notin E$. Thus -1 is not an interior point of E and so E is not open.

b) We now show E is not closed. Recall by Proposition 1.2.15b, E is closed if and only if E contains all of its adherent points. So, it suffices to find an adherent point of E not contained in E. We claim 1 is such a point. Indeed, define the sequence $(x^{(n)})_{n=1}^{\infty}$ by $x^{(n)} = 1 - 1/n$. Then

$$0 = 1 - 1/1 \leq 1 - 1/n = x^{(n)} < 1, \tag{31}$$

and so $x^{(n)} \in E$ for each $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then the Archimedean property of $\mathbb{R}$ asserts there is $N \in \mathbb{N}$ such that $1/N < \varepsilon$, and so

$$|1 - x^{(n)}| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon \ \ \forall\, n > N. \tag{32}$$

This shows $x^{(n)} \longrightarrow 1 \notin E$. Thus 1 is an adherent point of E not contained in E. This completes the proof.

c) By Proposition 1.3.4 in our text, a subset $E \subseteq Y \subseteq X$ is open with respect to Y if and only if there is a subset $V \subset X$ such that V is open with respect to X and $E = V \cap Y$. Let $V = (-1/2, 1/2)$ and observe V is open since this is precisely an open ball about 0 of radius $1/2$, i.e., $V = B(0, 1/2)$. Furthermore,

$$V \cap Y = (-1/2, 1/2) \cap [0, 2] = [0, 1/2) = E. \tag{33}$$

Therefore E is open with respect to Y.

■

Example 10: Let $X = \mathbb{R}^2$, $d = d_{\ell^2}$, $Y = (0,1) \times [0,1]$, and $E = (0,1) \times [0,1/2]$. Note (X,d) forms a metric space. Is E relatively closed in Y?

Proof:

We claim the answer is *yes.* Proposition 1.3.4b asserts E is relatively closed in Y if and only if there is a subset $K \subseteq X$ that is closed in X and satisfies $E = K \cap Y$. Let $K = [0,1] \times [0,1/2]$. Since K is the Cartesian product of two closed intervals, it is closed in X. Moreover,

$$\begin{aligned} K \cap Y &= ([0,1] \times [0,1/2]) \bigcap ((0,1) \times [0,1]) \\ &= ([0,1] \cap (0,1)) \times ([0,1/2] \cap [0,1]) \\ &= (0,1) \times [0,1/2] \\ &= E. \end{aligned} \tag{34}$$

The claim then follows from Proposition 1.3.4b. ■

Cauchy Sequences:

Example 11: Let (X, d) be a metric space. Prove every Cauchy sequence in (X, d) has at most one limit point.

Proof:

Let $(x^{(n)})_{n=1}^{\infty} \subseteq X$ be a Cauchy sequence. If this sequence has zero limit points, then we are done (since $0 < 1$). Now suppose this is not the case, i.e., the sequence has at least one limit point. Then let y and z be limits points of $(x^{(n)})$. It suffices to show $y = z$, which we do as follows.

Let $\varepsilon > 0$. Since $(x^{(n)})$ is Cauchy, there is N_1 such that $m, n > N_1$ implies $d(x^{(m)}, x^{(n)}) < \varepsilon$. Also because y is a limit point of $(x^{(n)})$, there is a subsequence $(x^{(n_k)}) \subseteq (x^{(n)})$ and N_2 such that $k > N_2$ implies $d(x^{(n_k)}, y) < \varepsilon$. Similarly, there is a subsequence $(x^{(m_k)}) \subseteq (x^{(n)})$ and N_3 such that $k > N_3$ implies $d(x^{(m_k)}, z) < \varepsilon$. Now let $N = \max\{N_1, N_2, N_3\}$. Noting $m_k \geq k$ and $n_k \geq k$, it follows that $k > N$ implies

$$\begin{aligned} d(y,z) &\leq d(y, x^{(n_k)}) + d(x^{(n_k)}, z) \\ &\leq d(y, x^{(n_k)}) + d(x^{(n_k)}, x^{(m_k)}) + d(x^{(m_k)}, z) \\ &< \varepsilon + \varepsilon + \varepsilon \\ &= 3\varepsilon. \end{aligned} \tag{35}$$

Consequently, $0 \leq d(y, z) < 3\varepsilon$ for each $\varepsilon > 0$. Taking the limit as $\varepsilon \longrightarrow 0$, the squeeze theorem asserts $d(y, z) = 0$, which implies $y = z$, as desired. ■

Example 12: Let (X, d) be a metric space. Suppose the sequence $(y^{(n)})_{n=1}^{\infty} \subset X$ converges to a limit y^* and the sequence $(x^{(n)})_{n=1}^{\infty}$ is Cauchy. Prove the sequence $(d(x^{(n)}, y^{(n)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$.

Proof:

First note the metric on $\mathbb{R}$ is implicitly given to be the absolute value of the difference of two numbers. Then let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$\left| d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) \right| < \varepsilon \quad \forall\ m, n > N. \tag{36}$$

Observe application of the triangle inequality yields

$$\begin{aligned} d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) &\le d(x^{(m)}, y^{(n)}) + d(y^{(n)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) \\ &\le d(x^{(m)}, x^{(n)}) + d(x^{(n)}, y^{(n)}) + d(y^{(n)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) \\ &= d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^{(m)}) \\ &= d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}). \end{aligned} \tag{37}$$

Swapping the positions of m and n and using the symmetry of the metric d, we obtain

$$d(x^{(n)}, y^{(n)}) - d(x^{(m)}, y^{(m)}) \le d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}), \tag{38}$$

from which we deduce

$$\left| d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) \right| \le d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}). \tag{39}$$

Now since $(x^{(n)})_{n=1}^{\infty}$ is Cauchy, there is $N_1 > 0$ such that

$$d(x^{(n)}, x^{(m)}) < \frac{\varepsilon}{3} \quad \forall\ m, n > N_1. \tag{40}$$

Similarly, by the convergence of our other sequence, there is $N_2 > 0$ such that

$$d(y^{(n)}, y^*) < \frac{\varepsilon}{3} \quad \forall\ n > N_2. \tag{41}$$

Letting $N := \max\{N_1, N_2\}$, we deduce $m, n > N$ implies

$$\left| d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) \right| \le d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \tag{42}$$

and we are done. ■

Example 13: Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^2$. Suppose there is $y \in \mathbb{R}^2$ such that $d_{\ell^2}(y, x)$ is monotonically decreasing. If $(x^{(n)})_{n=1}^{\infty}$ has a convergent subsequence, does the entire sequence $(x^{(n)})_{n=1}^{\infty}$ necessarily converge in $(\mathbb{R}^2, d_{\ell^2})$? Prove your answer.

Proof:

We claim the sequence $(x^{(n)})_{n=1}^{\infty}$ need not converge. For example, consider the sequence defined by

$$x^{(n)} := \big((-1)^n, (-1)^{n+1}\big) \quad \forall\, n \in \mathbb{N}. \tag{43}$$

Now let $y = (0, 0)$ and observe

$$d_{\ell^2}(x^{(n)}, y) = \sqrt{((-1)^n - 0)^2 + ((-1)^{n+1} - 0)^2} = \sqrt{2} \;\; \forall\, n \in \mathbb{N}, \tag{44}$$

and so $d_{\ell^2}(x^{(n+1)}, y) \leq d_{\ell^2}(x^{(n)}, y)$ for each $n \in \mathbb{N}$. Next note $x^{(2n)} = x^{(n)}$ for each $n \in \mathbb{N}$, and so the subsequence $(x^{(4n)})_{n=1}^{\infty}$ converges to x^1. However, $(x^{(n)})_{n=1}^{\infty}$ does *not* converge. Indeed,

$$d_{\ell^2}(x^{(n)}, x^{(m)}) = 2\sqrt{2} \;\; \text{whenever } m \text{ mod} 2 \neq n \text{ mod} 2. \tag{45}$$

From this, we deduce the sequence $(x^{(n)})_{n=1}^{\infty}$ cannot be Cauchy. Indeed, if it were, then there would exist $N > 0$ such that $m, n > N$ implies

$$d_{\ell^2}(x^{(n)}, x^{(m)}) < 1, \tag{46}$$

a contradiction to (45). Because every convergent sequence is Cauchy and $(x^{(n)})_{n=1}^{\infty}$ is not Cauchy, this sequence does not converge. ■

Example 14: Let $C[a,b]$ denote the collection of all continuous functions on $[a,b]$, i.e.,

$$C[a,b] := \{f : [a,b] \to \mathbb{R} \ : \ f \text{ is continuous}\}. \tag{47}$$

Define the sup metric $d : C[a,b] \times C[a,b] \to \mathbb{R}$ by

$$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|. \tag{48}$$

Prove d is in fact a metric on $C[a,b]$.

Proof:

First observe the sup metric is well-defined since every continuous function obtains its extreme values on a closed interval $[a,b]$ by the extreme value theorem. (Recall the sum of continuous functions is continuous.) Then we see

$$\forall \ f \in C[a,b], \quad d(f,f) = \sup_{x\in[a,b]} |f(x) - f(x)| = \sup_{x\in[a,b]} 0 = 0. \tag{49}$$

We also see the metric is symmetric since

$$\forall \ f,g \in C[a,b], \quad d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)| = \sup_{x\in[a,b]} |g(x) - f(x)| = d(g,f). \tag{50}$$

Additionally, the triangle inequality holds since

$$\begin{aligned} \forall \ f,g,h \in C[a,b], \quad d(f,g) &= \sup_{x\in[a,b]} |f(x) - g(x)| \\ &\leq \sup_{x\in[a,b]} \left(|f(x) - h(x)| + |h(x) - g(x)|\right) \\ &\leq \sup_{x\in[a,b]} |f(x) - h(x)| + \sup_{x\in[a,b]} |h(x) - g(x)| \\ &= d(f,h) + d(h,g). \end{aligned} \tag{51}$$

Lastly, suppose $f \neq g$. Then there is $z \in [a,b]$ such that $f(z) \neq g(z)$, which implies $f(z) - g(z) \neq 0$. Consequently,

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)| \geq |f(z) - g(z)| > 0. \tag{52}$$

We have now verified each fo the four axioms of a metric and conclude. ■

Example 15: Define $f : [2,4] \to \mathbb{R}$ by $f(x) = x^2 + 10x - 2$ and $g : [2,4] \to \mathbb{R}$ by $g(x) = 2x^2 + 4x + 2$. Compute $d_\infty(f,g)$.

Proof:
Observe

$$d_\infty(f,g) = \sup_{x\in[2,4]} |f(x) - g(x)| = \sup_{x\in[2,4]} \left|(x^2 + 10x - 2) - (2x^2 + 4x + 2)\right| = \sup_{x\in[2,4]} \left|-x^2 + 6x - 4\right|. \tag{53}$$

Then by completing the square yields

$$d_\infty(f,g) = \sup_{x\in[2,4]} \left|5 - (x^2 - 6x + 9)\right| = \sup_{x\in[2,4]} \left|5 - (x-3)^2\right| = \sup_{x\in[2,4]} 5 - (x-3)^2 = \boxed{5.} \tag{54}$$

■

Example 16: Define the sequence of functions $(f_n : [0,2] \to \mathbb{R})_{n=1}^{\infty}$ by

$$f_n(x) := \begin{cases} 1 - nx & \text{if } x \in [0, 1/n]|, \\ 0 & \text{if } x \in (1/n, 2]. \end{cases} \tag{55}$$

Prove this sequence converges point-wise to the function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0, 2]. \end{cases} \tag{56}$$

Proof:
Let $\varepsilon > 0$ be given and choose any $x \in (0,2]$. Then by the Archimedean property of $\mathbb{R}$ there is $N \in \mathbb{N}$ such that $1/N < x$. This implies $x \in (1/n, 2]$ for $n > N$, and so

$$|f_n(x) - 0| = |0 - 0| = 0 < \varepsilon \quad \forall\, n > N. \tag{57}$$

This shows $\lim_{n\to\infty} f_n(x) = 0 = f(x)$ for $x \in (0,2]$. Now observe

$$|f_n(0) - 1| = |1 - n0 - 1| = 0 < \varepsilon \quad \forall\, n \in \mathbb{N}. \tag{58}$$

This shows $\lim_{n\to\infty} f_n(0) = 1 = f(0)$ and completes the proof. ■

Remark 8: The above result will be used as a point of reference in the following examples. Remember if a sequence converges, then it converges to a unique limit. In this case, if $(f_n)_{n=1}^{\infty}$ converges, then $f_n \longrightarrow f$. However, such convergence is dependent upon the metric space in which we speak of this convergence, as the following examples illustrate. ◇

Example 17: Let $X = C[0,2]$ and d be the sup norm metric defined on $C[0,2]$. Prove the sequence of functions defined in Example 16 is not Cauchy in the metric space (X, d).

Proof:

Let $\varepsilon > 0$ be given and, by way of contradiction, suppose $(f_n)_{n=1}^{\infty}$ is Cauchy. Then there is $N > 0$ such that

$$d(f_m, f_n) < \varepsilon \ \ \forall\, m, n > N. \tag{59}$$

Now let $m, n \in \mathbb{N}$ and assume $m > n$ without loss of generality. Then

$$d(f_m, f_n) = \sup_{x\in[0,2]} |f_m(x) - f_n(x)| = \sup_{x\in[0,2]} |f_m(1/m) - f_n(1/m)| = f_n(1/m) = 1 - \frac{n}{m}. \tag{60}$$

The second equality follows from the fact

$$f_n(x) - f_m(x) = \begin{cases} (m-n)x & \text{for } x \in [0, 1/m], \\ 1 - nx & \text{for } x \in (1/m, 1/n], \\ 0 & \text{for } x \in (1/n, 2]. \end{cases} \tag{61}$$

(The reader is also encouraged to make a simple plot of f_n and f_m to gain more intuition.) The result in (60) implies

$$d(f_n, f_{3n}) = 1 - \frac{n}{3n} = 1 - \frac{1}{3} = \frac{2}{3} > \frac{1}{2} \ \ \forall\, n \in \mathbb{N}, \tag{62}$$

a contradiction to (59). Thus $(f_n)_{n=1}^{\infty}$ is *not* Cauchy in (X, d). ■

Remark 9: The above result should not be too surprising. Why? Remember that if $(f_n)_{n=1}^{\infty}$ converges in X, then its limit is in X (i.e., $f \in X$). However, the function f is not continuous, and so $f \notin X$. Also, the space (X, d) in the above example is complete. So, a sequence converges in X if and only if it is Cauchy. Thus, having the result of Example 16, we should be able to intuitively guess the sequence $(f_n)_{n=1}^{\infty}$ is not Cauchy. ◇

Example 18: Define the space of functions X by

$$X := \left\{ f : [0,2] \to \mathbb{R} \ : \ \int_0^1 |f(x)| \mathrm{d}x < \infty \right\}. \tag{63}$$

Then define the metric $d : X \times X \to \mathbb{R}$ by

$$d(f,g) = \int_0^2 |f(x) - g(x)| \ \mathrm{d}x. \tag{64}$$

Then (X,d) forms a metric space (which we shall not prove here). Show the sequence of functions defined in Example 16 converges in the metric space (X,d).

Proof:
First note each bounded function defined on $[0,2]$ is contained in X. This implies $f_n \in X$ for each $n \in \mathbb{N}$ and $f \in X$ where f is defined as in (56). Let $\varepsilon > 0$ be given. We claim $f_n \longrightarrow f$ and to show this it suffices to verify there is $N > 0$ such that

$$d(f_n, f) < \varepsilon \ \ \forall \ n > N. \tag{65}$$

Observe we have[2]

$$d(f_n, f) = \int_0^2 |f_n(x) - f(x)| \ \mathrm{d}x = \int_0^2 |f_n(x)| \ \mathrm{d}x = \frac{1}{2n}. \tag{66}$$

Note the single point $x = 0$ where $f(x) = 1$ does not affect the value of the integral.[3] By the Archimedean property of $\mathbb{R}$, there is $N > 0$ such that $1/N < \varepsilon$. Consequently, we deduce

$$d(f_n, f) = \frac{1}{2n} < \frac{1}{n} < \frac{1}{N} < \varepsilon \ \ \forall \ n > N, \tag{67}$$

which completes the proof. ■

Remark 10: This example shows we the sequence $(f_n)_{n=1}^{\infty}$ converges in one metric space, but does not converge in another. ◇

[2]Here we do not provide the details of this calculation. The reader should note the area of a triangle is half its with multiplied by its height, and the integral yields precisely the integral of triangle of width $1/n$ and height 1.

[3]This can be shown by looking at the upper and lower Darboux sums.

Compactness:

Here we provide a few examples related to Section 1.5 of our text.

Example 19: Let $S = [0,1]$. Show there exists a finite open cover of S and give an example.

Proof:

First note S is a closed interval and is bounded since $[0,1] \subset B(0,2) = (-2,2)$. By the Heine-Borel theorem, it follows that S is compact. Then Theorem 1.5.8 asserts for each open cover $\{V_\alpha\}_{\alpha \in I}$ of S, there exits a finite subcover of S. In other words, there is a finite subset $F \subset I$ such that

$$[0,1] = S \subset \bigcup_{i \in F} V_i. \tag{68}$$

Now let us provide an explicit example. For each $x \in [0,1]$, let $V_x = B(x, 1/2)$. Then the collection of all V_x with $x \in [0,1]$ forms a open cover of $[0,1]$. Our result above tells us there is some finite subset of $[0,1]$, denoted F, such that

$$[0,1] \subset \bigcup_{x \in F} V_x. \tag{69}$$

For example, observe we could use $F = \{0, 1/2, 1\}$ and obtain

$$B(0,1/2) \cup B(1/2,1/2) \cup B(1,1/2) = (-1/2,1/2) \cup (0,1) \cup (1/2,3/2) = (-1/2,3/2) \supset [0,1]. \tag{70}$$

■

Example 20: Explicitly show the set $Y \subset \mathbb{R}$ defined by $Y := \{(a,b) \ : \ a \in [-3,5],\ b \in [3,12]\}$ is bounded in $(\mathbb{R}^2, d_{\ell^2})$.

Proof:

We say $Y \subset \mathbb{R}^2$ is bounded if and only if there exists a ball $B(x,r) \subseteq \mathbb{R}^2$ which contains Y. Take $x = (0,0) \in \mathbb{R}^2$. Then observe

$$\forall\ (a,b) \in Y, \quad d_{\ell^2}((a,b),(0,0)) = \sqrt{a^2+b^2} \leq \sqrt{5^2+b^2} \leq \sqrt{5^2+12^2} = \sqrt{169} = 13. \tag{71}$$

This implies for each $(a,b) \in Y$, $(a,b) \in B(x,14)$. Letting $r = 14$, we deduce $Y \subset B(x,r) \subset \mathbb{R}^2$ and thus we conclude Y is bounded. ■

Example 21: Give an example of an open cover of $(-1,1)$ for which there is no finite subcover.

Solution:

Consider the collection of open intervals $\{V_n\}_{n=1}^{\infty}$ with $V_n := (-1, 1-1/n)$. We claim

$$(-1,1) = \bigcup_{n=1}^{\infty} (-1, 1-1/n). \tag{72}$$

For each n, it directly follows that $(-1,1-1/n) \subset (-1,1)$. Conversely, suppose $x \in (-1,1)$. Then there is $r > 0$ such that $B(x,r) \subseteq (-1,1)$. This implies $x + r < 1$, and so $1 - (x+r) > 0$. By the Archimedean property of $\mathbb{R}$, we can pick $N \in \mathbb{N}$ such that $1/N < 1 - (x+r)$. For such N, we see

$$x \in B(x,r) \subseteq (-1, 1-1/N) \subset \bigcup_{n=1}^{\infty} (-1, 1-1/n). \tag{73}$$

This shows $(-1,1)$ is contained in the open cover.

Now note that for each finite subcollection $\{V_{n_j}\}_{j=1}^{J}$ of $\{V_n\}_{n=1}^{\infty}$ we have

$$\bigcup_{j=1}^{J} V_{n_j} = (-1, 1-1/n_J). \tag{74}$$

Consequently,

$$\left(1 - \frac{1}{2n_J}\right) \notin (-1, 1-1/n_J) = \bigcup_{j=1}^{J} V_{n_j}. \tag{75}$$

Thus there is no finite subcover of $\{V_n\}_{n=1}^{\infty}$ containing $(-1,1)$. $\square$

Example 22: Prove if (X, d) is a compact metric space, then X is complete.

Proof:

Let $(x^{(n)})_{n=1}^{\infty} \subseteq X$ be Cauchy. We most prove this sequence converges. Because X is compact, every sequence in X has a convergent subsequence. This implies there is a subsequence $(x^{(n_k)})_{k=1}^{\infty} \subseteq (x^{(n)})_{n=1}^{\infty}$ that converges to a limit $z \in X$. We claim $x^{(n)} \longrightarrow z$ and verify this as follows.

Let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$d(x^{(n)}, z) < \varepsilon \quad \forall\, n > N. \tag{76}$$

By hypothesis, there is $N_1 > 0$ such that

$$d(x^{(n_k)}, z) < \frac{\varepsilon}{2} \quad \forall\, k > N_1. \tag{77}$$

Similarly, there is $N_2 > 0$ such that

$$d(x^{(n)}, x^{(m)}) < \frac{\varepsilon}{2} \quad \forall\, n, m > N_2. \tag{78}$$

Now let $N = \max\{N_1, N_2\}$ and note $n_{N+1} \geq N + 1 > N$. Together with (77) and (78), this implies

$$d(x^{(m)}, z) \leq d(x^{(m)}, x^{(n_{N+1})}) + d(x^{(n_{N+1})}, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, m > N. \tag{79}$$

This verifies (76) and completes the proof. ■

Example 23: We say a metric space (X, d) is *totally bounded* provided for each $\varepsilon > 0$, there is $n \in \mathbb{N}$ and a finite collection $\{B(x_i, \varepsilon)\}_{i=1}^n$ which covers X, i.e.,

$$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon). \tag{80}$$

Prove every totally bounded space (X, d) is bounded.

Proof:

Suppose (X, d) is totally bounded. We must show there is a ball $B(x, r) \subseteq X$ such that $X \subseteq B(x, r)$. By hypothesis, there is $n \in \mathbb{N}$ and a collection $\{B(x_i, 1)\}_{i=1}^n$ such that

$$X = \bigcup_{i=1}^{n} B(x_i, 1). \tag{81}$$

This implies for each $z \in X$, there is an index $j_z \in \{1, 2, \dots, n\}$ such that $z \in B(x_{j_z}, 1)$. Now set

$$\alpha := \max_{1 \leq i \leq n} d(x_1, x_i). \tag{82}$$

Then application of the triangle inequality yields

$$d(x_1, z) \leq d(x_1, x_{j_z}) + d(x_{j_z}, z) \leq \alpha + d(x_{j_z}, z) < \alpha + 1. \tag{83}$$

Consequently, $z \in B(x_1, \alpha+1)$. Because z was chosen arbitrarily in X, we deduce $X \subseteq B(x_1, \alpha+1)$. Thus we conclude X is bounded. ■

Continuity:

Example 24: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 0 & \text{if } x = y = 0, \\ \dfrac{2xy}{x^2 + y^2} & \text{otherwise.} \end{cases} \tag{84}$$

Prove f is not continuous with respect to the Euclidean metric d_{ℓ^2}.

Proof:

By Theorem 2.1.5, f is continuous if and only if whenever a sequence $(x^{(n)}) \subseteq \mathbb{R}^2$ converges to a point $x \in \mathbb{R}^2$ with respect to d_{ℓ^2} the sequence $(f(x^{(n)}))$ converges to $f(x)$. Consequently, to verify f is not continuous, it suffices to construct a sequence converging to $(0, 0) \in \mathbb{R}^2$ for which the limit of the function values do not converge to $f(0, 0) = 0$. For the sequence $((1/n, 1/n))_{n=1}^{\infty} \subseteq \mathbb{R}^2$ observe

$$d_{\ell^2}\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right) = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2} = \frac{\sqrt{2}}{n}. \tag{85}$$

Thus

$$\lim_{n\to\infty} d_{\ell^2}\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{\sqrt{2}}{n} = 0, \tag{86}$$

which implies

$$\lim_{n\to\infty} \left(\frac{1}{n}, \frac{1}{n}\right) = (0, 0). \tag{87}$$

Moreover,

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{2(1/n)^2}{(1/n)^2 + (1/n)^2} = 1 \quad \text{for each } n \in \mathbb{N}. \tag{88}$$

This shows

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} 1 = 1 \neq 0 = f(0, 0), \tag{89}$$

and we conclude f is not continuous. ■

Example 25: Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is uniformly continuous. Prove if the sequence $(x^{(n)}) \subseteq X$ is Cauchy, then $\big(f(x^{(n)})\big) \subseteq Y$ is Cauchy.

Proof:

Suppose the sequence $(x^{(n)}) \subseteq X$ is Cauchy. Let $\varepsilon > 0$ be given. Then to show $\big(f(x^{(n)})\big)$ is Cauchy, we must show there is N such that

$$m, n > N \quad \implies \quad d_Y(f(x^{(n)}), f(x^{(m)})) < \varepsilon. \tag{90}$$

By the uniform continuity of f, there is $\delta > 0$ such that for $x, y \in X$

$$d_X(x, y) < \delta \quad \implies \quad d_Y(f(x), f(y)) < \varepsilon. \tag{91}$$

And because the sequence $(x^{(n)})$ is Cauchy, there is N such that

$$m, n > N \quad \implies \quad d_X(x^{(n)}, x^{(m)}) < \delta. \tag{92}$$

Combining (91) and (92) reveals if $m, n > N$, then

$$d_X(x^{(n)}, x^{(m)}) < \delta \quad \implies \quad d_Y(f(x^{(n)}), f(x^{(m)})) < \varepsilon. \tag{93}$$

This verifies (90) and completes the proof. ■

Example 26: Let $f : X \to Y$ be continuous for some metric spaces (X, d_X) and (Y, d_Y). Fix $\alpha \in Y$. Prove the set $S = \{x \in X : f(x) = \alpha\}$ is closed.

Proof:

By Proposition 1.2.15b, S is closed if and only if it contains all its adherent pints. So, let $(x^{(n)}) \subseteq S$ be a sequence converging to a limit $x \in X$. Then the continuity of f reveals

$$\alpha = \lim_{n\to\infty} \alpha = \lim_{n\to\infty} f(x^{(n)}) = f\left(\lim_{n\to\infty} x^{(n)}\right) = f(x), \tag{94}$$

which implies $x \in S$. Because the sequence $(x^{(n)})$ was chosen arbitrarily in S, this shows S contains its adherent points and we conclude S is closed. ■

We now present an alternative proof.

Proof:

Note $S = \{x \in X : f(x) = \alpha\} = f^{-1}(\{\alpha\})$. Because $\{\alpha\}$ is a singleton set, Proposition 1.2.15d asserts $\{\alpha\}$ is closed. Then Theorem 2.1.5d asserts $f^{-1}(\{\alpha\})$ is closed, which completes the proof. ■

Example 27:

a) Prove the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.

b) Prove the function $g : [0, 2] \to \mathbb{R}$ defined by $g(x) = x^2$ is uniformly continuous.

Proof:

a) By way of contradiction, suppose f is uniformly continuous. Then there is $\delta > 0$ such that for $x, y \in \mathbb{R}$

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < 1. \tag{95}$$

Fix any $x \in \mathbb{R}$ and choose $y = x + \delta/3$. Then $|x - y| = \delta/3 < \delta$ and so

$$1 > |f(x) - f(y)| = \left|x^2 - y^2\right| = \left|x^2 - \left(x + \frac{\delta}{3}\right)^2\right| = \left|-\frac{2\delta x}{3} + \frac{\delta^2}{9}\right| = \frac{\delta}{3}\left|\frac{\delta}{3} - 2x\right|. \tag{96}$$

This implies

$$\frac{\delta}{3}\left(\frac{\delta}{3} - 2x\right) < -1 \quad \Longrightarrow \quad \frac{\delta}{3} - 2x < -\frac{3}{\delta} \quad \Longrightarrow \quad x > \frac{1}{2}\left(\frac{\delta}{3} + \frac{3}{\delta}\right) > 0. \tag{97}$$

Since x was chosen arbitrarily, this implies for each $x > 0$ for each $x \in \mathbb{R}$, a contradiction. This contradiction shows the initial assumption was false and the result follows.

b) Let $\varepsilon > 0$ be given. We must show there is $\delta > 0$ such that for $x, y \in [0, 2]$

$$|x - y| < \delta \quad \Longrightarrow \quad |g(x) - g(y)| < \varepsilon. \tag{98}$$

Observe for $x, y \in [0, 2]$

$$|g(x) - g(y)| = |x^2 - y^2| = |x - y||x + y| \leq |x - y|\left(|x| + |y|\right) \leq |x - y|\left(2 + 2\right) = 4|x - y|. \tag{99}$$

Letting $\delta = \varepsilon/4$, (99) reveals

$$|x - y| < \delta \quad \Longrightarrow \quad |g(x) - g(y)| \leq 4|x - y| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon. \tag{100}$$

This verifies (98) and completes the proof.

■

Example 28: Let X be $\mathbb{R}^2$ equipped with the d_{ℓ^1} metric and Y be $\mathbb{R}$ with the standard metric. Prove $f : X \to Y$ defined by $f(x) = \alpha x_1 - x_2$ is uniformly continuous, where $\alpha \in \mathbb{R}$ is nonzero.

Proof:

Let $\varepsilon > 0$ be given. We must show there is $\delta > 0$ such that for $x, y \in |X$

$$d_{\ell^1}(x,y) < \delta \quad\implies\quad |f(x) - f(y)| < \varepsilon. \tag{101}$$

Observe for $x, y \in X$ that

$$\begin{aligned} |f(x) - f(y)| &= |(\alpha x_1 - x_2) - (\alpha y_1 - y_2)| \\ &= |\alpha(x_1 - y_1) - (x_2 - y_2)| \\ &\leq |\alpha||x_1 - y_1| + |x_2 - y_2| \\ &\leq \max\{1, \alpha\}\,[|x_1 - y_1| + |x_2 - y_2|] \\ &= \max\{1, \alpha\} \cdot d_{\ell^1}(x, y). \end{aligned} \tag{102}$$

Letting $\delta = \varepsilon / \max\{\alpha, 1\}$, we see for $x, y \in X$

$$d_{\ell^1}(x,y) < \delta \quad\implies\quad |f(x) - f(y)| \leq \max\{\alpha, 1\} \cdot d_{\ell^1}(x,y) < \max\{\alpha, 1\} \cdot \delta = \varepsilon. \tag{103}$$

This verifies (101) and completes the proof. ■

Example 29: Let A and B be two nonempty open subsets of $\mathbb{R}$. Prove if $A \cup B$ and $A \cap B$ are connected, then A is connected.

Proof:

By way of contradiction, suppose A is not connected. Then there are nonempty subsets $S_1, S_2 \subset A$ relatively open in A such that $S_1 \cap S_2 = \emptyset$ and $A = S_1 \cup S_2$. For $i \in \{1, 2\}$, Proposition 1.3.4a asserts S_i is relatively open in A if and only if there is an open set $V_i \subset \mathbb{R}$ such that $S_i = A \cap V_i$. Then since $A \cap V_1$ and $A \cap V_2$ are open in $\mathbb{R}$, so also are S_1 and S_2. Furthermore, we have

$$A \cap B = (S_1 \cup S_2) \cap B = (S_1 \cap B) \cup (S_2 \cap B). \tag{104}$$

Note $S_1 \cap B$ and $S_2 \cap B$ are relatively open in $A \cap B$ since $S_i \cap B = S_i \cap (A \cap B)$ for each i and S_i is open in $\mathbb{R}$. Also observe

$$(S_1 \cap B) \cap (S_2 \cap B) \subseteq S_1 \cap S_2 = \emptyset, \tag{105}$$

and so $S_1 \cap B$ and $S_2 \cap B$ are disjoint. We claim $S_1 \cap B$ and $S_2 \cap B$ are nonempty. Consequently, (104) and (105) together show $A \cap B$ is disconnected, a contradiction. Thus A must be connected.

All that remains is to verify the claim $S_1 \cap B$ and $S_2 \cap B$ are nonempty. By way of contradiction and without loss of generality, suppose $S_1 \cap B = \emptyset$ and note $S_2 \cap B$ is nonempty since $A \cap B \neq \emptyset$. Then observe

$$A \cup B = (S_1 \cup S_2) \cup B = S_1 \cup (S_2 \cup B) \tag{106}$$

and

$$S_1 \cap (S_2 \cup B) = S_1 \cap S_2 \cup S_1 \cup B = \emptyset \cup \emptyset = \emptyset. \tag{107}$$

Note S_1 is relatively open in $A \cup B$ since $S_1 = S_1 \cap (A \cup B)$ and S_1 is open in $\mathbb{R}$. Similarly, $S_2 \cup B$ is relatively open in $A \cup B$ since $S_2 \cup B = (S_2 \cup B) \cap (A \cup B)$ and $S_2 \cup B$ is open in $\mathbb{R}$. Hence $A \cup B$ is disconnected, a contradiction. This completes the proof. ■

Remark 11: The above approach seems a bit wordy since in each case we verified somes sets were relatively open to another set. To summarize, we took the following steps.

1. Assume A is disconnected.
2. Show this implies $A \cap B$ is disconnected when $S_1 \cap B$ and $S_2 \cap B$ are nonempty, a contradiction.
3. Assume $S_1 \cap B$ is empty and show this implies $A \cup B$ is disconnected, a contradiction.

◇

Uniform Convergence

In this section we discuss different notions of convergence. We informally introduce the material as follows and refer the reader to our text for formal definitions. Suppose we have a sequence of functions $(f_n : X \to Y)$ and there is a function $f : X \to Y$ such that for each $x \in X$ we have $\lim\limits_{n\to\infty} f_n(x) = f(x)$. Then we say $f_n \longrightarrow f$ pointwise. This is a useful notion. However, there is another form of convergence known as *uniform* convergence. In this case, we can effectively say all parts of the function are converging to the limit at a comparable rate. In other words, if for every $\varepsilon > 0$ there is $N > 0$ such that

$$d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n > N,\ x \in X, \tag{108}$$

then we say $f_n \longrightarrow f$ uniformly. This notion is stronger and preserves continuity. Below we extend an example previously shown.

Example 30: Define the sequence of functions $(f_n : [0,2] \to \mathbb{R})_{n=1}^{\infty}$ by

$$f_n(x) := \begin{cases} 1 - nx & \text{if } x \in [0, 1/n]|, \\ 0 & \text{if } x \in (1/n, 2]. \end{cases} \tag{109}$$

Define the function $f : [0,2] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0, 2]. \end{cases} \tag{110}$$

a) Prove $f_n \longrightarrow f$ pointwise.

b) Prove does not converge to f uniformly.

Proof:

a) Let $\varepsilon > 0$ be given and choose any $x \in (0,2]$. Then by the Archimedean property of $\mathbb{R}$ there is $N \in \mathbb{N}$ such that $1/N < x$. This implies $x \in (1/n, 2]$ for $n > N$, and so

$$|f_n(x) - 0| = |0 - 0| = 0 < \varepsilon \quad \forall\, n > N. \tag{111}$$

This shows $\lim\limits_{n\to\infty} f_n(x) = 0 = f(x)$ for $x \in (0,2]$. Now observe

$$|f_n(0) - 1| = |1 - n0 - 1| = 0 < \varepsilon \quad \forall\, n \in \mathbb{N}. \tag{112}$$

This shows $\lim\limits_{n\to\infty} f_n(0) = 1 = f(0)$ and completes the proof.

b) By way of contradiction, suppose $f_n \longrightarrow f$ uniformly and let $\varepsilon \in (0,1)$. Then there is $N \in \mathbb{N}$ such that $n > N$ implies

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in [0,1]. \tag{113}$$

In particular,

$$|f_{N+1}(x) - f(x)| < \varepsilon \quad \forall\, x \in [0,1]. \tag{114}$$

However, for $x \in (0, (1-\varepsilon)/N)$ we have $1 - \varepsilon > Nx$ and so $1 - Nx > \varepsilon$, which implies

$$|f_{N+1}(x) - f(x)| = |(1 - Nx) - 0| > \varepsilon, \tag{115}$$

a contradiction.

■

REMARK 12: To provide some intuition, we illustrate the above example below. ◇

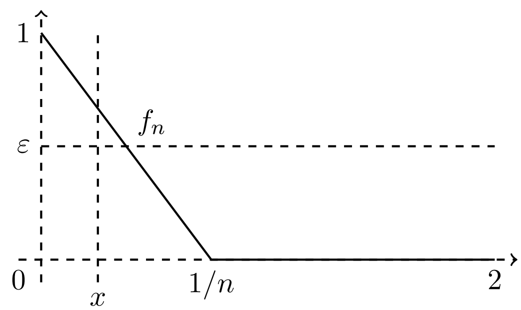

Figure 1: A plot f_n and $\varepsilon \in (0,1)$. Note $f_n(x) > \varepsilon$ for the choice of x shown.

REMARK 13: The following example is quite lengthy. We need not expect such a long problem to show up on an exam; however, individual tricks and ideas from individual parts of this example may be useful. Things that may be worth noting include the part about having a finite subcover since $[0,1]$ is compact and there being an index j such that $z \in B(x_j, \alpha)$. Also, note how we set N to be the max of a finite number of N_i's. Also note the layout of the proof. Namely, the proof starts by essentially stating the definition of what we have to prove and by providing the definitions of each of the hypotheses. Having these, we obtain a collection of inequalities and then combine these to obtain the initial equality that we set out to verify. ◇

Example 31: Let $(f_n : [0,1] \to \mathbb{R})$ be a sequence of functions converging pointwise to a continuous function $f : [0,1] \to \mathbb{R}$. Assume the sequence (f_n) is uniformly Lipschitz. Prove $f_n \longrightarrow f$ uniformly.

Proof:

Let $\varepsilon > 0$ be given. We must show there is N such that $n > N$ implies

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in [0,1]. \tag{116}$$

Since $[0,1]$ is closed and bounded, the Heine-Borel theorem asserts it is compact. Then because f is continuous on a compact set, f is uniformly continuous. Thus there is $\delta > 0$ such that for $x, y \in [0,1]$

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{3}. \tag{117}$$

Also since (f_n) is uniformly Lipschitz, there is $L > 0$ such that for $x, y \in [0,1]$

$$|f_n(x) - f_n(y)| \le L|x - y| \;\; \forall\, n \in \mathbb{N}. \tag{118}$$

let $\alpha = \min\{\varepsilon/3L, \delta\}$ and observe the collection $\{B(x,\alpha)\}_{x\in[0,1]}$ forms an open cover of $[0,1]$. Since $[0,1]$ is compact, there is a finite subcollection $\{x_i\}_{i=1}^m \subset [0,1]$ such that

$$[0,1] \subseteq \bigcup_{i=1}^{m} B(x_i, \alpha). \tag{119}$$

Since $f_n \longrightarrow f$, for each $i \in \{1, 2, \ldots, m\}$ there is N_i such that $n > N_i$ implies

$$|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}. \tag{120}$$

Now set $N = \max\{N_1, N_2, \ldots, N_m\}$ and pick $z \in [0,1]$. Then there is $j \in \{1, 2, \ldots, m\}$ such that $z \in B(x_j, \alpha)$. This implies for $n > N$

$$\begin{aligned}
|f_n(z) - f(z)| &\le |f_n(z) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(z)| \\
&\le L|z - x_j| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(z)| \\
&< L\left(\frac{\varepsilon}{3L}\right) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon,
\end{aligned} \tag{121}$$

where we have applied the above results and note $|x_j - z| < \delta$ by choice of α. Since $z \in [0,1]$ was arbitrarily chosen, we conclude (116) holds and so $f_n \longrightarrow f$ uniformly. ■

Example 32: Consider a decreasing sequence of functions ($f_n : [0,1] \to \mathbb{R}$) obeying the uniform bound $|f_n| \leq M$ for some $M \in (0,1)$. Suppose $f_n \longrightarrow f$ pointwise for some continuous function $f : [0,1] \to \mathbb{R}$. Prove $f_n \longrightarrow f$ uniformly.

Proof:
Let $\varepsilon > 0$ be given. We must show there is N such that $n > N$ implies

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in [0,1]. \tag{122}$$

For each $n \in \mathbb{N}$ set $g_n := f_n - f$, note $g_n(x) \geq 0$ for each $x \in [0,1]$, and observe the set $g_n^{-1}((-\infty, \varepsilon))$ is open since it is the preimage of an open set and $g_n = f_n - f$ is continuous. Since $f_n \longrightarrow f$ pointwise, $g_n \longrightarrow 0$ pointwise, which implies the collection $\left\{g_n^{-1}((-\infty,\varepsilon))\right\}_{n\in\mathbb{N}}$ forms an open cover of $[0,1]$. Because $[0,1]$ is closed and bounded, the Heine-Borel theorem implies this set is compact. Thus, there is a finite subset $F \subset \mathbb{N}$ such that $\{g_i^{-1}((-\infty,\varepsilon))\}_{i\in F}$ forms an open cover of $[0,1]$, i.e.,

$$[0,1] \subseteq \bigcup_{i\in F} g_i^{-1}\left((-\infty,\varepsilon)\right). \tag{123}$$

But, because (f_n) is monotonically decreasing (i.e., $f_{n+1}(x) \leq f_n(x)$ for each $x \in [0,1]$ and $n \in \mathbb{N}$), (g_n) is also monotonically decreasing, which implies $g_n^{-1}((-\infty,\varepsilon)) \subseteq g_{n+1}^{-1}((-\infty,\varepsilon))$ for each $n \in \mathbb{N}$. Consequently, letting $N = \max F$ yields

$$[0,1] \subseteq \bigcup_{i\in F} g_i^{-1}\left((-\infty,\varepsilon)\right) = g_N^{-1}\left((-\infty,\varepsilon)\right) \subseteq g_n^{-1}\left((-\infty,\varepsilon)\right) \quad \forall\, n > N. \tag{124}$$

Together (124) and the fact $g_n \geq 0$ show $n > N$ implies

$$-\varepsilon < 0 \leq g_n(x) = f_n(x) - f(x) < \varepsilon \quad \forall\, x \in [0,1], \tag{125}$$

and so (122) holds. This completes the proof. ∎

Example 33: Let $X \subset \mathbb{R}$ have multiple elements and be finite. For each $n \in \mathbb{N}$ define $f_n : X \to \mathbb{R}$ by

$$\forall\ x \in X,\ \ f_n(x) = \frac{x^n}{n!}. \tag{126}$$

Prove $\sum f_n$ converges uniformly and do not directly assume polynomials are continuous.

Proof:
We seek to apply the Weierstrass M-test. First we show each f_n is continuous. Then we verify $\|f_n\|_\infty < \infty$ for each $n \in \mathbb{N}$ and $\sum \|f_n\|_\infty < \infty$. The result will then follow from the Weierstrass M-test.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Let $z \in X$. we must show there is $\delta > 0$ such that $x \in X$ and $d(x,z) < \delta$ implies $|f(x) - f(z)| < \varepsilon$. Let $\delta := \frac{1}{2}\min\{|x - z|\ :\ x \in X \text{ and } x \neq z\}$. Then if $x \in X$ and $d(x,z) < \delta$, it follows that $x = z$, which implies

$$|f(x) - f(z)| = |f(z) - f(z)| = 0 < \varepsilon, \tag{127}$$

and so f_n is continuous.

Now let $\alpha = \max\{|x| : x \in X\}$ and note $\alpha > 0$ since X has multiple elements, thereby implying one of these is nonzero. Then for each $n \in \mathbb{N}$

$$\|f_n\|_\infty = \frac{\alpha^n}{n!}. \tag{128}$$

Observe

$$\lim_{n\to\infty} \frac{\|f_{n+1}\|_\infty}{\|f_n\|_\infty} = \lim_{n\to\infty} \frac{\alpha^{n+1}}{(n+1)!} \cdot \frac{n!}{\alpha^n} = \lim_{n\to\infty} \frac{\alpha}{n+1} = 0 < 1. \tag{129}$$

Thus the ratio test asserts $\sum \|f_n\|_\infty < \infty$, and we conclude $\sum f_n$ converges uniformly. ■

Remark 14: It is possible to have a differentiable sequence of functions (f_n) such that $f_n \to f$ uniformly for some function f, but for which there is z such that

$$\lim_{n\to\infty} f_n'(z) \neq f'(z). \tag{130}$$

We illustrate this with the following example. ◇

Example 34: For[4] each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := \frac{x}{1+nx^2}. \tag{131}$$

Prove $f_n \longrightarrow 0$ uniformly and there is $z \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n'(z) \neq 0. \tag{132}$$

You may assume each f_n is differentiable.

Proof:
First note for each $n \in \mathbb{N}$ we have $f_n(0) = 0$ and for $x \neq 0$

$$|f_n(x)| = \frac{|x|}{1+nx^2} = \frac{|x|}{(1-2\sqrt{n}|x|+nx^2)+2\sqrt{n}|x|} = \frac{|x|}{(1-\sqrt{n}|x|)^2+2\sqrt{n}|x|} \leq \frac{|x|}{2\sqrt{n}|x|} = \frac{1}{2\sqrt{n}}. \tag{133}$$

Let $\varepsilon > 0$ be given. Since $1/2\sqrt{n} \longrightarrow 0$, there is $N > 0$ such that $n > N$ implies $0 < 1/2\sqrt{n} < \varepsilon$. Consequently, $n > N$ implies

$$|f_n(x) - 0| \leq \frac{1}{2\sqrt{n}} < \varepsilon \quad \forall\, x \in \mathbb{R}. \tag{134}$$

Thus $f_n \longrightarrow 0$ uniformly.

Now, using the quotient rule, observe for each $n \in \mathbb{N}$

$$f_n'(x) = \frac{(1+nx^2)\cdot 1 - (2nx)\cdot x}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}. \tag{135}$$

However, for each $n \in \mathbb{N}$

$$f_n'(0) = \left.\frac{1-nx^2}{(1+nx^2)^2}\right|_{x=0} = \frac{1-0}{(1+0)^2} = 1, \tag{136}$$

and so

$$\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} 1 = 1 \neq 0. \tag{137}$$

This verifies the claim, taking $z = 0$. ■

[4]This problem is due to Rudin's analysis text commonly called *Baby Rudin.*

Power Series

Example 35: Show the function $\exp : \mathbb{R} \to \mathbb{R}$ defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{138}$$

is absolutely convergent for each $x \in \mathbb{R}$ and has infinite radius of convergence.

Proof:

Let $x \in \mathbb{R}$. We shall proceed by applying the ratio test. Observe

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n} = |x| \lim_{n\to\infty} \frac{1}{n} = |x| \cdot 0 = 0 < 1. \tag{139}$$

Consequently, the ratio test asserts the series converges absolutely, and so $\exp(x)$ exists and is real for each $x \in \mathbb{R}$. Now let R be the radius of convergence. By a previous homework problem, if $|x - 0| > R$, then the series diverges. Thus if the series converges, then $|x| \leq \mathbb{R}$. The above result shows the series converges for each $x \in \mathbb{R}$. Whence

$$R \geq \lim_{x\to\infty} |x| = \infty, \tag{140}$$

and we conclude $R = \infty$, as desired.

$\blacksquare$