Review Notes for the Applied Differential Equations Qualifying Exam

Spring 2018 – Spring 2019

Written by

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To my princess and my big mama

Thanks for everything

Purpose: This document is a compilation of notes generated to prepare for the Applied Differential Equations (ADE) Qualifying Exam at UCLA. I have documented the followings solutions as part of my review process. These are incomplete and certainly contain typos and errors, for which I apologize in advance. Much credit is due to Peter Cheng and Zane Li, Jeffrey Hellrung, and Alejandro Canteraro for their excellent notes! Several of my solutions follow in likewise fashion to theirs; however, plenty others are distinct. I would also like to extend many gracious thanks to Victoria Kala for her aid in producing many of the solutions to problems from recent years, and also thanks to Bohyun Kim. To anyone that embarks on the journey of preparing for this exam, I hope these notes prove valuable to you. I also hope you pass the exam early on in the program (rather than fail twice and be told if you don't pass next time then you are out of the program, as was my situation).

Contents

1 Introduction

Many solutions from old exams are contained herein. First we provide some background material that I found relevant for the solutions. Several example problems are drawn from textbooks, as I believe they reflect the types of problems that have shown up in the past and may show up in the near future. After this, over two hundred qual problem solutions are presented. I attempted to make these self-contained, as if to model what I would hope to write on the actual exam. Consequently, some of the solutions may seem repetitive.

The primary references we have found useful are as follows. First, the PDE text by Evans is of utmost value. Particularly, we suggest studying Chapters 2, 3, 4, 5, 6, and 8. We also recommend Chapters 1 and 3 of Bender and Orszag's text for the Frobenius and asymptotics methods. Chapters 5, 6, and 7 of Strogatz's text are also useful for the ODE problems (in addition to the similarity solution portion of Chapter 4 of Evans' text). I have attempted to include as many relevant problems and examples from these as possible.

Remark: Because the Bender and Orzsag portion of the notes has been removed from the online version of these notes, the links in the document were removed (as part of the process). Kindly email me if you seek the complete set of notes (with links). $\triangle$

2 Evans

Chapter 2

Problem 2.1. Write down an explicit formula fo ra function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{1}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Solution:

This is the transport equation with an added cu term. We claim $u = g(x-bt)e^{-ct}$. Indeed, this gives $u = g$ on $\mathbb{R}^n \times \{t=0\}$ and

$$u_t + b \cdot Du + cu = \left(-b \cdot Dg(x-bt)e^{-ct} - cg(x-bt)e^{-ct}\right) + b \cdot Dg(x-bt)e^{-ct} + cg(x-bt)e^{-ct} = 0. \tag{2}$$

□

Problem 2.2 Prove Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$ for $x \in \mathbb{R}^n$, then $\Delta v = 0$.

Proof:

Set $g(x) := Ox$ so that $v(x) = u(g(x))$. Then

$$v_{x_i}(x) = Du(g(x)) \cdot g_{x_i}(x) = \sum_{j=1}^{n} u_{x_j}(g(x)) \partial_{x_i} (Ox)_j = \sum_{j=1}^{n} u_{x_j}(g(x)) O_{ji} \quad \text{for } i = 1, 2, \ldots, n, \tag{3}$$

where

$$\partial_{x_i}(Ox)_j = \partial_{x_i} \sum_{k=1}^{n} O_{jk} x_k = \sum_{k=1}^{n} O_{jk} \delta_{ik} = O_{ji} \quad \text{for } i = 1, 2, \ldots, n. \tag{4}$$

Differentiating once more yields

$$v_{x_i x_i}(x) = \sum_{j=1}^{n} O_{ji} \partial_{x_i} u(g(x)) = \sum_{j=1}^{n} O_{ji} \sum_{k=1}^{n} u_{x_j x_k}(g(x)) O_{ki}. \tag{5}$$

Note

$$\sum_{i=1}^{n} O_{ji} O_{ki} = \sum_{i=1}^{n} O_{ji} O_{ik}^T = (OO^T)_{jk} = \delta_{jk}, \tag{6}$$

where the final equality holds since O is orthogonal. Then

$$\Delta v(x) = \sum_{i=1}^{n} \sum_{j,k=1}^{n} O_{ji} O_{ki} u_{x_j x_k}(g(x)) = \sum_{j,k=1}^{n} \delta_{jk} u_{x_j x_k}(g(x)) = \sum_{j=1}^{n} u_{x_j x_j}(g(x)) = \Delta u(g(x)) = 0. \tag{7}$$

This completes the proof. $\square$

Problem 2.3. Modify the proof of the mean-value formulas to show for $n \geq 3$ that[1]

$$u(0) = \fint_{\partial B(0,r)} g \ \mathrm{d}\sigma + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \ \mathrm{d}x, \tag{8}$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r), \\ \quad\ u = g & \text{on } \partial B(0,r). \end{cases} \tag{9}$$

Proof:

Assume u is a solution to (9). Then define $\phi : [0,\infty) \to \mathbb{R}$ by

$$\phi(r) := \fint_{\partial B(0,r)} u \ \mathrm{d}\sigma = \fint_{\partial B(0,r)} g(y) \ \mathrm{d}\sigma(y) = \fint_{\partial B(0,1)} g(rz) \ \mathrm{d}\sigma(z). \tag{10}$$

Then differentiating yields

$$\phi'(r) = \fint_{\partial B(0,1)} Dg(rz) \cdot z \ \mathrm{d}\sigma(z) = \fint_{\partial B(0,r)} Dg(y) \cdot \frac{y}{r} \ \mathrm{d}\sigma(y) = \fint_{\partial B(0,r)} \frac{\partial u}{\partial \nu} \ \mathrm{d}\sigma. \tag{11}$$

Using Green's formula and (9), we then deduce

$$\phi'(r) = \frac{r}{n} \fint_{B(0,r)} \Delta u(x) \ \mathrm{d}x = -\frac{r}{n} \fint_{B(0,r)} f(x) \ \mathrm{d}x \tag{12}$$

By the fundamental theorem of calculus, we see

$$\phi(r) - \phi(0) = \int_0^r \phi'(s) \ \mathrm{d}s \quad \Longrightarrow \quad \phi(0) = \phi(r) - \int_0^r \phi'(s) \ \mathrm{d}s, \tag{13}$$

and so the fact $u(0) = \phi(0)$ implies

$$u(0) = \fint_{\partial B(0,r)} u \ \mathrm{d}\sigma + \int_0^r \frac{s}{n} \fint_{B(0,s)} f(x) \ \mathrm{d}x\mathrm{d}s. \tag{14}$$

[1]The solution here is due to a helpful conversation from Bohyun Kim.

All that remains is to verify

$$\int_0^r \frac{s}{n} \fint_{B(0,s)} f(x)\ \mathrm{d}x\mathrm{d}s = \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f\ \mathrm{d}x. \tag{15}$$

Let $\varepsilon > 0$. Then observe integration by parts yields

$$\begin{aligned}
&\int_\varepsilon^r \frac{s}{n} \fint_{B(0,s)} f(x)\ \mathrm{d}x\mathrm{d}s \\
&= \int_\varepsilon^r \frac{1}{n\alpha(n)s^{n-1}} \int_{B(0,s)} f(x)\ \mathrm{d}x\ \mathrm{d}s \\
&= \frac{1}{n\alpha(n)} \left(\left[\frac{1}{(2-n)s^{n-2}} \int_{B(0,s)} f(x)\ \mathrm{d}x \right]_{s=-\varepsilon}^r - \int_\varepsilon^r \frac{1}{(2-n)s^{n-2}} \int_{\partial B(0,s)} f(y)\ \mathrm{d}\sigma(y) \right) \\
&= \frac{1}{n(n-2)\alpha(n)} \left[\underbrace{\frac{1}{\varepsilon^{n-2}} \int_{B(0,\varepsilon)} f(x)\ \mathrm{d}x}_{J(\varepsilon)} - \frac{1}{r^{n-2}} \int_{B(0,r)} f(x)\ \mathrm{d}x + \underbrace{\int_\varepsilon^r \int_{\partial B(0,s)} \frac{f(y)}{s^{n-2}}\ \mathrm{d}\sigma(y)\mathrm{d}s}_{K(\varepsilon)} \right],
\end{aligned} \tag{16}$$

where we let $J(\varepsilon)$ and $K(\varepsilon)$ be the underbraced quantities. Note

$$\lim_{\varepsilon\to 0^+} K(\varepsilon) = \int_0^r \int_{\partial B(0,s)} \frac{f(y)}{s^{n-2}}\ \mathrm{d}\sigma(y)\mathrm{d}s = \int_{B(0,r)} \frac{f(x)}{|x|^{n-2}}\ \mathrm{d}x. \tag{17}$$

Furthermore,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} |J(\varepsilon)| &= \lim_{\varepsilon\to 0^+} \left| \frac{1}{\varepsilon^{n-2}} \int_{B(0,\varepsilon)} f(x)\ \mathrm{d}x \right| \\
&\leq \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{n-2}} \int_{B(0,\varepsilon)} \|f\|_{L^\infty(B(0,r))}\ \mathrm{d}x \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{n-2}} \cdot \alpha(n)\varepsilon^n \|f\|_{L^\infty(B(0,r))} \\
&= \|f\|_{L^\infty(B(0,r))} \cdot \lim_{\varepsilon\to 0^+} \varepsilon^2 \\
&= 0
\end{aligned} \tag{18}$$

Therefore, taking the limit as $\varepsilon \to 0^+$, (16), (17), and (18) together imply (15) holds, which completes the proof. □

Problem 2.4. Give a direct proof that if $u \in C^2(U) \cap C(\overline{U})$ is harmonic within a bounded open set U, then

$$\max_{\overline{U}} u = \max_{\partial U} u. \tag{19}$$

Proof:

Because $\overline{U} \subset \mathbb{R}^n$ is closed and bounded, it is compact. Since u is continuous on the compact set $\overline{U}$, it attains its supremum. Similarly, since ∂U is closed and bounded u attains its supremum on ∂U. Thus the use of max in (19) is well-defined. Now let $\varepsilon > 0$ and set $u_\varepsilon := u + \varepsilon |x|^2$, and note u_ε is continuous on $\overline{U}$. By way of contradiction, suppose u_ε attains its maximum at an interior point $z \in \text{int}(U)$. This implies

$$0 \geq \Delta u_\varepsilon(z) = \left[\sum_{i=1}^n \partial_{x_i x_i} \left(u(x) + \varepsilon |x|^2 \right) \right]_{x=z} = \Delta u(z) + 2n\varepsilon = 2n\varepsilon > 0. \tag{20}$$

This implies $0 > 0$, a contradiction. Consequently, $\max_{\overline{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon$. Then observe

$$\max_{\overline{U}} u \leq \max_{x \in \overline{U}} \left(u(x) + \varepsilon |x|^2 \right) = \max_{x \in \overline{U}} u_\varepsilon(x) = \max_{x \in \partial U} u_\varepsilon(x) = \max_{x \in \partial U} \left(u(x) + \varepsilon |x|^2 \right). \tag{21}$$

Since $\overline{U}$ is bounded, there is $M > 0$ such that $|x|^2 \leq M$ for all $x \in \overline{U}$. Thus,

$$\max_{\overline{U}} u \leq \max_{x \in \partial U} u(x) + \varepsilon |x|^2 \leq \left(\max_{\partial U} u \right) + \varepsilon M. \tag{22}$$

Letting $\varepsilon \longrightarrow 0$, we deduce $\max_{\overline{U}} u \leq \max_{\partial U} u$. And, because $\partial U \subset \overline{U}$, $\max_{\overline{U}} u \geq \max_{\partial U} u$. Combining our inequalities, we conclude (19) holds. $\square$

Problem 2.5. We say $v \in C^2(\overline{U})$ is *subharmonic* if $-\Delta v \leq 0$ in U.[2]

a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v(y)\ \mathrm{d}y \quad \forall\ B(x,r) \subset U. \tag{23}$$

b) Prove that therefore $\max_{\overline{U}} v = \max_{\partial U} v$.

c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

d) Prove $v := |Du|^2$ is subharmonic whenever u is harmonic.

Proof:

a) Let $B(x,r) \subset U$. Then define $\phi : [0,\infty) \to \mathbb{R}$ by

$$\phi(r) := \int_{\partial B(x,r)} u(y)\ \mathrm{d}\sigma(y) = \int_{\partial B(0,1)} u(x+rz)\ \mathrm{d}\sigma(z). \tag{24}$$

Then

$$\begin{aligned} \phi'(r) &= \int_{\partial B(0,1)} Du(x+rz)\cdot z\ \mathrm{d}\sigma(z) \\ &= \int_{\partial B(0,1)} \frac{\partial u}{\partial \nu}(x+rz)\ \mathrm{d}\sigma(z) \\ &= \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\ \mathrm{d}\sigma \\ &= \frac{r}{n}\int_{B(x,r)} \Delta u\ \mathrm{d}\sigma \\ &\geq 0. \end{aligned} \tag{25}$$

This shows ϕ is monotonically increasing. Whence

$$v(x) = \lim_{t\to 0} \fint_{\partial B(x,t)} u\ \mathrm{d}\sigma = \lim_{t\to 0}\phi(t) =\leq \phi(r) = \fint_{B(x,r)} v(y)\ \mathrm{d}y. \tag{26}$$

b) Because $\partial U \subset U$, we know

$$\max_{x\in\overline{U}} v(x) \geq \max_{x\in\partial\overline{U}} v(x). \tag{27}$$

[2]The solution to b) was constructed through reference to stack exchange.

By way of contradiction, now suppose

$$\max_{x\in\overline{U}} v(x) > \max_{x\in\partial\overline{U}} v(x). \tag{28}$$

This implies v attains its maximum at an interior point $z \in \text{int}(U)$. From a), we know for any $r > 0$ such that $B(z,r) \subset U$

$$v(z) \leq \fint_{B(z,r)} v(x)\ \mathrm{d}x \quad \Longrightarrow \quad 0 \leq \fint_{B(z,r)} v(x) - v(z)\ \mathrm{d}x, \tag{29}$$

but note the integrand on the right hand size is nonpositive, which implies $v(x) = v(z)$ for all $x \in B(z,r)$.

Let $S = \{r > 0 : B(z,r) \subset U\}$ and note S is bounded since U is bounded. Moreover, by definition of s, we know

$$B(z,s) = \bigcup_{r\in S} B(z,r) \subset U. \tag{30}$$

Thus $v(x) = v(z)$ for all $x \in B(z,s)$. And, because v is continuous on $\overline{U}$, it follows that v is continuous on $\overline{B(z,s)}$. Whence $v(y) = v(z)$ for all $y \in \partial B(z,s)$. We claim $\partial B(z,s) \cap \partial U \neq \emptyset$. Consequently, there is a point $y \in \partial B(z,s) \cap \partial U$, which implies there is a point $y \in \partial U$ for which $v(y) = v(z)$, a contradiction to (28). This contradiction together with (27) imply $\max_{\overline{U}} v = \max_{\partial U} v$, as desired.

All that remains is to verify $\partial B(z,s) \cap \partial U \neq \emptyset$. By way of contradiction, suppose $\partial B(z,s) \cap \partial U = \emptyset$, which implies $\partial B(z,s) \cap (\mathbb{R}^n \backslash U) = \emptyset$. Thus letting

$$d := \inf\{|x-y| : x \in \overline{B(z,s)},\ y \in (\mathbb{R}^n \backslash U)\} \tag{31}$$

yields $d > 0$. Consequently, $B(z, s+d/2)$ and $(\mathbb{R}^n \backslash U)$ are disjoint. This implies $B(z, s+d/2) \subset U$ and so $s + d/2 \in S$, contradicting the fact $s = \sup S$. The result directly follows.

c) First note $v(x) = \phi(u)$ implies

$$v_{x_i} = \phi'(u) u_{x_i} \quad \Longrightarrow \quad v_{x_i x_i} = \phi''(u) u_{x_i}^2 + \phi'(u) u_{x_i x_i} \ \text{ for } i = 1, 2, \ldots, n. \tag{32}$$

Thus

$$\Delta u = \sum_{i=1}^{n} v_{x_i x_i} = \sum_{i=1}^{n} \phi''(u) u_{x_i}^2 + \phi'(u) u_{x_i x_i} = \phi''(u)|Du|^2 + \phi'(u)\Delta u = \phi''(u)|Du|^2 \geq 0, \quad (33)$$

using the fact $\phi''(u) \geq 0$. This shows $-\Delta v \leq 0$, as desired.

d) Observe u_{x_j} is harmonic for $j = 1, 2, \ldots, n$ since

$$\Delta u_{x_j} = \sum_{i=1}^{n} u_{x_i x_i x_j} = \partial_{x_j} (\Delta u) = \partial_{x_j} 0 = 0. \quad (34)$$

Then note $\phi(x) = x^2$ is convex and so $-\Delta \phi(u_{x_i}) \leq 0$ for $i = 1, 2, \ldots, n$. Whence

$$-\Delta \left(|Du|^2\right) = -\Delta \left(\sum_{i=1}^{n} u_{x_i}^2 \right) = \sum_{i=1}^{n} -\Delta \phi(u_{x_i}) \leq \sum_{i=1}^{n} 0 = 0. \quad (35)$$

□

Problem 2.6. Let $U \subset \mathbb{R}^n$ be bounded and open. Prove there exists a constant C, depending only on U, such that

$$\max_{\overline{U}} |u| \leq C \left(\max_{\partial U} |g| + \max_{\overline{U}} |f| \right) \tag{36}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{37}$$

Proof:

Let $\lambda := \max_{\overline{U}} |f|$. Then set

$$v(x) := u + \frac{|x|^2}{2n}\lambda \tag{38}$$

and observe

$$-\Delta v = -\Delta \left(u + \frac{|x|^2}{2n}\lambda \right) = -\Delta u - \lambda \leq -\Delta u - f = 0. \tag{39}$$

Thus v is subharmonic. By a previous exercise, it follows that $\max_{\overline{U}} v = \max_{\partial U} v$. Thus

$$\max_{\overline{U}} |u| \leq \max_{\overline{U}} |u| + \frac{\lambda |x|^2}{2n} = \max_{\overline{U}} v = \max_{\partial U} v = \max_{\partial U} |u| + \frac{\lambda |x|^2}{2n}. \tag{40}$$

Because U is bounded, there exists $M > 0$ such that $|x|^2 \leq M$ for all $x \in U$. Thus taking $C = \max\{1, M/2n\}$ reveals

$$\max_{\overline{U}} |u| \leq \max_{\partial U} |u| + \frac{\lambda M}{2n} \leq C \left(\max_{\partial U} |u| + \lambda \right) = C \left(\max_{\partial U} |g| + \max_{\overline{U}} |f| \right). \tag{41}$$

Noting C depends only on U, we see we have obtained the desired result. □

Problem 2.7. Use Poisson's formula for the ball to prove for $x \in B^0(0,r)$

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0) \tag{42}$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Proof:

First note Poisson's formula for the ball is is given by

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B^0(0,r)} \frac{g(y)}{|x-y|^n}\ \mathrm{d}\sigma(y), \tag{43}$$

which implies

$$u(0) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B^0(0,r)} g(y)\ \mathrm{d}\sigma(y). \tag{44}$$

Now fix any $x \in B^0(0,r)$. Then for each $y \in \partial B^0(0,r)$ it holds

$$|x-y| \leq |x|+|y| = |x|+r \quad \text{and} \quad |x-y| \geq |y|-|x| = r-|x|. \tag{45}$$

Whence

$$\begin{aligned} u(x) &\leq \frac{(r+|x|)(r-|x|)}{n\alpha(n)r}\int_{\partial B^0(0,r)} \frac{g(y)}{(r-|x|)^n}\ \mathrm{d}\sigma(y) \\ &= r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}\cdot\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B^0(0,r)} g(y)\ \mathrm{d}\sigma(y) \\ &= r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0). \end{aligned} \tag{46}$$

In similar fashion, we see

$$\begin{aligned} u(x) &\geq \frac{(r+|x|)(r-|x|)}{n\alpha(n)r}\int_{\partial B^0(0,r)} \frac{g(y)}{(r+|x|)^n}\ \mathrm{d}\sigma(y) \\ &= r^{n-2}\frac{r-|x|}{(r+|x|)^{n+1}}\cdot\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B^0(0,r)} g(y)\ \mathrm{d}\sigma(y) \\ &= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0). \end{aligned} \tag{47}$$

Combining our results, we obtain the given form of Harnack's inequality. □

Problem 2.9. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = g & \text{on } \partial R^n_+, \end{cases} \tag{48}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$ satisfying $|x| \leq 1$. Show Du is unbounded near $x = 0$.

Proof:

We verify Du is unbounded near $x = 0$ as follows. First we claim

$$\lim_{\lambda \to 0^+} \left| \frac{u(\lambda e_n) - u(0)}{\lambda} \right| = \infty. \tag{49}$$

Since u is harmonic, $u \in C^\infty(\mathbb{R}^n_+)$. By way of contradiction, suppose Du is bounded in some open ball about the origin, i.e., there is $M > 0$ such that $|Du(x)| \leq M$ for all $x \in \mathbb{R}^n_+ \cap B_r(0)$ for some $r > 0$. Then for any $\lambda, \varepsilon \in (0, r)$ with $\lambda > \varepsilon$ we see

$$|u(\lambda e_n) - u(\varepsilon e_n)| = \left| \int_\varepsilon^\lambda u_{x_n}(te_n)\ \mathrm{d}t \right| \leq \int_\varepsilon^\lambda |u_{x_n}(te_n)|\ \mathrm{d}t \leq M(\lambda - \varepsilon). \tag{50}$$

From a theorem in Evans text[3], we deduce $u(\varepsilon e_n) \longrightarrow u(0)$ as $\varepsilon \longrightarrow 0^+$. Consequently,

$$M \geq \lim_{\lambda \to 0^+} \lim_{\varepsilon \to 0^+} \left| \frac{u(\lambda e_n) - u(\varepsilon e_n)}{\lambda - \varepsilon} \right| = \lim_{\lambda \to 0^+} \left| \frac{u(\lambda e_n) - u(0)}{\lambda - 0} \right| = \infty, \tag{51}$$

a contradiction to the fact M is finite. Thus Du is unbounded near the origin.

All that remains is to verify (49). Using Poisson's formula for the half-space, we see for $\lambda > 0$

$$\frac{u(\lambda e_n) - u(0)}{\lambda} = \frac{u(\lambda e_n)}{\lambda} = \frac{1}{\lambda} \cdot \frac{2\lambda}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|\lambda e_n - y|^n}\ \mathrm{d}y = \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{(\lambda^2 + |y|^2)^{n/2}}\ \mathrm{d}y, \tag{52}$$

where the final equality holds as the n-th component of y is zero since $y \in \partial\mathbb{R}^n_+$. Let $J(\lambda)$ and

[3] See Theorem 14iii on page 37.

$K(\lambda)$ be the corresponding integrals over $R_1 = \partial\mathbb{R}^n_+ \cap B(0,1)$ and $R_2 = \partial\mathbb{R}^n_+ \backslash B(0,1)$, respectively, so that $u(\lambda e_n) = J(\lambda) + K(\lambda)$. The fact g is bounded implies there is $B > 0$ such that $\|g\|_\infty \le B$, and so

$$|K(\lambda)| \le \int_{R_2} \frac{B}{(\lambda^2 + |y|^2)^{n/2}}\ \mathrm{d}y \le B \int_{R_2} \frac{1}{|y|^n}\ \mathrm{d}y = BC(n) \int_1^\infty \frac{1}{r^n} \cdot r^{n-2}\mathrm{d}r = BC(n) < \infty. \tag{53}$$

The first equality follows from using polar coordinates, where $C(n)$ is a scalar dependent only on n. Next observe

$$\begin{aligned} J_\lambda(x) &= \frac{2}{n\alpha(n)} \int_{R_1} \frac{|y|}{(\lambda^2 + |y|^2)^{n/2}}\ \mathrm{d}y \\ &= \frac{2}{n\alpha(n)} \int_{\tilde{B}(0,1)} \frac{|y|}{(\lambda^2 + |y|^2)^{n/2}}\ \mathrm{d}y \\ &= \frac{2}{n\alpha(n)} \int_{\tilde{B}(0,1/\lambda)} \frac{|\lambda z|}{\lambda^n(1 + |z|^2)^{n/2}}\ \lambda^{n-1}\mathrm{d}z \\ &= \frac{2}{n\alpha(n)} \int_{\tilde{B}(0,1/\lambda)} \frac{|z|}{(1 + |z|^2)^{n/2}}\ \mathrm{d}z \end{aligned} \tag{54}$$

Note here we use $\tilde{B}$ for balls in $\mathbb{R}^{n-1}$ and use y in $\mathbb{R}^n$ in the first line and $\mathbb{R}^{n-1}$ from the second line onward. Employing polar coordinates, for some scalar $A(n)$, dependent only on n, for $\lambda \in (0,1)$ the integral becomes

$$\begin{aligned} J_\lambda(x) &= A(n) \int_0^{1/\lambda} \frac{r}{(1 + r^2)^{n/2}} \cdot r^{n-2}\mathrm{d}r \\ &\ge A(n) \int_1^{1/\lambda} \frac{r^{n-1}}{(1 + r^2)^{n/2}}\ \mathrm{d}r \\ &\ge \frac{A(n)}{2^{n/2}} \int_1^{1/\lambda} \frac{r^{n-1}}{r^n}\ \mathrm{d}r \\ &= -\frac{A(n)}{2^{n/2}} \ln(\lambda). \end{aligned} \tag{55}$$

The third line follows since $1 + r^2 \le 2r^2$ for $r \ge 1$. Consequently,

$$\lim_{\lambda \to 0^+} J(\lambda) \ge \lim_{\lambda \to 0^+} -\frac{A(n)}{2^{n/2}} \ln(\lambda) = \infty. \tag{56}$$

Note since $K(\lambda)$ is bounded as $\lambda \longrightarrow 0^+$ while $J(\lambda) \to \infty$, compiling our results yields

$$\lim_{\lambda \to 0^+} \frac{u(\lambda e_n) - u(0)}{\lambda} = \lim_{\lambda \to 0^+} J(\lambda) + K(\lambda) = \infty, \tag{57}$$

as desired. This verifies (49) and completes the proof. □

Problem 2.10

Proof:

Let U^- denote the open hall-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n < 0\}$. Observe $v_{x_i x_i}$ is continuous in U^+ for each $i \in [n]$ since $v_{x_i x_i} = u_{x_i x_i}$ in U^+. Also $v_{x_i x_i}$ is continuous in U^- for each $i \in [n]$ since $-u_{x_i x_i}$ and $\phi(z_1, \ldots, z_{n-1}, z_n) := (z_1, \ldots, z_{n-1}, -z_n)$ are continuous in U^- and the composition $v_{x_i x_i} = -u_{x_i x_i} \circ \phi$ is continuous. All that remains is to verify $v_{x_i x_i}$ is continuous along $\partial U^+ \cap \{x_n = 0\}$. For $i \in [n-1]$ we directly deduce $v_{x_i x_i} = u_{x_i x_i} = 0$ since $u = 0$ in this set. For each $z \in B^0(0,1)$

$$\lim_{\alpha \to 0^+} v_{x_n x_n}(z_1, \ldots, z_{n-1}, \alpha) = \lim_{\alpha \to 0^+} u_{x_n x_n}(z_1, \ldots, z_{n-1}, \alpha) = u_{x_n x_n}(z_1, \ldots, z_{n-1}, 0) = 0. \tag{58}$$

The first equality holds by the definition of v, and the second by the continuity of $u_{x_n x_n}$, and the third since, using the facts $u_{x_i x_i} = 0$ on $\partial U^+ \cap \{x_n = 0\}$ for $i \in [n-1]$ and $u \in C^2(\overline{U^+})$ and u is harmonic in U^+,

$$0 = \lim_{\alpha \to 0^+} 0 = \lim_{\alpha \to 0^+} \Delta u = \lim_{\alpha \to 0^+} \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} u_{x_i x_i}(x_1, \ldots, x_{n-1}, 0) = u_{x_n x_n}(x_1, \ldots, x_{n-1}, 0). \tag{59}$$

Similarly, we deduce

$$\begin{aligned} \lim_{\alpha \to 0^-} v_{x_n x_n}(z_1, \ldots, z_{n-1}, \alpha) &= \lim_{\alpha \to 0^+} (-u_{x_n x_n} \circ \phi)(z_1, \ldots, z_{n-1}, \alpha) \\ &= (-u_{x_n x_n} \circ \phi)(z_1, \ldots, z_{n-1}, 0) \\ &= -u_{x_n x_n}(z_1, \ldots, z_{n-1}, 0) \\ &= 0. \end{aligned} \tag{60}$$

Since the right and left hand limits exist and are equal, it follows for each $z \in B^0(0,1)$

$$\lim_{\alpha \to 0} v_{x_n x_n}(z_1, \ldots, z_{n-1}, \alpha) = 0 = v_{x_n x_n}(z_1, \ldots, z_{n-1}, 0), \tag{61}$$

which implies $v_{x_n x_n} \in C^2(U)$, from which we conclude $u \in C^2(U)$. $\square$

Problem 2.15. Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} \exp\left(-\frac{x^2}{4(t-s)}\right) g(s)\ \mathrm{d}s \tag{62}$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty), \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\}, \\ u = g & \text{on } \{x = 0\} \times [0, \infty). \end{cases} \tag{63}$$

Solution:

Define $v(x,t) := u(x,t) - g(t)$. Let us momentarily assume $g(t)$ is differentiable. Then define

$$\tilde{f}(x,t) := \begin{cases} -g'(t) & \text{if } x \geq 0, \\ g'(t) & \text{if } x < 0. \end{cases} \tag{64}$$

and define the odd reflection $\tilde{v} : \mathbb{R} \times [0, \infty)$ by

$$\tilde{v}(x,t) := \begin{cases} v(x,t) & \text{if } x \geq 0, \\ -v(-x,t) & \text{if } x < 0. \end{cases} \tag{65}$$

Then note, for $x \geq 0$,

$$\tilde{v}_t(x,t) - \tilde{v}_{xx}(x,t) = \underbrace{u_t(x,t) - u_{xx}(x,t)}_{=0} - g'(t) = \tilde{f}(x,t), \tag{66}$$

and, for $x < 0$,

$$\tilde{v}_t(x,t) - \tilde{v}_{xx}(x,t) = -\left(v_t(-x,t) - v_{xx}(-x,t)\right) = \underbrace{-\left(u_t(-x,t) - u_{xx}(-x,t)\right)}_{=0} + g'(t) = \tilde{f}(x,t). \tag{67}$$

Because $g(0) = 0$, we further see, for $x \geq 0$, $v(x,0) = u(x,0) - g(0) = u(x,0)$. This implies $\tilde{v} =$

0 on $\mathbb{R}\times\{t=0\}$. Compiling the previous results, we may write

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = \tilde{f} & \text{in } \mathbb{R}\times(0,\infty), \\ \tilde{v} = 0 & \text{on } \mathbb{R}\times\{t=0\}, \\ \tilde{v} = 0 & \text{on } \{x=0\}\times[0,\infty). \end{cases} \tag{68}$$

For fixed $s\in(0,t)$, we see

$$\tilde{v}(x,t;s) := \int_{-\infty}^{\infty} \Phi(x-\xi,t-s) f(\xi,s)\ \mathrm{d}\xi \tag{69}$$

solves the PDE

$$\begin{cases} \tilde{v}_t(\cdot;s) - \tilde{v}_{xx}(\cdot;s) = 0 & \text{in } \mathbb{R}\times(s,\infty), \\ \tilde{v}(\cdot;s) = \tilde{f}(\cdot;s) & \text{on } \mathbb{R}\times\{t=s\}, \end{cases} \tag{70}$$

where Φ is the funadmental solution to the heat equation:

$$\Phi(x,t) := (4\pi t)^{-1/2}\exp\left(-\frac{x^2}{4t}\right). \tag{71}$$

Duhamels' principle then asserts

$$\begin{aligned} \tilde{v}(x,t) &= \int_0^t \tilde{v}(x,t;s)\ \mathrm{d}s \\ &= \int_0^t \int_{-\infty}^{\infty} \Phi(x-\xi,t-s) f(\xi,s)\ \mathrm{d}\xi\mathrm{d}s \\ &= \int_0^t g'(s)\left[-\int_0^{\infty} \Phi(x-\xi,t-s)\ \mathrm{d}\xi + \int_{-\infty}^0 \Phi(x-\xi,t-s)\ \mathrm{d}\xi\right]\ \mathrm{d}s \\ &= \int_0^t g'(s)\left[-\int_{-\infty}^{\infty} \Phi(x-\xi,t-s)\ \mathrm{d}\xi + 2\int_{-\infty}^0 \Phi(x-\xi,t-s)\ \mathrm{d}\xi\right]\ \mathrm{d}s \\ &= -g(t) + 2\int_0^t g'(s)\int_{-\infty}^0 \Phi(x-\xi,t-s)\ \mathrm{d}\xi\mathrm{d}s, \end{aligned} \tag{72}$$

where we have substituted in the expression for $\tilde{f}$ and note the integral of Φ over $\mathbb{R}$ is unity and $g(0)=0$.

Now define

$$q(s) := \int_{-\infty}^{0} \Phi(x-\xi, t-s)\ \mathrm{d}\xi = \frac{1}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{0} \exp\left(-\frac{(x-\xi)^2}{4(t-s)}\right)\ \mathrm{d}\xi = \frac{1}{\sqrt{4\pi}} \int_{x/\sqrt{4(t-s)}}^{\infty} \exp(-y^2)\ \mathrm{d}y, \tag{73}$$

where the final equality holds by using the change of variables $y = (x-\xi)/\sqrt{4(t-s)}$. This implies

$$q'(s) = \frac{1}{\sqrt{4\pi}} \cdot - \exp\left(-\frac{x^2}{\sqrt{4(t-s)}}\right) \cdot -\frac{x}{2}\left(4(t-s)\right)^{-3/2} \cdot -4 = -\frac{x}{4\sqrt{\pi}(t-s)^{3/2}} \exp\left(-\frac{x^2}{\sqrt{4(t-s)}}\right). \tag{74}$$

Using integration by parts, it follows that

$$\int_0^t g'(s)q(s)\ \mathrm{d}s = [gq]_0^t - \int_0^t q'(s)g(s)\ \mathrm{d}s = 0 - \frac{x}{4\sqrt{\pi}} \int_0^t - \exp\left(-\frac{x^2}{\sqrt{4(t-s)}}\right) \frac{g(s)}{(t-s)^{3/2}}\ \mathrm{d}s, \tag{75}$$

where the boundary terms vanish since $g(0) = 0$ and

$$\lim_{s\to t^-} q(s) = \lim_{s\to t^-} \frac{1}{\sqrt{4\pi}} \int_{x/\sqrt{4(t-s)}}^{\infty} \exp(-y^2)\ \mathrm{d}y = \frac{1}{\sqrt{4\pi}} \int_{\infty}^{\infty} \exp(-y^2)\ \mathrm{d}y = 0. \tag{76}$$

Therefore,

$$\tilde{v}(x,t) = -g(t) + 2\int_0^t g'(s)q(s)\ \mathrm{d}s = -g(t) + \frac{x}{2\sqrt{\pi}} \int_0^t \exp\left(-\frac{x^2}{\sqrt{4(t-s)}}\right) \frac{g(s)}{(t-s)^{3/2}}\ \mathrm{d}s, \tag{77}$$

from which we conclude, for $(x,t) \in \mathbb{R} \times (0,\infty)$,

$$u(x,t) = \tilde{v}(x,t) + g(t) = \frac{x}{\sqrt{4\pi}} \int_0^t \exp\left(-\frac{x^2}{\sqrt{4(t-s)}}\right) \frac{g(s)}{(t-s)^{3/2}}\ \mathrm{d}s, \tag{78}$$

as desired. Finally, note our final solution makes sense even if g is merely bounded (rather than differentiable). □

Chapter 3

Remark: The following is an interesting example, with a brief solution sketch provided on page 134 of the text by Evans. △

Example. Solve

$$\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = |x| & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{79}$$

Solution:

The Hamiltonian H for this PDE is $H(p) = |p|^2/2$, for which the Legendre/Fenchel transform is the Lagrangian

$$L(v) = H^*(v) = \sup_{p\in\mathbb{R}^n} v \cdot p - H(p) = \sup_{p\in\mathbb{R}^n} v \cdot p - \frac{|p|^2}{2}. \tag{80}$$

Differentiating the expression to be minimized reveals the critical $p^\star$ satisfies the optimality condition $0 = v - p^\star$, and so

$$L(v) = v \cdot v - \frac{|v|^2}{2} = \frac{|v|^2}{2}. \tag{81}$$

The Hopf-Lax formula then gives

$$u(x,t) = \min_{y\in\mathbb{R}^n} tL\left(\frac{x-y}{t}\right) + |y| = \min_{y\in\mathbb{R}^n} \frac{|x-y|^2}{2t} + |y|. \tag{82}$$

Differentiating yields the optimality condition for optimal $y^\star$ to be

$$0 = \frac{y^\star - x}{t} + \frac{y^\star}{|y^\star|} \quad \Longrightarrow \quad y^\star = x - t\frac{y^\star}{|y^\star|} \quad \Longrightarrow \quad y^\star\left(1 + \frac{t}{|y^\star|}\right) = x, \quad \text{if } y^\star \neq 0. \tag{83}$$

If $|x| > t$, then $y^\star \neq 0$ and the sign of $y^\star$ is the same as that of x, and so

$$y^\star = (|x| - t)\,\frac{x}{|x|}, \tag{84}$$

which implies

$$u(x,t) = \frac{t^2}{2t} + |x| - t = |x| - \frac{t}{2}. \tag{85}$$

Now suppose $|x| \leq t$. The result (83) shows $y^\star/|y^\star| = \pm x/|x|$ if $y^\star \neq 0$. For such a choice, we see

$$y^\star = (|x| \mp t)\frac{x}{|x|}, \tag{86}$$

which implies

$$u(x,t) = \frac{t^2}{2t} + (t \mp |x|) = \frac{3t}{2} \mp |x| \geq \frac{t}{2}. \tag{87}$$

However, if $|x| \leq t$ and $y^\star = 0$, then

$$u(x,t) = \frac{|x|^2}{2t} \leq \frac{t}{2}. \tag{88}$$

Because $y^\star$ is the minimizer, (87) and (88) show $y^\star = 0$ must hold when $|x| \leq t$. In summary, we conclude

$$u(x,t) = \begin{cases} |x| - t/2 & \text{if } |x| > t, \\ |x|^2/2t & \text{if } |x| \leq t. \end{cases} \tag{89}$$

□

Example. Solve

$$\begin{cases} u_t + \dfrac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = -|x| & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{90}$$

Solution:

The Hamiltonian H for this PDE is $H(p) = |p|^2/2$, for which the Legendre/Fenchel transform is the Lagrangian

$$L(v) = H^*(v) = \sup_{p\in\mathbb{R}^n} v\cdot p - H(p) = \sup_{p\in\mathbb{R}^n} v\cdot p - \frac{|p|^2}{2}. \tag{91}$$

Differentiating the expression to be minimized reveals the critical $p^\star$ satisfies the optimality condition $0 = v - p^\star$, and so

$$L(v) = v\cdot v - \frac{|v|^2}{2} = \frac{|v|^2}{2}. \tag{92}$$

The Hopf-Lax formula then gives

$$u(x,t) = \min_{y\in\mathbb{R}^n} tL\left(\frac{x-y}{t}\right) - |y| = \min_{y\in\mathbb{R}^n} \frac{|x-y|^2}{2t} - |y|. \tag{93}$$

Differentiating yields the optimality condition for optimal $y^\star$ to be

$$0 = \frac{y^\star - x}{t} - \frac{y^\star}{|y^\star|} \quad\Longrightarrow\quad y^\star = x + t\frac{y^\star}{|y^\star|} \quad\Longrightarrow\quad y^\star\left(1 - \frac{t}{|y^\star|}\right) = x, \quad \text{if } y^\star \neq 0. \tag{94}$$

This shows $y^\star/|y^\star| = \pm x/|x|$. Minimizing the candidate values for u among all possible cases for $y^\star$ reveals

$$\begin{aligned} \min\left\{\underbrace{\frac{|x|^2}{2t}}_{y^\star=0},\ \underbrace{\frac{t}{2} - (|x|+t)}_{y^\star=x+tx/|x|},\ \underbrace{\frac{t}{2} - (\pm(|x|-t))}_{y^\star=x-tx/|x|}\right\} &= \min\left\{\frac{|x|^2}{2t},\quad -|x| - \frac{t}{2},\quad \frac{3t}{2} - |x|,\quad |x| - \frac{t}{2}\right\} \\ &= -|x| - \frac{t}{2}, \end{aligned} \tag{95}$$

from which we conclude $y^\star = 0$ and

$$u(x,t) = -|x| - \frac{t}{2}. \tag{96}$$

□

Problem 3.5a. Solve using characteristics: $x_1 u_{x_1} + x_2 u_{x_2} = 2u$, $u(x_1, 1) = g(x_1)$.

Solution:

We proceed by using the method of characteristics. Define $F(p, z, x) := x \cdot p - 2z$. Taking $p = Du$ and $z = u$ yields $F = 0$ and gives rise to the system of characteristic ODE:

$$\begin{cases} \dot{x}(s) = F_p = x, \quad x(0) = (x_1^0, 1), \\ \dot{z}(s) = F_p \cdot p = x \cdot p = 2z, \quad z(0) = g(x_1^0). \end{cases} \tag{97}$$

This implies

$$(x_1, x_2) = x(s) = x(0)\exp(s) = (x_1^0, 1)\exp(s) \quad \Longrightarrow \quad x_1^0 = x_1 \exp(-s) = \frac{x_1}{x_2}, \tag{98}$$

and so

$$u(x) = z(s) = z(0)\exp(2s) = g(x_1^0)\exp(2s) = g\left(\frac{x_1}{x_2}\right)(x_2)^2. \tag{99}$$

□

Problem 3.5b. Solve using characteristics: $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution:

We proceed by using the method of characteristics. Define $F(p, z, x) := p \cdot r - 3z$, where $r(x) := (x_1, 2x_2, 1)$. Taking $p = Du$ and $z = u$ yields $F = 0$ and gives rise to the system of characteristic ODE:

$$\begin{cases} \dot{x}(s) = F_p = r, \quad x(0) = (x_1^0, x_2^0, 0), \\ \dot{z}(s) = F_p \cdot p = x \cdot p = 3z, \quad z(0) = g(x_1^0, x_2^0). \end{cases} \tag{100}$$

This implies $\dot{x}_3 = 1$ and $x_3^0 = 0$. Thus $x_3 = s$ and

$$(x_1, x_2, x_3) = (x_1^0 \exp(s), x_2^0 \exp(2s), s) = (x_1^0 \exp(x_3), x_2^0 \exp(2x_3), s), \tag{101}$$

which implies

$$(x_1^0, x_2^0) = (x_1 \exp(-x_3), x_2 \exp(-2x_3)). \tag{102}$$

Then

$$u(x) = z(s) = z(0) \exp(2s) = g(x_1^0, x_2^0) \exp(3s) = g\left(x_1 \exp(-x_3), x_2 \exp(-2x_3)\right) \exp(3x_3). \tag{103}$$

□

Problem 3.5c. Solve using characteristics: $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = x_1/2$.

Solution:

We proceed by using the method of characteristics. Define $F(p, z, x) := p \cdot (z, 1) - 1$. Taking $p = Du$ and $z = u$ yields $F = 0$ and gives rise to the system of characteristic ODE:

$$\begin{cases} \dot{x}(s) = F_p = (z, 1), \quad x(0) = (\alpha, \alpha), \\ \dot{z}(s) = F_p \cdot p = x \cdot p = 1, \quad z(0) = \alpha/2. \end{cases} \tag{104}$$

This implies

$$z = s + \frac{\alpha}{2} \quad \text{and} \quad x_2 = s + \alpha, \tag{105}$$

and so

$$s = 2z - x_2 \quad \text{and} \quad \alpha = 2x_2 - 2z. \tag{106}$$

Then

$$\begin{aligned} x_1 &= \alpha + \int_0^s \dot{x}(\tau) \, \mathrm{d}\tau \\ &= \alpha + \int_0^s \tau + \frac{\alpha}{2} \, \mathrm{d}\tau \\ &= \alpha + \frac{s^2}{2} + \frac{\alpha s}{2} \\ &= (2x_2 - 2z) + \frac{4z^2 - 4zx_2 + x_2^2}{2} + (x_2 - z)(2z - x_2) \\ &= 2x_2 - \frac{x_2^2}{2} + z(x_2 - 2), \end{aligned} \tag{107}$$

which yields

$$u(x) = z(s) = \frac{x_1 - 2x_1 + x_2^2/2}{x_2 - 2}. \tag{108}$$

□

Problem 3.17. Show that[4]

$$u(x,t) := \begin{cases} -\frac{2}{3}\left(t+\sqrt{3x+t^2}\right) & \text{if } 4x+t^2>0, \\ 0 & \text{if } 4x+t^2<0 \end{cases} \tag{109}$$

is an (unbounded) entropy solution of $u_t + (u^2/2)_x = 0$.

Solution:

We proceed in the following manner. First we show u is a smooth solution to the PDE to the left and right of the shock curve $4x + t^2 = 0$. Then we show the shock curve satisfies the Rankine-Hugoniot condition. These two facts prove u is a weak solution to the PDE. Lastly, we must verify the entropy condition holds.

For $4x + t^2 > 0$, observe

$$u_t = -\frac{2}{3}\left(1+\frac{1}{2}\left(3x+t^2\right)^{-1/2}\cdot 2t\right) = -\frac{2}{3}\left(1+t\left(3x+t^2\right)^{-1/2}\right), \tag{110}$$

and

$$u_x = -\frac{2}{3}\left(0+\frac{1}{2}\left(3x+t^2\right)^{-1/2}\cdot 3\right) = -\left(3x+t^2\right)^{-1/2}, \tag{111}$$

which imply

$$uu_x = -\frac{2}{3}\left(t+\left(3x+t^2\right)^{1/2}\right)\left(-\left(3x+t^2\right)^{-1/2}\right) = \frac{2}{3}\left(t\left(3x+t^2\right)^{-1/2}-1\right) = -u_t. \tag{112}$$

Thus,

$$u_t + \left(\frac{u^2}{2}\right)_x = u_t + uu_x = 0 \quad \text{if } 4x+t^2>0. \tag{113}$$

Additionally, we see $u_t + uu_x = 0$ if $4x + t^2 < 0$ since the derivative of 0 is 0.

Let $f(u) = u^2/2$. The RH condition is that along a shock, parametrized by $(x(t), t)$, we have

$$\sigma := \dot{x}(t) = \frac{f(u_\ell)-f(u_r)}{u_\ell - u_r} = \frac{0-u_r^2/2}{0-u_r} = \frac{u_r}{2}, \tag{114}$$

[4] This showed up as S13.8.

where u_ℓ and u_r are the limiting values of u to the left and right of the shock, respectively. Because

$$u_r = \lim_{x\to(-t^2/4)^+} u(x,t) = -\frac{2}{3}\left(t + \left(3\cdot\frac{-t^2}{4} + t^2\right)^{1/2}\right) = -\frac{2}{3}\left(t+\frac{t}{2}\right) = -t, \tag{115}$$

we deduce $\sigma = -t/2$. This implies, with the fact $x(0) = 0$,

$$x(t) = \int_0^t u_r \ \mathrm{d}\tau = \int_0^t -\frac{\tau}{2}\ \mathrm{d}\tau = -\frac{t^2}{4} \quad\Longrightarrow\quad 4x + t^2 = 0, \tag{116}$$

as desired. Hence u is a weak solution of the PDE. Then observe $f'(u) = u$ and

$$f'(u_\ell) = 0 > \sigma = -\frac{t}{2} > -t = f'(u_r), \tag{117}$$

from which it follows that the entropy condition is satisfied. This completes the proof. □

Chapter 6

Problem 6.1. Consider Laplace's equation with the potential function c:

$$-\Delta u + cu = 0, \tag{118}$$

and the divergence structure equation:

$$-\operatorname{div}(aDv) = 0, \tag{119}$$

where the function a is positive.

a) Show that if u solves (118) and $w > 0$ also solves (118), then $v := u/w$ solves (119) for $a := w^2$.

b) Conversely, show that if v solves (119), then $u := va^{1/2}$ solves (118) for some potential c.

Solution:

a) Observe

$$v_{x_i} = \frac{u_{x_i} w - w_{x_i} u}{w^2}, \quad \text{for } i = 1, 2, \ldots, n. \tag{120}$$

Then

$$\operatorname{div}(aDv) = \sum_{i=1}^{n} \partial_{x_i}\left(w^2 v_{x_i}\right) = \sum_{i=1}^{n} \partial_{x_i}\left(u_{x_i} w - w_{x_i} u\right) = \sum_{i=1}^{n} u_{x_i x_i} w - w_{x_i x_i} u. \tag{121}$$

Rewriting this in terms of the Laplacian operator reveals

$$\operatorname{div}(aDv) = \sum_{i=1}^{n} u_{x_i x_i} w - w_{x_i x_i} u = w\Delta u - u\Delta w = w(cu) - u(cw) = 0, \tag{122}$$

and the result follows.

b) Observe

$$\begin{aligned}
\Delta u = \sum_{i=1}^{n} \partial_{x_i x_i}\left[va^{1/2}\right] &= \sum_{i=1}^{n} \partial_{x_i}\left[\frac{a_{x_i} v}{2a^{1/2}} + v_{x_i} a^{1/2}\right] \\
&= \sum_{i=1}^{n} -\frac{a_{x_i}}{4a^{3/2}}\left[a_{x_i} v\right] + \frac{1}{2a^{1/2}}\left[a_{x_i x_i} v + a_{x_i} v_{x_i}\right] + \frac{a_{x_i} v_{x_i}}{2a^{1/2}} + a^{1/2} v_{x_i x_i} \\
&= va^{1/2}\left[-\frac{|Da|^2}{4a^2} + \frac{\Delta a}{2a}\right] + a^{-1/2}\left[Da \cdot Dv + a\Delta v\right].
\end{aligned} \tag{123}$$

Additionally,

$$0 = \operatorname{div}(aDv) = \sum_{i=1}^{n} \partial_{x_i} (av_{x_i}) = \sum_{i=1}^{n} a_{x_i} v_{x_i} + av_{x_i x_i} = Da \cdot Dv + a\Delta v. \tag{124}$$

Defining

$$c := -\left[-\frac{|Da|^2}{4a^2} + \frac{\Delta a}{2a} \right] \tag{125}$$

and utilizing (124), we see (123) becomes

$$\Delta u = u(-c) + a^{-1/2} \left[Da \cdot Dv + a\Delta v \right] = -cu, \tag{126}$$

and the proof is complete.

□

Problem 6.2. Let

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu. \tag{127}$$

Prove there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram theorem, provided

$$c(x) \geq -\mu \quad \text{for all } x \in U. \tag{128}$$

Solution:

Let $H := H_0^1(U)$. The associated bilinear form $B[u,v]$ is given by

$$B[u,v] := \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + cuv \ \mathrm{d}x = \int_U \langle ADu, Dv\rangle + cuv \ \mathrm{d}x. \tag{129}$$

We seek to show B is bounded and coercive. Boundedness follows since

$$\begin{aligned} |B[u,v]| &\leq \int_U \|ADu\|\|Dv\| + c|u||v| \ \mathrm{d}x \\ &\leq \||A|\|_{L^\infty(U)} \||Du||Dv|\|_{L^1(U)} + c\|uv\|_{L^1(U)} \\ &\leq C_1 \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} + \|c\|_{L^\infty(U)}\|u\|_{L^2(U)}\|v\|_{L^2(U)} \\ &\leq \left[C_1 + \|c\|_{L^\infty(U)}\right] \|u\|_H \|v\|_H, \end{aligned} \tag{130}$$

where $C_1 := \||A|\|_{L^\infty(U)}$, $|A|$ denotes the induced Euclidean norm of $A(x)$ in U, and we have made use of Hölder's inequality. Poincaré's inequality asserts there exists $C_2 > 0$, dependent only on U, such that

$$\|u\|_{L^2(U)}^2 \leq C_2 \|Du\|_{L^2(U)}^2, \quad \text{for all } u \in H. \tag{131}$$

And, because we assume A is uniformly elliptic, there exists $\theta > 0$ such that

$$\langle A\xi, \xi\rangle \geq \theta|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{132}$$

Assuming (128) holds with $\mu < C_2/\theta$, our combined results imply

$$B[u,u] = \int_U \langle ADu, Du\rangle + cu^2 \text{ d}x \geq \theta\|Du\|^2_{L^2(U)} - \mu\|u\|^2_{L^2(U)} \geq [\theta - \mu C_2]\,\|Du\|^2_{L^2(U)}. \tag{133}$$

However,

$$\|Du\|^2_{L^2(U)} = \frac{1}{2}\left[\|Du\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)}\right] \geq \frac{\min\{C_2,1\}}{2}\left[\|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)}\right] = \frac{\min\{C_2,1\}}{2}\|u\|^2_H, \tag{134}$$

and so

$$B[u,u] \geq [\theta - \mu C_2]\cdot\frac{\min\{C_2,1\}}{2}\cdot\|u\|^2_H, \tag{135}$$

where we note the leading scalar is positive, by our choice of μ. This proves B is coercive, and we are done.

□

Problem 6.3. A function $H_0^2(U)$ is a weak solution of this boundary-value problem for the *biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in } U, \\ u = u_n = 0 & \text{on } \partial U, \end{cases} \tag{136}$$

provided

$$\int_U \Delta u \Delta v \, \mathrm{d}x = \int_U fv \, \mathrm{d}x, \quad \text{for all } v \in H_0^2(U). \tag{137}$$

Given $f \in L^2(U)$, prove that there exists a unique weak solution of (136).

Solution:

Set $H := H_0^2(U)$, let $u \in H_0^2(U)$, and let $\varepsilon > 0$ be given. Since H is the closure of $C_c^\infty(U)$ in $H^2(U)$, there exists $w \in C_c^\infty(U) \cap H^2(U)$ such that

$$\|u - w\|_H < \varepsilon. \tag{138}$$

Integrating by parts, we see

$$\|\Delta v\|_{L^2(U)}^2 = \sum_{i,j=1}^n \int_U v_{x_i x_i} v_{x_j x_j} \, \mathrm{d}x = \sum_{i,j=1}^n \int_U -v_{x_i x_i x_j} v_{x_j} \, \mathrm{d}x = \sum_{i,j=1}^n \int_U v_{x_i x_j}^2 \, \mathrm{d}x = \|D^2 v\|_{L^2(U)}^2, \tag{139}$$

where the boundary terms vanish since v has compact support. Together with the triangle inequality, this implies

$$\begin{aligned} \left| \|D^2 u\|_{L^2(U)} - \|\Delta u\|_{L^2(U)} \right| &\leq \left| \|D^2(u - v)\|_{L^2(U)} + \|D^2 v\|_{L^2(U)} - \|\Delta u\|_{L^2(U)} \right| \\ &\leq \|D^2(u - v)\|_{L^2(U)} + \left| \|D^2 v\|_{L^2(U)} - \|\Delta u\|_{L^2(U)} \right| \\ &\leq \varepsilon + \left| \|\Delta v\|_{L^2(U)} - \|\Delta u\|_{L^2(U)} \right| \\ &\leq \varepsilon + \|\Delta(v - u)\|_{L^2(U)} \\ &\leq 2\varepsilon, \end{aligned} \tag{140}$$

where the fourth line is an application of the reverse triangle inequality. Since ε was arbitrary, letting $\varepsilon \to 0^+$ reveals

$$\|D^2 u\|_{L^2(U)} = \|\Delta u\|_{L^2(U)}. \tag{141}$$

Since u was chosen arbitrarily, this result holds for all $u \in H$. Now define the bilinear form $B : H \times H \to \mathbb{R}$

and the linear form $\ell : H \to \mathbb{R}$ via

$$B[u,v] := \int_U \Delta u \Delta v \, \mathrm{d}x \quad \text{and} \quad \ell(v) := \int_U fv \, \mathrm{d}x. \tag{142}$$

We claim the Lax-Milgram theorem asserts there exists a unique $\overline{u} \in H$ such that

$$B[\overline{u}, v] = \ell(v), \quad \text{for all } v \in H, \tag{143}$$

from which the result follows.

All that remains is to verify the assumptions of the Lax-Milgram theorem hold. Namely, we must show ℓ and B are bounded and B is coercive. Observe ℓ and B are bounded since, for all $u, v \in H$,

$$|\ell(v)| \leq \|fv\|_{L^1(U)} \leq \|f\|_{L^2(U)} \|v\|_{L^2(U)} \leq \|f\|_{L^2(U)} \|v\|_H, \tag{144}$$

and

$$|B[u,v]| \leq \|\Delta u \Delta v\|_{L^1(U)} \leq \|\Delta u\|_{L^2(U)} \|\Delta v\|_{L^2(U)} \leq \|u\|_H \|v\|_H. \tag{145}$$

Additionally, by Poincaré's inequality, there exists $C_1 > 0$, dependent only upon U, such that

$$\|u\|^2_{L^2(U)} \leq C_1 \|Du\|^2_{L^2(U)}, \quad \text{for all } u \in H^1_0(U). \tag{146}$$

However, since $u_{x_i} \in H^1_0(U)$ for each index i, it follows from (146) that there exists $C_2 > 0$ such that

$$\|Du\|^2_{L^2(U)} \leq C_2 \|D^2 u\|_{L^2(U)}, \quad \text{for all } u \in H. \tag{147}$$

Combining (141), (146), and (147), we see

$$\begin{aligned} B[u,u] &= \|\Delta u\|^2_{L^2(U)} \\ &= \|D^2 u\|^2_{L^2(U)} \\ &\geq \frac{1}{3}\left[\frac{1}{C_1 \cdot C_2}\|u\|^2_{L^2(U)} + \frac{1}{C_2}\|Du\|^2_{L^2(U)} + \|D^2 u\|^2_{L^2(U)}\right] \\ &\geq \frac{1}{3}\min\{1/(C_1 \cdot C_2), 1/C_2, 1\}\|u\|^2_H, \end{aligned} \tag{148}$$

from which we deduce B is coercive. □

Problem 6.4. Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Neumann's problem

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u_n = 0 & \text{on } \partial U, \end{cases} \tag{149}$$

if

$$\int_U Du \cdot Dv \ \mathrm{d}x = \int_U fv \ \mathrm{d}x, \quad \text{for all } v \in H^1(U). \tag{150}$$

Suppose $f \in L^2(U)$. Prove (149) has a weak solution if and only if

$$\int_U f \ \mathrm{d}x = 0. \tag{151}$$

Solution:

Suppose (149) has a weak solution $u^\star$. Let v be the constant function with value unity in U. Since U is bounded, $v \in H^1(U)$. Thus, $Dv = 0$ in U and

$$\int_U f \ \mathrm{d}x = \int_U fv \ \mathrm{d}x = \int_U Du^\star \cdot Dv \ \mathrm{d}x = \int_U Du^\star \cdot 0 \ \mathrm{d}x = 0. \tag{152}$$

Conversely, suppose (151) holds. Set $H := \{v \in H^1(U) : \overline{v} \ \mathrm{d}x = 0\}$, where we set

$$\overline{v} := \int_U v \ \mathrm{d}x. \tag{153}$$

Note H is a closed subspace of $H^1(U)$. Then define the bilinear form $B : H \times H \to \mathbb{R}$ and $\ell : H \to \mathbb{R}$ by

$$B[u, v] := \int_U Du \cdot Dv \ \mathrm{d}x \quad \text{and} \quad \ell(v) := \int_U fv \ \mathrm{d}x. \tag{154}$$

We claim B is bounded and coercive and ℓ is bounded. Thus, the Lax-Milgram theorem asserts there exists a unique $u^\star \in H$ such that

$$B[u^\star, v] = \ell(v), \quad \text{for all } v \in H. \tag{155}$$

Therefore, for all $v \in H^1(U)$,

$$\ell(v) = \ell(v - \overline{v}) + \ell(\overline{v}) = B[u^\star, v - \overline{v}] + 0 = B[u^\star, v] - B[u^\star, \overline{v}] = B[u^\star, v], \tag{156}$$

where the second equality holds since $(v - \overline{v}) \in H$ and (151) holds, and the final equality holds since $D\overline{v} = 0$. This proves $u^\star$ is a weak solution of the PDE.

All that remains is to verify our claims. Observe B and ℓ are bounded since, for all $u, v \in H$,

$$|\ell(v)| \leq \|fv\|_{L^1(U)} \leq \|f\|_{L^2(U)}\|v\|_{L^2(U)} \leq \|f\|_{L^2(U)}\|v\|_H, \tag{157}$$

and

$$|B[u,v]| \leq \|Du \cdot Dv\|_{L^1(U)} \leq \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} \leq \|u\|_H\|v\|_H. \tag{158}$$

We say B is coercive provided there exists $\beta > 0$ such that

$$\beta\|u\|_H^2 \leq B[u,u], \quad \text{for all } u \in H. \tag{159}$$

By way of contradiction, suppose this is not the case. This implies there exists a sequence of nonzero functions $\{u^k\} \subset H$ such that

$$B[u^k, u^k] \leq \frac{1}{k} \cdot \|u^k\|_H^2. \tag{160}$$

Define $v^k := u^k/\|u^k\|$. Then $\|v^k\| = 1$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \|Dv^k\|_{L^2(U)}^2 = \lim_{k\to\infty} B[v^k, v^k] \leq \lim_{k\to\infty} \frac{1}{k} = 0. \tag{161}$$

Since the sequence $\{v^k\}$ is bounded, the Rellich-Kondrachov compactness theorem asserts there exists a subsequence $\{v^{n_k}\} \subseteq \{v^k\}$ and $v^\star \in L^2(U)$ such that

$$\lim_{k\to\infty} \|v^{n_k} - v^\star\|_{L^2(U)} = 0. \tag{162}$$

Letting α be any multi-index with $|\alpha| = 1$, we see, for each $\phi \in C_c^\infty(U)$,

$$\lim_{k\to\infty} \left| \int_U v^{n_k} \partial^\alpha \phi \, \mathrm{d}x \right| = \lim_{k\to\infty} \left| -\int_U \partial^\alpha v^{n_k} \phi \, \mathrm{d}x \right| \leq \lim_{k\to\infty} \|\partial^\alpha v^{n_k} \phi\|_{L^1(U)} \leq \lim_{k\to\infty} \|Dv^{n_k}\|_{L^2(U)}\|\phi\|_{L^2(U)} = 0, \tag{163}$$

where we have utilized (161) and Hölder's inequality. Additionally, (162) and Hölder's inequality imply

$$\lim_{k\to\infty} \left| \int_U (v^\star - v^{n_k}) \partial^\alpha \phi \, \mathrm{d}x \right| \leq \lim_{k\to\infty} \|(v^\star - v^{n_k})\partial^\alpha \phi\|_{L^1(U)} \leq \lim_{k\to\infty} \|v^\star - v^{n_k}\|_{L^2(u)}\|\partial^\alpha \phi\|_{L^2(U)} = 0. \tag{164}$$

Together, these facts reveal

$$\int_U v^\star \partial^\alpha \phi \ \mathrm{d}x = \lim_{k\to\infty} \int_U (v^\star - v^{n_k} + v^{n_k}) \partial^\alpha \phi \ \mathrm{d}x = \lim_{k\to\infty} \int_U (v^\star - v^{n_k}) \partial^\alpha \phi \ \mathrm{d}x + \int_U v^{n_k} \partial^\alpha \phi \ \mathrm{d}x = 0. \quad (165)$$

Since α was an arbitrary multi-index with $|\alpha| = 1$ and ϕ was arbitrary, this implies $v^\star$ has a weak derivative in U and $Dv^\star = 0$ a.e. in U. Because U is also connected, $v^\star = C$ a.e. in U for some $C \in \mathbb{R}$. However, by the definition of H,

$$0 = \lim_{k\to\infty} \int_U v^{n_k} \ \mathrm{d}x = \lim_{k\to\infty} \int_U v^\star \ \mathrm{d}x + \int_U v^{n_k} - v^\star \ \mathrm{d}x = \int_U v^\star \ \mathrm{d}x = C|U| \quad \Longrightarrow \quad C = 0. \quad (166)$$

Therefore, $v^\star = 0$ a.e. in U and

$$1 = \lim_{k\to\infty} \|v^{n_k}\|_H^2 = \lim_{k\to\infty} \|v^{n_k}\|_{L^2(U)}^2 + \|Dv^{n_k}\|_{L^2(U)}^2 = \lim_{k\to\infty} \|v^{n_k} - v^\star\|_{L^2(U)}^2 + \|Dv^{n_k}\|_{L^2(U)}^2 = 0, \quad (167)$$

a contradiction. Consequently, B is coercive and the proof is complete. □

Problem 6.5. Explain how to define $u \in H^1(U)$ to be a weak solution of Poisson's equation with *Robin boundary conditions*:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u + \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U. \end{cases} \tag{168}$$

Discuss the existence and uniqueness of a weak solution for a given $f \in L^2(U)$.

Solution:

Let us momentarily assume u is a smooth solution to the PDE. For each test function $v \in C_c^\infty(U)$, we see

$$0 = \int_U (-\Delta u - f)\, v \ \mathrm{d}x = \int_U Du \cdot Dv - fv \ \mathrm{d}x - \int_{\partial U} \frac{\partial u}{\partial \nu} v \ \mathrm{d}\sigma = \int_U Du \cdot Dv - fv \ \mathrm{d}x + \int_{\partial U} uv \ \mathrm{d}\sigma. \tag{169}$$

The final expression makes sense even if u and v are merely in $H := H^1(U)$, evaluating the boundary terms in the trace sense. Consequently, we say u is a weak solution of the PDE provided

$$B[u, v] = \ell(v), \quad \text{for all } v \in H, \tag{170}$$

where the bilinear form $B : H \times H \to \mathbb{R}$ and the linear form $\ell : H \to \mathbb{R}$ are defined by

$$B[u, v] := \int_U Du \cdot Dv \ \mathrm{d}x + \int_{\partial U} uv \ \mathrm{d}\sigma \quad \text{and} \quad \ell(v) := \int_U fv \ \mathrm{d}x. \tag{171}$$

Suppose $f \in L^2(U)$. We claim B is bounded and coercive and ℓ is bounded, from which the Lax-Milgram theorem asserts there exists a unique weak solution $u^\star$ to (170).

All that remains is to verify our three claims. First note ℓ is bounded since

$$|\ell(v)| \leq \|fv\|_{L^1(U)} \leq \|f\|_{L^2(U)} \|v\|_{L^2(U)} \leq \|f\|_{L^2(U)} \|v\|_H, \quad \text{for all } v \in H. \tag{172}$$

The first inequality follows from the triangle inequality and the second inequality is an application of Hölder's ienquality. Similarly, B is bounded since, by application of the trace theorem, there exists $C > 0$

such that

$$\begin{aligned} |B[u,v]| &\leq \|Du\cdot Dv\|_{L^1(U)} + \|uv\|_{L^1(U)} \\ &\leq \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} + C\|uv\|_{L^1(U)} \\ &\leq \|u\|_H\|v\|_H + C\|u\|_{L^2(U)}\|v\|_{L^2(U)} \\ &\leq (1+C)\|u\|_H\|v\|_H, \qquad \text{for all } u,v\in H. \end{aligned} \tag{173}$$

The form B is coercive provided there exists $\beta > 0$ such that

$$B[u,u] \geq \beta\|u\|_H^2, \quad \text{for all } u \in H. \tag{174}$$

By way of contradiction, suppose this is not the case. This implies there exists a nonzero sequence $\{u^k\} \subset H$ such that

$$B[u^k,u^k] \leq \frac{1}{k}\|u^k\|_H^2, \quad \text{for all } k\in\mathbb{N}. \tag{175}$$

Since each u^k is nonzero, we may define the sequence $\{v^k\}$ such that $v^k := u^k/\|u^k\|_H$, for all $k\in\mathbb{N}$. Then

$$\lim_{k\to\infty}\|Dv^k\|_{L^2(U)}^2 + \|v^k\|_{L^2(\partial U)}^2 = \lim_{k\to\infty} B[v^k,v^k] \leq \lim_{k\to\infty}\frac{1}{k} = 0. \tag{176}$$

Because the sequence $\{v^k\}$ is bounded, the Rellich-Kondrachov compactness embedding theorem asserts there exists $v^\star \in L^2(U)$ and a subsequence $\{v^{n_k}\} \subseteq \{v^k\}$ such that

$$\lim_{k\to\infty}\|v^{n_k} - v^\star\|_{L^2(U)} = 0. \tag{177}$$

Now let α be any multi-index with $|\alpha| = 1$ and let $\phi \in C_c^\infty(U)$. Then observe

$$\lim_{k\to\infty}\left|\int_U v^{n_k}\partial^\alpha\phi \ \mathrm{d}x\right| = \lim_{k\to\infty}\left|-\int_U \phi\partial^\alpha v^{n_k} \ \mathrm{d}x\right| \leq \lim_{k\to\infty}\|\phi\partial^\alpha v^{n_k}\|_{L^1(U)} \leq \lim_{k\to\infty}\|\phi\|_{L^2(U)}\|Dv^{n_k}\|_{L^2(U)} = 0, \tag{178}$$

where the final equality follows from (176). Similarly,

$$\lim_{k\to\infty}\left|\int_U (v^\star - v^{n_k})\partial^\alpha\phi \ \mathrm{d}x\right| \leq \lim_{k\to\infty}\|(v^\star - v^{n_k})\partial^\alpha\phi\|_{L^1(U)} \leq \lim_{k\to\infty}\|v^\star - v^{n_k}\|_{L^2(U)}\|\partial^\alpha\phi\|_{L^2(U)} = 0. \tag{179}$$

Together, these facts reveal

$$\int_U v^\star \partial^\alpha \phi \ \mathrm{d}x = \lim_{k\to\infty} \int_U (v^\star - v^{n_k} + v^{n_k}) \partial^\alpha \phi \ \mathrm{d}x = \lim_{k\to\infty} \int_U (v^\star - v^{n_k}) \partial^\alpha \ \mathrm{d}x + \int_U v^{n_k} \partial^\alpha \phi \ \mathrm{d}x = 0. \quad (180)$$

Because this holds for an arbitrary multi-index α with $|\alpha| = 1$ and an arbitrary $\phi \in C_c^\infty(U)$, we deduce $v^\star$ has a weak derivative and $Dv^\star = 0$ a.e. in U, and so $v^\star$ is piecewise constant. Moreover,

$$\begin{aligned} \|v^\star\|_{L^2(\partial U)} &\leq \lim_{k\to\infty} \|v^\star - v^{n_k} + v^{n_k}\|_{L^2(\partial U)} \\ &\leq \lim_{k\to\infty} \|v^\star - v^{n_k}\|_{L^2(\partial U)} + \|v^{n_k}\|_{L^2(\partial U)} \\ &\leq \lim_{k\to\infty} C\|v^\star - v^{n_k}\|_{L^2(U)} + \|v^{n_k}\|_{L^2(\partial U)} \\ &= 0, \end{aligned} \quad (181)$$

which reveals $v^\star = 0$ a.e. in U. Consequently,

$$1 = \lim_{k\to\infty} \|v^{n_k}\|_H^2 = \lim_{k\to\infty} \|v^{n_k}\|_{L^2(U)}^2 + \|Dv^{n_k}\|_{L^2 U}^2 = \lim_{k\to\infty} \|v^{n_k} - v^\star\|_{L^2(U)}^2 + \|Dv^{n_k}\|_{L^2 U}^2 = 0, \quad (182)$$

which implies $1 = 0$, a contradiction. This shows B is, in fact, coercive, and the proof is complete. □

Problem 6.6. Suppose U is connected and ∂U consists of two disjoint, closed sets Γ_1 and Γ_2. Define what it measn for u to be a weak solution of Poisson's equation with *mixed Dirichlet-Neumann boundary conditions:*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } \Gamma_1, \\ u_n = 0 & \text{on } \Gamma_2. \end{cases} \tag{183}$$

Discuss the existence and uniqueness of solutions.

Solution:

Let us momentarily assume u is a smooth solution to the PDE. Set $H := \{w \in H^1(U) : w|_{\Gamma_1} = 0\}$ and observe H is a closed subspace of the Hilbert space $H^1(U)$. And, for all $v \in H$,

$$\begin{aligned} 0 = \int_U (\Delta u + f)v \ \mathrm{d}x &= \int_U -Du \cdot Dv + fv \ \mathrm{d}x - \int_{\partial U} v \frac{\partial u}{\partial n} \ \mathrm{d}\sigma \\ &= \int_U -Du \cdot Dv + fv \ \mathrm{d}x - \int_{\Gamma_1} v \frac{\partial u}{\partial n} \ \mathrm{d}\sigma - \int_{\Gamma_2} v \frac{\partial u}{\partial n} \ \mathrm{d}\sigma \\ &= \int_U -Du \cdot Dv + fv \ \mathrm{d}x, \end{aligned} \tag{184}$$

where the first boundary term vanishes due to the fact $v = 0$ on Γ_1 in the trace sense and the second boundary term vanishes since u solves the PDE. The final line makes sense even if u is merely in $H^1(U)$. Consequently, we say $u^\star \in H$ is a weak solution to the PDE provided

$$B[u^\star, v] = \ell(v), \quad \text{for all } v \in H, \tag{185}$$

where we define the bilinear form $B : H \times H \to \mathbb{R}$ and the lienar form $\ell : H \to \mathbb{R}$ by

$$B[u, v] := \int_U Du \cdot Dv \ \mathrm{d}x \quad \text{and} \quad \ell(v) := \int_U fv \ \mathrm{d}x. \tag{186}$$

We claim B is coercive and bounded and ℓ is bounded, from which the Lax-Milgram theorem asserts there exists a unique $u^\star \in H$ such that (185) holds.

All that remains are to verify our claims. Observe ℓ and B are bounded since, for all $u, v \in H$,

$$|\ell(v)| \leq \|fv\|_{L^1(U)} \leq \|f\|_{L^2(U)}\|v\|_{L^2(U)} \leq \|f\|_{L^2(U)}\|v\|_H, \tag{187}$$

and

$$|B[u,v]| \leq \|Du \cdot Dv\|_{L^1(U)} \leq \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} \leq \|u\|_H\|v\|_H, \tag{188}$$

where we have made use of Hölder's ienquality and the definition of the norm $\|\cdot\|_H$ induced from $H^1(U)$. The functional B is coercive provided there exists $\beta > 0$ such that

$$\beta\|u\|_H^2 \leq B[u,u], \quad \text{for all } u \in H. \tag{189}$$

By way of contradiction, suppose this is not the case. Then there exists a nonzero sequence of functions $\{u^k\}_{k\in\mathbb{N}}$ such that

$$B[u,u] \leq \frac{1}{k} \cdot \|u\|_H^2, \quad \text{for all } k \in \mathbb{N}. \tag{190}$$

Setting $v^k := u^k/\|u^k\|_H$ yields $\|v^k\|_H = 1$ and

$$\|Dv^k\|_{L^2(U)} = B[v^k, v^k] \leq \frac{1}{k}, \quad \text{for all } k \in \mathbb{N}. \tag{191}$$

Furthermore, because $\{v^k\}$ is bounded, the Rellich-Kondrachov compactness theorem asserts there exists $v^\star \in L^2(U)$ and a subsequence $\{v^{n_k}\} \subseteq \{v^k\}$ such that

$$\lim_{k\to\infty} \|v^{n_k} - v^\star\|_{L^2(U)} = 0. \tag{192}$$

Let α be any multi-index with $|\alpha| = 1$ and $\phi \in C_c^\infty(U)$. Then (192) implies

$$\lim_{k\to\infty}\left|\int_U (v^\star - v^{n_k})\partial^\alpha\phi \ \mathrm{d}x\right| \leq \lim_{k\to\infty}\|(v^\star - v^{n_k})\partial^\alpha\phi\|_{L^1(U)} \leq \lim_{k\to\infty}\|v^\star - v^{n_k}\|_{L^2(U)}\|\partial^\alpha\phi\|_{L^2(U)} = 0. \tag{193}$$

And, by (191),

$$\lim_{k\to\infty}\left|\int_U v^{n_k}\partial^\alpha\phi \ \mathrm{d}x\right| = \lim_{k\to\infty}\left|-\int_U \phi\partial^\alpha v^{n_k} \ \mathrm{d}x\right| \leq \lim_{k\to\infty}\|\phi\partial^\alpha v^{n_k}\|_{L^1(U)} \leq \lim_{k\to\infty}\|\phi\|_{L^2(U)}\|Dv^{n_k}\|_{L^2(U)} = 0. \tag{194}$$

Consequently,

$$\int_U v^\star \partial^\alpha \ \mathrm{d}x = \lim_{k\to\infty} \int_U (v^\star - v^{n_k} + v^{n_k})\partial^\alpha \phi \ \mathrm{d}x = \lim_{k\to\infty} \int_U (v^\star - v^{n_k})\partial^\alpha \phi \ \mathrm{d}x - \int_U v^{n_k} \partial^\alpha \phi \ \mathrm{d}x = 0, \quad (195)$$

which implies $v^\star$ has a weak derivative $Dv^\star$ and $Dv^\star = 0$ a.e. in U. Since U is connected, it further follows that there exists $C \in \mathbb{R}$ such that $v = c$ a.e. in U. However, since $v_{\Gamma_1} = 0$ in the trace sense and the measure of Γ_1 is positive (i.e., $|\Gamma_1| > 0$), it follows that $c = 0$. Whence $v^\star = 0$. Therefore,

$$1 = \lim_{k\to\infty} \|v^{n_k}\|_H^2 = \lim_{k\to\infty} \|v^{n_k}\|_{L^2(U)}^2 + \|Dv^{n_k}\|_{L^2(U)}^2 = \lim_{k\to\infty} \|v^{n_k} - v^\star\|_{L^2(U)}^2 + \|Dv^{n_k}\|_{L^2(U)}^2 = 0, \quad (196)$$

where the final equality holds by our previous results. However, this shows $1 = 0$, a contradiction. Thus, B is coercive and the result follows. □

Strogatz

Chapter 5

Problem 5.1.1. Consider the harmonic oscillator $\dot{x} = v$ and $\dot{v} = -\omega^2 x$.

a) Show the orbits are given by ellipses $\omega^2 x^2 + v^2 = C$, where C is a nonnegative constant.

b) Show this condition is equivalent to the conservation of energy.

Solution:

a) Observe

$$\frac{\mathrm{d}x}{\mathrm{d}v} = \frac{\dot{x}}{\dot{v}} = \frac{v}{-\omega^2 x} \qquad \Longrightarrow \qquad -\omega^2 x \ \mathrm{d}x = v \ \mathrm{d}v. \tag{197}$$

Integrating reveals there exists $\alpha \in \mathbb{R}$ such that, for all times,

$$-\frac{\omega^2 x^2}{2} = \frac{v^2}{2} + \alpha \qquad \Longrightarrow \qquad C = 2\alpha = \omega^2 x^2 + v^2, \tag{198}$$

where we define $C := 2\alpha$. Since the right hand side above is nonnegative as each quantity is squared, C is also nonnegative. This equation is the form of an ellipse.

b) Define the energy

$$e(t) := \omega^2 x^2 + v^2. \tag{199}$$

Then

$$\dot{e}(t) = 2\omega^2 x\dot{x} + 2v\dot{v} = 2\omega^2 x\dot{v} + 2v\left[-\omega^2 x\right] = 0, \tag{200}$$

which shows $e(t)$ is constant in time, i.e.,

$$e(t) = e(0), \quad \text{for all } t \in [0, \infty). \tag{201}$$

Taking $C = e(0)$, we see (198) and (201) are identical, from which the result follows.

□

Problem 5.1.2. Consider the system $\dot{x} = ax$ and $\dot{y} = -y$, where $a < -1$. Show all trajectories becomes parallel to the y-direction as $t \to \infty$ and parallel to the x-direction as $t \to -\infty$.

Solution:

Consider any trajectory in the system, parameterized by $(x(t), y(t))$, that does not originate along a null-cline. This implies $(x_0, y_0) := (x(0), y(0))$ satisfies $x_0 \neq 0$ and $y_0 \neq 0$ since the null-clines are $x = 0$ and $y = 0$. We seek to show

$$\lim_{t\to\infty} \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\infty \quad \text{and} \quad \lim_{t\to-\infty} \frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{202}$$

Observe we may write

$$x = x_0 e^{at} \quad \text{and} \quad y = y_0 e^{-t}. \tag{203}$$

Differentiating reveals

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{-y_0 e^{-t}}{a x_0 e^{at}} = -\frac{y_0}{x_0} e^{(-1-a)t} = -\frac{y_0}{x_0} e^{bt}, \tag{204}$$

where we note $b := -1 - a > 0$, by hypothesis. Thus,

$$\lim_{t\to\infty} \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{t\to\infty} -\frac{y_0}{x_0} e^{bt} = -\frac{y_0}{x_0} \underbrace{\lim_{t\to\infty} e^{bt}}_{=\infty} = \pm\infty, \tag{205}$$

as desired. Similarly,

$$\lim_{t\to-\infty} \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{t\to-\infty} -\frac{y_0}{x_0} e^{bt} = -\frac{y_0}{x_0} \underbrace{\lim_{t\to-\infty} e^{bt}}_{=0} = 0. \tag{206}$$

This verifies (202), and we are done. □

Remark: We presume in the previous example that $(x_0, y_0) \neq (x_0, 0)$. Otherwise, $\mathrm{d}y/\mathrm{d}x = 0$ for all time, and the result would not hold. △

Problem 5.1.10. Consider a fixed point $x^\star$ of a system $\dot{x} = f(x)$. We say $x^\star$ is *attracting* if there is a $\delta > 0$ such that $\lim_{t\to\infty} x(t) = x^\star$ whenever $\|x(0) - x^\star\| < \delta$. In other words, any trajectory that starts within a distance δ of $x^\star$ is guaranteed to converge to $x^\star$ eventually. In contrast, Liapunov stability requires that nearby trajectories remain close for *all* time. We say that $x^\star$ is *Liapunov stable* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x(t) - x^\star\| < \varepsilon$ whenever $t \geq 0$ and $\|x(0) - x^\star\| < \delta$. Thus, trajectories that start within δ of $x^\star$ remain within ε of $x^\star$ for all positive time. Finally, $x^\star$ is asymptotically stable if it is both attracting and Liapunov stable.

For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

a) $\dot{x} = y$, $\dot{y} = -4x$.

b) $\dot{x} = 2y$, $\dot{y} = x$.

c) $\dot{x} = 0$, $\dot{y} = x$.

d) $\dot{x} = 0$, $\dot{y} = -y$.

e) $\dot{x} = -x$, $\dot{y} = -5y$.

f) $\dot{x} = x$, $\dot{y} = y$.

Solution:

a) Observe

$$\ddot{x} + 4x = \dot{y} + 4x = 0, \tag{207}$$

which implies x and y are of the form

$$x = a\sin(2t) + b\cos(2t) \qquad \Longrightarrow \qquad y = 2a\cos(2t) - 2b\sin(2t), \tag{208}$$

for some $a, b \in \mathbb{R}$. In particular, we see

$$\|(x(0), y(0))\|^2 = 4a^2 + b^2. \tag{209}$$

Then, for all times t,

$$\begin{aligned}\|(x,y)\|^2 &= [a\sin(2t)+b\cos(2t)]^2 + 4\left[a\cos(2t)-b\sin(2t)\right]^2 \\ &\leq [|a|+|b|]^2 + 4[|a|+|b|]^2 \\ &= 5\left[a^2+b^2+2|a||b|\right] \\ &\leq 10\left[a^2+b^2\right] \\ &\leq 10\left[4a^2+b^2\right].\end{aligned} \tag{210}$$

Let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon/\sqrt{10}$, the above result reveals if

$$\|(x(0),y(0))\| < \delta \quad \Longrightarrow \quad \|(x,y)\| \leq \sqrt{10}\|(x(0),y(0))\| < \varepsilon, \tag{211}$$

from which we conclude $(0,0)$ is Liapunov stable. Note $(0,0)$ is not attracting since x and y are periodic.

b) Let $z=(x,y)$ and observe $\dot{z} = Az$, where

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \tag{212}$$

which has eigenvalues $\lambda = \pm\sqrt{2}$. Thus, the origin forms a saddle and, thus, is neither Liapunov stable nor attracting.

This may also be shown as follows. Observe

$$\ddot{x} = \dot{y} = x \quad \Longrightarrow \quad x = ae^t + be^{-t} \quad \Longrightarrow \quad y = ae^t - be^{-t} \quad \Longrightarrow \quad z = ae^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{213}$$

for some scalars $a, b \in \mathbb{R}$. This implies, for each point originating along the line $y = x$, we have $b = 0$, $a \neq 0$, and

$$\lim_{t\to\infty} \|z(t)\| = \lim_{t\to\infty} |a|e^t\sqrt{2} = +\infty. \tag{214}$$

Whence the origin is neither Liapunov stable nor attracting.

c) Let (x,y) be any trajectory. From the ODE, $\dot{x} = 0$ implies there exists $a \in \mathbb{R}$ such that $x(t) = a$ for all time. This implies $\dot{y} = a$, and so $y = at + b$, for some $b \in \mathbb{R}$. For each trajectory originating at $(a, 0)$, we see

$$\lim_{t\to\infty} \|(x,y)\| = \lim_{t\to\infty} \sqrt{a^2 + (at)^2} = |a| \lim_{t\to\infty} \sqrt{1+t^2} = +\infty, \tag{215}$$

from which we conclude the origin is neither attracting nor Liapunov stable.

d) We claim the origin is Liapunov stable, but not attracting. For each trajectory, parameterized by (x, y) and originating at (x_0, y_0), the ODE implies $x = x_0$ and $y = y_0e^{-t}$. Consequently,

$$\|(x,y)\| = \sqrt{x_0^2 + (y_0e^{-t})^2} \leq \sqrt{x_0^2 + y_0^2} = \|(x_0, y_0)\|, \quad \text{for all } t \in [0, \infty). \tag{216}$$

This shows, for all $\varepsilon > 0$,

$$\|(x_0, y_0)\| < \varepsilon \quad \Longrightarrow \quad \|(x, y)\| < \varepsilon, \tag{217}$$

and so the origin is Liapunov stable. In the limit as $t \longrightarrow \infty$, we see

$$\lim_{t\to\infty} (x, y) = \lim_{t\to\infty} (x_0y_0e^{-t}) = (x_0, 0). \tag{218}$$

Thus, for all trajectories not originating along the y-axis, the trajectory will converge to a point on the x-axis other than the origin. This proves the origin is not attracting, and we are done.

e) Let (x, y) be the parameterization of any trajectory and let (x_0, y_0) denote the starting point of the trajectory. Integrating the ODE reveals

$$(x, y) = (x_0e^{-t}, y_0e^{-5t}) \quad \Longrightarrow \quad \|(x,y)\| = \sqrt{(x_0e^{-t})^2 + (y_0e^{-5t})^2} \leq \sqrt{x_0^2 + y_0^2} = \|(x_0, y_0)\|. \tag{219}$$

Let $\varepsilon > 0$ be given. The previous result shows if $\|(x_0, y_0)\| \leq \varepsilon$, then $\|(x, y)\| \leq \varepsilon$. Thus the origin is Liapunov stable. Moreover, it is attracting since

$$\lim_{t\to\infty} \|(x,y)\| = \lim_{t\to\infty} \sqrt{(x_0e^{-t})^2 + (y_0e^{-5t})^2} = \sqrt{x_0^2 \cdot 0 + y_0^2 \cdot 0} = 0. \tag{220}$$

Therefore, we conclude the origin is asymptotically stable.

f) We claim the origin is unstable. Let (x, y) be the parameterization of a trajectory originating at

(x_0, y_0). Integrating the ODE reveals $(x, y) = (x_0, y_0)e^t$. Therefore, if (x_0, y_0) is not the origin, then

$$\lim_{t\to\infty} \|(x, y)\| = \lim_{t\to\infty} \sqrt{x_0^2 + y_0^2} \cdot e^t = \sqrt{x_0^2 + y_0^2} \cdot \underbrace{\lim_{t\to\infty} e^t}_{=\infty} = \infty. \tag{221}$$

This shows the origin is neither asymptotically stable nor Liapunov stable, as claimed.

□

Problem 5.2.12. Consider the circuit equation $L\ddot{I} + R\dot{I} + I/C = 0$, where $L, C > 0$ and $R \geq 0$.

a) Rewrite the equation as a two-dimensional linear system.

b) Show that the origin is asymptotically stable if $R > 0$ and Liapunov stable if $R = 0$.

c) Classify the fixed point at the origin, depending on whether $R^2C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.

Solution:

a) Letting $x = I$ and $y = \dot{x}$ and $z = (x, y)$, we see $\dot{z} = Az$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1/LC & -R/L \end{pmatrix}. \tag{222}$$

b) Momentarily assume $R = 0$ so that we may identify a conserved quantity corresponding to the associated undamped system. Let $\alpha := 1/LC$ so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = -\frac{\alpha x}{y} \quad \Longrightarrow \quad y\ \mathrm{d}x = -\alpha x\ \mathrm{d}x \quad \Longrightarrow \quad \frac{1}{2}\left(\alpha x^2 + y^2\right) = C, \tag{223}$$

for some constant $C \in \mathbb{R}$. This shows that trajectories form an ellipse when $R = 0$. Consequently, the origin is Liapunov stable. However, the origin is not asymptotically stable in this case since there are trajectories arbitrarily close to the origin that are periodic.

Now suppose $R > 0$ and, in light of above, define the Liapunov function $V(x, y)$ via

$$V(x, y) := \frac{1}{2}\left(\alpha x^2 + y^2\right). \tag{224}$$

Then observe

$$\dot{V} = \alpha x\dot{x} + y\dot{y} = \alpha xy + y(-\alpha x - \beta y) = -\beta y^2 \leq 0, \tag{225}$$

where $\beta := R/L > 0$. Additionally, $V > 0$ everywhere other than the origin. And, the only point that is a fixed point of the system and yields $\dot{V} = 0$ is the origin. This verifies the assumptions of Lasalle's theorem, from which we conclude the origin is asymptotically stable.

c) The characteristic polynomial for the system is

$$\chi(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ \alpha & \lambda + \beta \end{vmatrix} = \lambda(\lambda + \beta) + \alpha = \lambda^2 + \beta\lambda + \alpha. \tag{226}$$

This implies the eigenvalues of the system satisfy

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2}. \tag{227}$$

Since

$$\beta^2 - 4\alpha = \frac{R^2}{L^2} - \frac{4}{LC} = \frac{1}{L^2 C}\left[R^2 C - L\right], \tag{228}$$

and so it suffices to consider the sign of $\beta^2 - 4\alpha$ since this matches the term given in the prompt. If $\beta^2 - 4\alpha > 0$, then $\lambda_1, \lambda_2 \in \mathbb{R}$ and are negative. In this case, the origin forms a stable node. If $\beta^2 - 4\alpha = 0$, then $\lambda_1 = \lambda_2 = -\beta < 0$, and so the origin forms a stable degenerate node??.

□

Chapter 6

Example 6.8.5. Show that closed orbits are impossible for the "rabbit vs. sheep" system

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - x - y). \tag{229}$$

Solution:

The fixed points of this system, written in the form (x, y), are $(0, 0)$, $(0, 2)$, $(3, 0)$, and $(1, 1)$. The Jacobian for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}. \tag{230}$$

This implies

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \tag{231}$$

which has eigenvalues 2 and 3, making the origin form an unstable node. Also,

$$J(, 1, 1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}, \tag{232}$$

which has eigenvalues that satisfy

$$0 = (\lambda + 1)^2 - 2 = \lambda^2 + 2\lambda - 1 \quad \Longrightarrow \quad \lambda = \frac{-2 \pm \sqrt{2^2 - 4(-1)}}{2} = -1 \pm \sqrt{2}, \tag{233}$$

and so $(1, 1)$ forms a saddle. Similarly,

$$J(3, 0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}, \tag{234}$$

which has eigenvalues -1 and -3, thereby implying $(3,0)$ is a stable node. Lastly,

$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}, \tag{235}$$

which has eigenvalues -1 and -2, thereby implying $(0,2)$ is a stable node.

Having the above results, we prove no closed orbits exist via index theory. A theorem in Strogatz's text[5] states any closed orbit in the phase plane must enclose fixed points whose indices sum to unity. Since the fixed point $(1,1)$ is a saddle point, its index is -1. And, as $(0,2)$ and $(3,0)$ are stable nodes, these each have index 1. Now suppose there exists a closed orbit in the xy-plane. By the theorem and the listed indices, the closed orbit must enclose either $(0,2)$ or $(3,0)$. However, this is not possible since no trajectories cross the null-clines $x=0$ and $y=0$. Therefore, the system does not admit any closed orbits. □

Example 6.8.6. Show the system $\dot{x} = xe^{-x}$, $\dot{y} = 1 + x + y^2$ has no closed orbits.

Solution:

Observe $\dot{x} = 0$ if and only if $x = 0$. Thus, if $\dot{x} = 0$, then $\dot{y} = 1 + y^2 \geq 1 \neq 0$. This shows the system has no fixed points. By a theorem in in Strogatz's text[6], if C is a closed orbit in the system, then it must enclose fixed points whose indices sum to unity. Since the system does not admit any fixed points, the contrapositive of the theorem implies the system does not possess any closed orbits. □

[5] See Theorem 6.8.2 on page 180.

[6] See Theorem 6.8.2 on page 180.

Problem 6.4.1. Consider the following "rabbit vs. sheep" problem, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

$$\dot{x} = x(3 - x - y), \quad \dot{y} = y(2 - x - y). \tag{236}$$

Solution:

The fixed points (x, y) for this system are given by $(0, 0)$, $(0, 2)$, and $(3, 0)$. The Jacobian matrix for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 3 - 2x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}. \tag{237}$$

This implies

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \tag{238}$$

for which the eigenvalues are both positive, thereby implying $(0, 0)$ forms an unstable node. Similarly,

$$J(0, 2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}, \tag{239}$$

which has eigenvalues 1 and -2, thereby implying $(0, 2)$ forms a saddle point. Lastly,

$$J(3, 0) = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix}, \tag{240}$$

which has eigenvalues -3 and -1, thereby implying $(3, 0)$ forms a stable node. The nullclines are given by $y = 3 - x$ (where $\dot{x} = 0$) and $y = 2 - x$ (where $\dot{y} = 0$). A phase portrait is given in Figure 1. As illustrated, the basin of attraction for the single stable node $(3, 0)$ is $\{x > 0\} \times \{y > 0\}$.

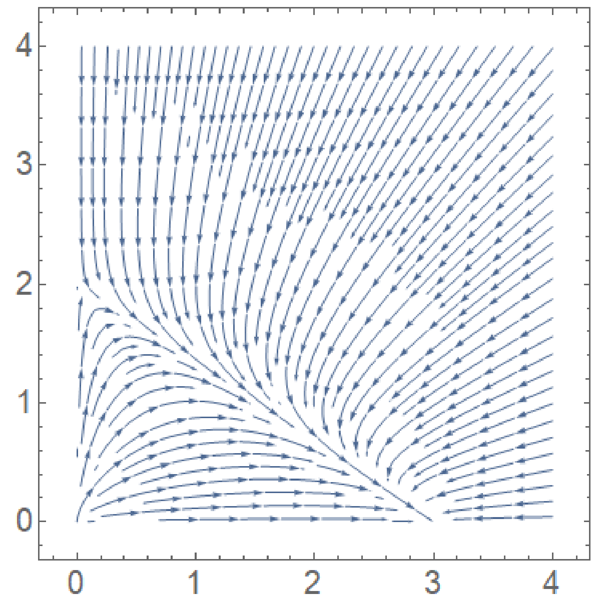

Figure 1: Phase portrait for Strogatz Problem 6.4.1.

□

Problem 6.4.2. Consider the following "rabbit vs. sheep" problem, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

$$\dot{x} = x(3 - 2x - y), \quad \dot{y} = y(2 - x - y). \tag{241}$$

Solution:

The fixed points (x, y) for this system are given by $(0, 0)$, $(0, 2)$, $(3/2, 0)$, and $(1, 1)$. The Jacobian matrix for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 3 - 4x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}. \tag{242}$$

This implies

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \tag{243}$$

for which the eigenvalues are both positive, thereby implying $(0, 0)$ forms an unstable node. Similarly,

$$J(0, 2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}, \tag{244}$$

which has eigenvalues 1 and -2, thereby implying $(0, 2)$ forms a saddle point. And,

$$J(3/2, 0) = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix}, \tag{245}$$

which has eigenvalues -3 and $1/2$, thereby implying $(3/2, 0)$ forms a saddle. Lastly,

$$J(1, 1) = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}, \tag{246}$$

which has eigenvalues that satisfy the characteristic equation

$$0 = (\lambda + 2)(\lambda + 1) - 1 = \lambda^2 + 3\lambda + 1 \quad \Longrightarrow \quad \lambda = \frac{-3 \pm \sqrt{3^2 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2}, \tag{247}$$

and so $(1, 1)$ forms a stable node. The nullclines are given by $y = 3 - 2x$ (where $\dot{x} = 0$) and $y = 2 - x$ (where $\dot{y} = 0$). A phase portrait is given in Figure 2. As illustrated, the basin of attraction for the single stable node $(1, 1)$ is $\{x > 0\} \times \{y > 0\}$.

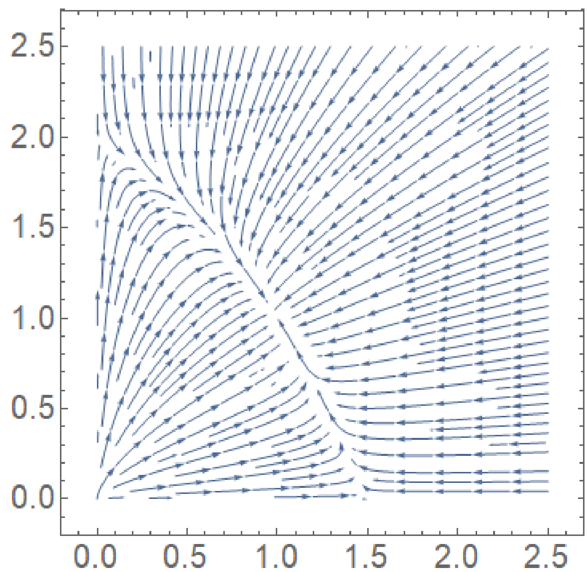

Figure 2: Phase portrait for Strogatz Problem 6.4.2.

□

Problem 6.4.3. Consider the following "rabbit vs. sheep" problem, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

$$\dot{x} = x(3 - 2x - 2y), \quad \dot{y} = y(2 - x - y). \tag{248}$$

Solution:

The fixed points (x, y) for this system are given by $(0, 0)$, $(0, 2)$, and $(3/2, 0)$. The Jacobian matrix for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 3 - 4x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}. \tag{249}$$

This implies

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \tag{250}$$

for which the eigenvalues are both positive, thereby implying $(0, 0)$ forms an unstable node. Similarly,

$$J(0, 2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}, \tag{251}$$

which has eigenvalues -1 and -2, thereby implying $(0, 2)$ forms a stable node. And,

$$J(3/2, 0) = \begin{pmatrix} -3 & -3 \\ 0 & 1/2 \end{pmatrix}, \tag{252}$$

which has eigenvalues -3 and $1/2$, thereby implying $(3/2, 0)$ forms a saddle. The nullclines are given by $y = (3 - 2x)/2$ (where $\dot{x} = 0$) and $y = 2 - x$ (where $\dot{y} = 0$). A phase portrait is given in Figure 3. As illustrated, the basin of attraction for the single stable node $(0, 2)$ is $\{x > 0\} \times \{y > 0\}$.

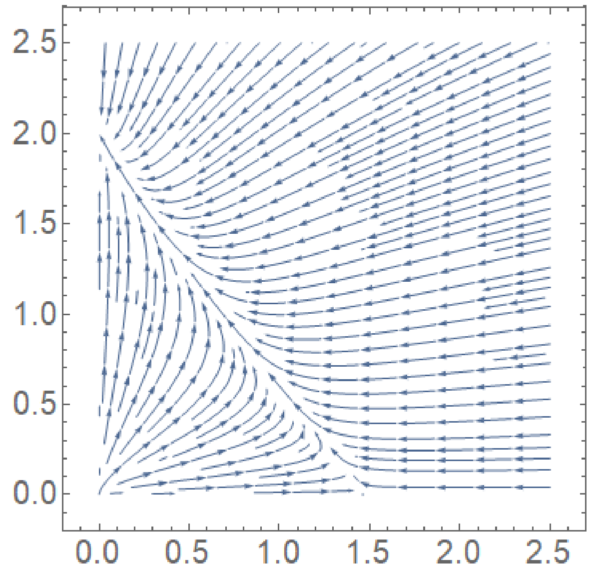

Figure 3: Phase portrait for Strogatz Problem 6.4.3.

□

Problem 6.5.1. Consider the system $\ddot{x} = x^3 - x$. Find all the equilibrium points and classify them. Find a conserved quantity. Sketch the phase portrait.

Solution:

We write the ODE as the system

$$\dot{x} = y, \qquad \dot{y} = x^3 - x. \tag{253}$$

The fixed points (x, y) of the system are $(0, 0)$, $(1, 0)$ and $(-1, 0)$. The system is Hamiltonian since

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial x}[y] + \frac{\partial}{\partial y}\left[x^3 - x\right] = 0. \tag{254}$$

This implies there exists a function $H(x, y)$ such that $H_x = -\dot{y}$ and $H_y = \dot{x}$. Integrating reveals

$$H = \int^x H_x(\tilde{x}, y)\ \mathrm{d}\tilde{x} = \int^x \tilde{x} - \tilde{x}^3\ \mathrm{d}\tilde{x} = \frac{x^2}{2} - \frac{x^4}{4} + g(y), \tag{255}$$

for some function $g(y)$. Similarly,

$$H = \int^y H_y(x, \tilde{y})\ \mathrm{d}\tilde{y} = \int^y \tilde{y}\ \mathrm{d}\tilde{y} = \frac{y^2}{2} + f(x), \tag{256}$$

for some function $f(x)$. Combining the previous two results, we may take

$$H(x, y) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{y^2}{2}. \tag{257}$$

Then

$$\dot{H} = (x - x^3)\dot{x} + y\dot{y} = (x - x^3)y + y\left(x^3 - x\right) = 0, \tag{258}$$

as desired. Furthermore, because the system is Hamiltonian, the only fixed points are centers and saddles. The Jacobian matrix for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{pmatrix}, \tag{259}$$

which has eigenvalues that satisfy

$$0 = \lambda^2 - (3x^2 - 1) \qquad \Longrightarrow \qquad \lambda = \pm\sqrt{3x^2 - 1}. \tag{260}$$

Consequently, $(-1, 0)$ and $(1, 0)$ form saddles and the eigenvalues of $J(0, 0)$ are entirely complex, making $(0, 0)$ form a center. The nullclines are given by $x = 0$, $x = 1$, $x = -1$ (where $\dot{y} = 0$) and $y = 0$ (where $\dot{x} = 0$). A phase plot is given in Figure 4.

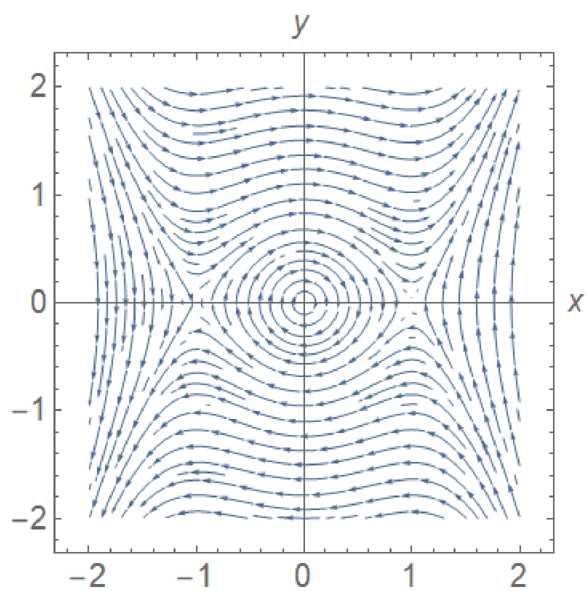

Figure 4: Phase portrait for Strogatz Problem 6.5.1.

□

Problem 6.5.2. Consider the system $\ddot{x} = x - x^2$. Find all the equilibrium points and classify them. Find a conserved quantity. Sketch the phase portrait. Find an equation of the homoclinic orbit that separates closed and nonclosed trajectories.

Solution:

We write the ODE as the system

$$\dot{x} = y, \qquad \dot{y} = x - x^2. \tag{261}$$

The fixed points (x, y) of the system are $(0,0)$ and $(1,0)$. The system is Hamiltonian since

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial x}[y] + \frac{\partial}{\partial y}\left[x - x^2\right] = 0. \tag{262}$$

This implies there exists a function $H(x, y)$ such that $H_x = -\dot{y}$ and $H_y = \dot{x}$. Integrating reveals

$$H = \int^x H_x(\tilde{x}, y)\ \mathrm{d}\tilde{x} = \int^x \tilde{x}^2 - \tilde{x}\ \mathrm{d}\tilde{x} = \frac{x^3}{3} - \frac{x^2}{2} + g(y), \tag{263}$$

for some function $g(y)$. Similarly,

$$H = \int^y H_y(x, \tilde{y})\ \mathrm{d}\tilde{y} = \int^y \tilde{y}\ \mathrm{d}\tilde{y} = \frac{y^2}{2} + f(x), \tag{264}$$

for some function $f(x)$. Combining the previous two results, we may take

$$H(x, y) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{y^2}{2}. \tag{265}$$

Then

$$\dot{H} = (x^2 - x)\dot{x} + y\dot{y} = (x^2 - x)y + y\left(x - x^2\right) = 0, \tag{266}$$

as desired. Furthermore, because the system is Hamiltonian, the only fixed points are centers and saddles. The Jacobian matrix for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 2x & 0 \end{pmatrix}, \tag{267}$$

which has eigenvalues that satisfy

$$0 = \lambda^2 - (1 - 2x) \qquad \Longrightarrow \qquad \lambda = \pm\sqrt{1 - 2x}. \tag{268}$$

Consequently, $(0,0)$ forms a saddle and $(1,0)$ forms a center. The nullclines are given by $x = 0$ and $x = 1$ (where $\dot{y} = 0$) and $y = 0$ (where $\dot{x} = 0$). A phase plot is given in Figure 5.

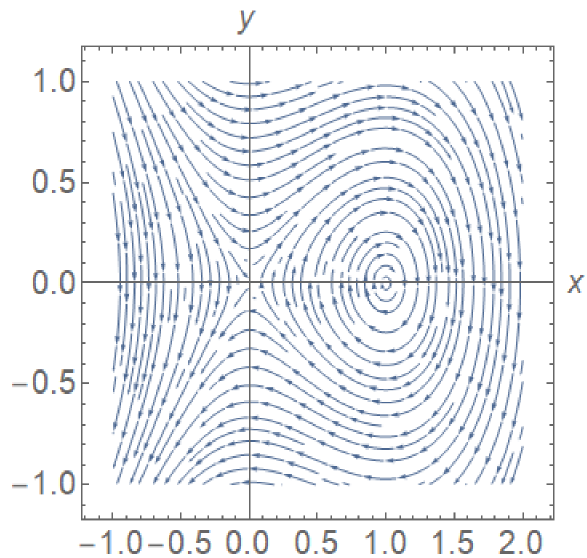

Figure 5: Phase portrait for Strogatz Problem 6.5.2.

Lastly, we identify the homoclinic orbit. This orbit originates at the origin and terminates at the origin in the limit as $t \longrightarrow \infty$. Because the Hamiltonian H is conserved in time, the set of all points along this orbit are given by

$$\left\{ (x, y) : \frac{x^3}{3} - \frac{x^2}{2} + \frac{y^2}{2} = H(x, y) = H(0, 0) = 0 \right\}. \tag{269}$$

□

Problem 6.5.4. Consider the system $\ddot{x} = ax - x^2$. Find all the equilibrium points and classify them (for the different cases of a). Find a conserved quantity. Sketch the phase portrait.

Solution:

We write the ODE as the system

$$\dot{x} = y, \qquad \dot{y} = ax - x^2. \tag{270}$$

The fixed points (x, y) of the system are $(0, 0)$ and $(a, 0)$. The system is Hamiltonian since

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial x}[y] + \frac{\partial}{\partial y}\left[ax - x^2\right] = 0. \tag{271}$$

This implies there exists a function $H(x, y)$ such that $H_x = -\dot{y}$ and $H_y = \dot{x}$. Integrating reveals

$$H = \int^x H_x(\tilde{x}, y)\ \mathrm{d}\tilde{x} = \int^x \tilde{x}^2 - \tilde{x}\ \mathrm{d}\tilde{x} = \frac{x^3}{3} - \frac{ax^2}{2} + g(y), \tag{272}$$

for some function $g(y)$. Similarly,

$$H = \int^y H_y(x, \tilde{y})\ \mathrm{d}\tilde{y} = \int^y \tilde{y}\ \mathrm{d}\tilde{y} = \frac{y^2}{2} + f(x), \tag{273}$$

for some function $f(x)$. Combining the previous two results, we may take

$$H(x, y) = \frac{x^3}{3} - \frac{ax^2}{2} + \frac{y^2}{2}. \tag{274}$$

Then

$$\dot{H} = (x^2 - ax)\dot{x} + y\dot{y} = (x^2 - ax)y + y\left(ax - x^2\right) = 0, \tag{275}$$

as desired. Furthermore, because the system is Hamiltonian, the only fixed points are centers and saddles. The Jacobian matrix for this system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a - 2x & 0 \end{pmatrix}, \tag{276}$$

which has eigenvalues that satisfy

$$0 = \lambda^2 - (a - 2x) \quad \implies \quad \lambda = \pm\sqrt{a - 2x}. \tag{277}$$

Consequently, when $a > 0$, the origin forms a saddle and $(a, 0)$ forms a center. When $a < 0$, the origin forms a center and $(a, 0)$ forms a saddle. When $a = 0$, the only fixed point is the origin, which forms a saddle (Return and explain). A phase plot is given in Figure 6.

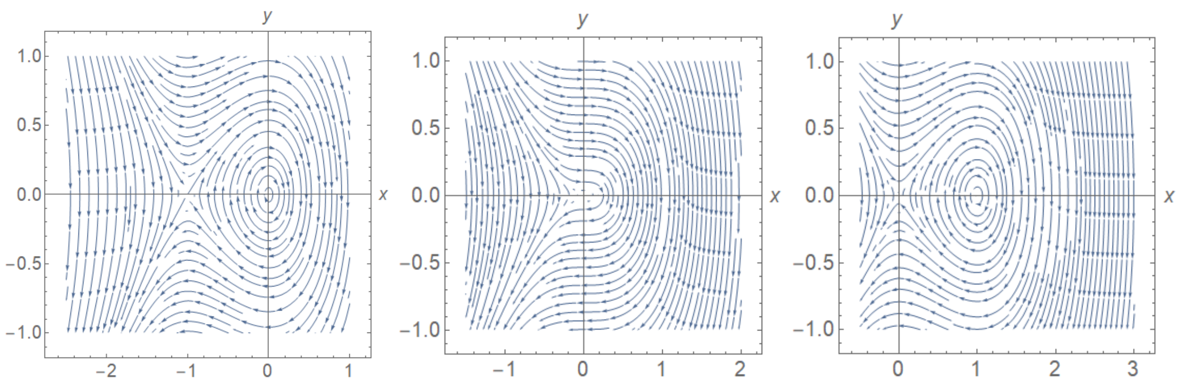

Figure 6: Phase portraits for Strogatz Problem 6.5.4. From left to right, a has values -1, 0, and 1.

$\square$

Problem 6.7.1. Find and classify the fixed points of $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ for all $b > 0$, and plot the phase portraits for qualitatively different cases.

Solution:

We may rewrite the given ODE as the ODE system, taking $x = \theta$,

$$\dot{x} = y, \quad \dot{y} = -by - \sin(x). \tag{278}$$

Since $\dot{x} = 0$ if and only if $y = 0$, we see the fixed points occur whenever $y = 0$ and $\sin(x) = 0$, i.e., at $(k\pi, 0)$ for all $k \in \mathbb{Z}$. The Jacobian matrix for the system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos(x) & -b \end{pmatrix}. \tag{279}$$

For even $k \in \mathbb{Z}$,

$$J(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}, \tag{280}$$

which has eigenvalues that satisfy

$$0 = \lambda(\lambda + b) + 1 = \lambda^2 + b\lambda + 1 \quad \Longrightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4}}{2}. \tag{281}$$

For odd $k \in \mathbb{Z}$,

$$J(k\pi, 0) = \begin{pmatrix} 0 & -1 \\ -1 & -b \end{pmatrix}, \tag{282}$$

which similarly yields eigenvalues

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4}}{2}. \tag{283}$$

From the above, we see $(k\pi, 0)$ forms a saddle whenever $k \in \mathbb{Z}$ is odd. And, when $k \in \mathbb{Z}$ is even, $(k\pi, 0)$ forms a stable node when $b > 2$ and a stable spiral when $b < 2$. The null-clines are given by $y = 0$, where $\dot{x} = 0$, and $y = \sin(x)/b$, where $\dot{y} = 0$. Below are plots of phase portraits. Note the system is periodic in x, with period 2π.

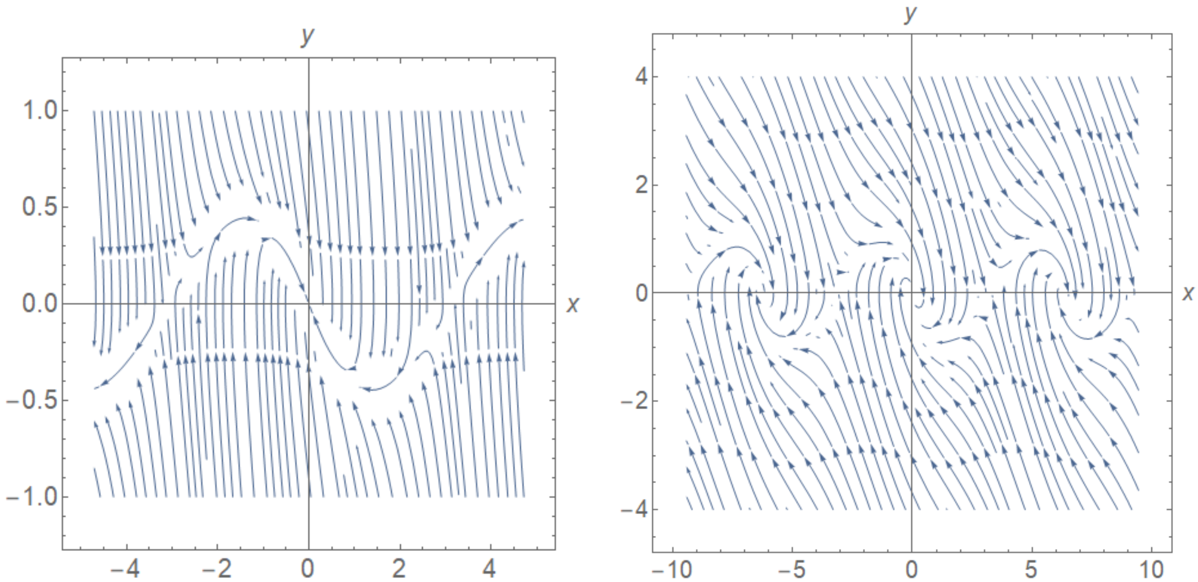

Figure 7: On the left is a plot with $b = 2.2$ and on the right with $b = 1$.

□

Problem 6.7.2.[7] The equation $\ddot{\theta} + \sin\theta = \gamma$ describes the dynamics of an undamped pendulum driven by a constant torque, or an undamped Josephson junction driven by a constant bias current. Find all the equilibrium points and classify them as γ varies. Sketch the phase portrait for qualitatively different γ.

Solution:

Taking $x = \theta$, we may rewrite the given ODE as the ODE system

$$\dot{x} = y, \quad \dot{y} = \gamma - \sin(x). \tag{284}$$

First note the system is Hamiltonian since

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial x}[y] + \frac{\partial}{\partial y}[\gamma - \sin(x)] = 0, \tag{285}$$

which implies all fixed points are either centers or saddles. If $\gamma \notin [-1,1]$, then the system does not admit any fixed points (since in such a case $\dot{y} \neq 0$ always). If $\gamma \in [-1,1]$, then the fixed points are given by all points (x,y) such that

$$(x,y) = (\arcsin(\gamma), 0). \tag{286}$$

The Jacobian matrix for this system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}. \tag{287}$$

Thus,

$$J(\arcsin(\gamma), 0) = \begin{pmatrix} 0 & 1 \\ -\cos(\arcsin(\gamma)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-\gamma^2} & 0 \end{pmatrix}, \tag{288}$$

which has eigenvalues $\lambda = i\sqrt{1-\gamma^2}$. Therefore, the fixed points are centers. The null-clines are given by $y = 0$, where $\dot{x} = 0$, and $x = \arcsin(\gamma)$, where $\dot{y} = 0$. Note that, because the sine function is periodic, there are countably many vertical lines where $\dot{y} = 0$. Phase plots are provided below.

[7] This is a modified form of the prompt, reflecting what we might expect on the qual.

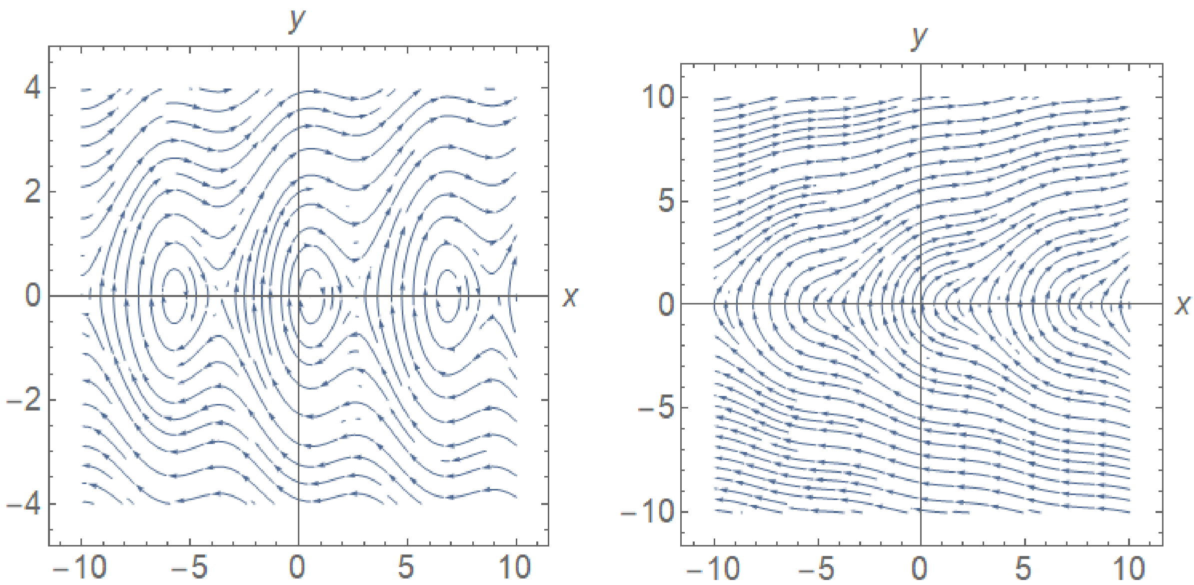

Figure 8: Phase plot with $\gamma = \pi/6$ on the left and $\gamma = 3/2$ on the right.

$\square$

Problem 6.8.7. Show the system $\dot{x} = x(4 - y - x^2)$, $\dot{y} = y(x - 1)$ has no closed orbits.

Solution:

We first identify the fixed points and their type. Written in the form (x, y), these are $(0, 0)$, $(2, 0)$, $(-2, 0)$, and $(1, 3)$. The Jacobian for the system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 4 - y - 3x^2 & -x \\ y & x - 1 \end{pmatrix}. \tag{289}$$

This implies

$$J(0, 0) = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}, \tag{290}$$

which has eigenvalues 4 and -1, and thus $(0, 0)$ forms a saddle. Also,

$$J(\pm 2, 0) = \begin{pmatrix} -8 & -2 \\ 0 & 1 \end{pmatrix}, \tag{291}$$

which has eigenvalues -8 and 1, and thus and $(-2, 0)$ and $(2, 0)$ form saddles. Similarly,

$$J(1, 3) = \begin{pmatrix} -24 & -1 \\ 3 & 0 \end{pmatrix}, \tag{292}$$

which has eigenvalues that satisfy

$$0 = (\lambda + 24)\lambda + 3 = \lambda^2 + 24\lambda + 3 \quad \Longrightarrow \quad \lambda = \frac{-24 \pm \sqrt{24^2 - 4 \cdot 3}}{2}, \tag{293}$$

and there are two distinct real-valued negative eigenvalues. Thus, $(1, 3)$ forms a stable spiral.

By a theorem in in Strogatz's text[8], if C is a closed orbit in the system, then C must enclose fixed points whose indices sum to unity. The index of a saddle is -1 and the index of a stable node is 1. Since there is precisely one stable node and the rest of the fixed points are saddles, if there is a closed orbit in the system, then it must enclose $(1, 3)$ and not enclose any other fixed points.

[8]See Theorem 6.8.2 on page 180.

Because there is a vertical null-cline along $x = 0$ and a horizontal null-cline along $y = 0$, each orbit enclosing $(1, 3)$ is restricted to the first quadrant. Now let (x, y) be a parameterization of any trajectory originating in the first quadrant not at $(1, 3)$. We claim $(x, y) \to (1, 3)$ as $t \to \infty$. This implies the trajectory is nonperiodic and, thus, not a closed orbit. Since this was an arbitrary trajectory in the first quadrant, we conclude no closed orbits exist.

All that remains is to verify asymptotic stability. Observe

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\dot{x}}{\dot{y}} = \frac{x(4 - y - x^2)}{y(x-1)} \quad \Longrightarrow \quad \frac{x-1}{x}\ \mathrm{d}x = \frac{4}{y} - 1 - \frac{x^2}{y}\ \mathrm{d}y \approx \frac{3}{y} - 1\ \mathrm{d}y, \tag{294}$$

for $x \approx 1$. Upon integrating, we see it fitting to define the Liapunov function $V(x, y)$ by

$$V(x, y) := y - 3\ln(y) + x - \ln(x) + 3\ln(3) - 4. \tag{295}$$

Note $V(1, 3) = 0$ and

$$\nabla V(x, y) = \left(\frac{x-1}{x},\ \frac{y^2 - 3}{y}\right). \tag{296}$$

This shows $V_x < 0$ for $x < 1$ and $V_x > 0$ for $x > 1$. Similarly, $V_y < 0$ for $y < 3$ and $V_y > 0$ for $y > 3$. Whence $(1, 3)$ is a strict local minimizer of V. Furthermore, for $x, y > 0$,

$$\begin{aligned} \dot{V} &= \left(\frac{y-3}{y}\right)\dot{y} + \left(\frac{x-1}{x}\right)\dot{x} \\ &= (y-3)(x-1) + (x-1)(4 - y - x^2) \\ &= (y-3)(x-1) + (x-1)(3-y) + (x-1)(1-x^2) \\ &= -(1-x)^2(1+x) \\ &\leq 0. \end{aligned} \tag{297}$$

The only fixed point for which $\dot{V} = 0$ is $(1, 3)$. We have verified the hypotheses of Lasalle's theorem, which asserts $(1, 3)$ is necessarily asymptotically stable. This completes the proof. □

Chapter 7

Example 7.2.4. Show the following system has no closed orbits in the positive portion of the first quadrant:

$$\dot{x} = x(2 - x - y), \quad \dot{y} = y(4x - x^2 - 3). \tag{298}$$

Solution:

We proceed by applying Dulac's Criterion, which asserts there are no closed orbits in the first quadrant provided there exists a continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla \cdot (g(z)\dot{z})$ is single-signed in the first quadrant, where here we write $z = (x, y)$. For $g(z) = x^a y^b$, we see

$$\begin{aligned} \nabla \cdot (g\dot{z}) &= \frac{\partial}{\partial x}\left[x^{a+1}y^b(2 - x - y)\right] + \frac{\partial}{\partial y}\left[x^a y^{b+1}(4x - x^2 - 3)\right] \\ &= \left[(a+1)x^a y^b(2 - x - y) - x^{a+1}y^b\right] + \left[(b+1)x^a y^b(4x - x^2 - 3) + 0\right]. \end{aligned} \tag{299}$$

Thus, taking $a = b = -1$, we obtain $g = 1/(xy)$ and

$$\nabla \cdot (g\dot{z}) = 0 - x^{-1+1}y^{-1} + 0 = -y^{-1} < 0. \tag{300}$$

Since this holds for all $z = (x, y) \in (0, \infty) \times (0, \infty)$, we conclude the ODE system admits no closed orbits.

□

Problem 7.3.1. Consider the ODE system[9]

$$\dot{x} = x - y - x(x^2 + 5y^2), \quad \dot{y} = x + y - y(x^2 + y^2). \tag{301}$$

Classify the fixed point at the origin and prove there exists a limit cycle.

Solution:

The Jacobian $J(x, y)$ matrix for this system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}, \tag{302}$$

and so

$$J(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{303}$$

which has eigenvalues that satisfy

$$0 = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2 \quad \implies \quad \lambda = \frac{2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = 1 \pm i. \tag{304}$$

Thus the origin forms an unstable spiral. We now convert to polar coordinates to find

$$\begin{aligned} 2r\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left[r^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[x^2 + y^2\right] \\ &= 2[x\dot{x} + y\dot{y}] \\ &= 2\left[x^2 - xy - x^4 - 5x^2y^2\right] + 2\left[xy + y^2 - x^2y^2 - y^4\right] \\ &= 2[(x^2 + y^2) - (x^2 + y^2)^2 - 4x^2y^2] \\ &= 2\left[r^2 - r^4\left[1 + 4\sin^2\cos^2\theta\right]\right] \\ &= 2r^2\left[1 - r^2\left[1 + \sin^2(2\theta)\right]\right]. \end{aligned} \tag{305}$$

Consequently, we see

$$\dot{r} = r\left[1 - r^2\left[1 + \sin^2(2\theta)\right]\right], \tag{306}$$

[9]The prompt has been modified to reflect the style that shows up on quals.

which has the nullclines $r = 0$ and

$$r = \frac{1}{\sqrt{1+\sin^2(2\theta)}}. \tag{307}$$

Similarly,

$$\begin{aligned}\dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \\ &= \frac{\left[x^2 + xy - x^3y - xy^3\right] - \left[xy - y^2 - x^3y - 5xy^3\right]}{r^2} \\ &= \frac{(x^2+y^2) + 4xy^3}{r^2} \\ &= 1 + \frac{4xy^3}{x^2+y^2} \\ &= 1 + 4r^2\cos(\theta)\sin^3(\theta) \\ &= 1 + r^2\sin(2\theta)[1-\cos(2\theta)].\end{aligned} \tag{308}$$

Along the nontrivial null-cline $\dot{r} = 0$ we see

$$\dot{\theta} = 1 + \frac{\sin(2\theta)[1-\cos(2\theta)]}{1+\sin^2(2\theta)} \geq 1 - \frac{2\sin(2\theta)}{1+\sin^2(2\theta)}. \tag{309}$$

Taking $f(x) := 1 - 2x/(1+x^2)$, we see

$$f'(x) = -2\cdot\frac{(1+x^2)-(2x)(x)}{(1+x^2)^2} = -2\cdot\frac{1-x^2}{(1+x^2)^2} \leq 0, \quad \text{for all } x \in [-1,1], \tag{310}$$

with the inequality strict wheenver $|x| \neq 1$. Thus,

$$\dot{\theta} \geq \inf_{\theta\in[0,\pi]} 1 - \frac{2\sin(2\theta)}{1+\sin^2(2\theta)} = \inf_{\theta\in[0,\pi]} f(\sin(2\theta)) = f(\sin(\pi/4)) = 1 - \frac{2}{1+1^2} = 0, \tag{311}$$

which shows the only possible angle θ at which $\dot{\theta} = 0$ along the null-cline would be at $\theta = \pi/4$. But, plugging in the exact expression for $\dot{\theta}$ reveals, along the null-cline,

$$\dot{\theta}\Big|_{\theta=\pi/4} = 1 + \frac{1[1-0]}{1+1^2} = \frac{3}{2} > 0. \tag{312}$$

Whence the system admits no fixed points.

We now proceed by constructing a trapping region. Observe

$$\sup_{\theta} \frac{1}{\sqrt{1+\sin^2(2\theta)}} = \frac{1}{\sqrt{1+0}} = 1, \tag{313}$$

which implies, by (306) and (307), $\dot{r} \leq 0$ for $r \geq 1$. Similarly,

$$\inf_{\theta} \frac{1}{\sqrt{1+\sin^2(2\theta)}} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} \tag{314}$$

implies $\dot{r} \geq 0$ for $r \leq 1/\sqrt{2}$. Therefore, the region $R := \{(r,\theta) : 1/\sqrt{2} \leq r \leq 1\}$ forms a closed subset of the plane $\mathbb{R}^2$. And, by our earlier work, R contains no fixed points. Additionally, the trajectory originating at $(r,\theta) = (1,0)$ is along the null-cline in (307) and, by our choice of R, is contained within R for all time. The Poincaré-Bendixson theorem then asserts R contains a closed orbit.[10] □

Remark: A phase plane plot for the previous problem is given below.

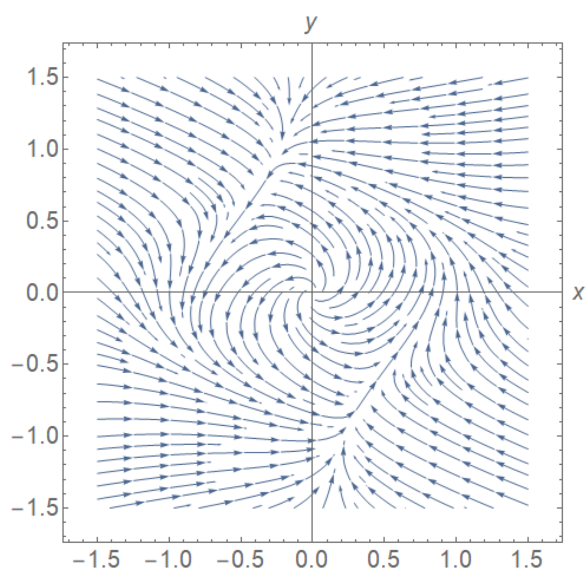

Figure 9: Phase plane for Strogatz Problem 7.3.1.

△

[10]In fact, the null-cline for r forms a closed orbit, and so we didn't actually need the Poincaré-Bendixson theorem here. However, we find it pedagogical to show its application.

Problem 7.3.3. Show the following ODE system has a periodic solution:

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3. \tag{315}$$

Solution:

We proceed by converting to polar coordinates and applying the Poincaré-Bendixson theorem. First observe differentiating $r^2/2$ reveals

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= \left[x^2 - xy - x^4\right] + \left[xy + y^2 - y^4\right] \\ &= (x^2 + y^2) - (x^4 + 2x^2y^2 + y^4) + 2x^2y^2 \\ &= r^2 - r^4 + 2r^4 \sin^2\theta\cos^2\theta \\ &= r^2\left[1 - r^2\left(1 - \frac{\sin^2(2\theta)}{2}\right)\right], \end{aligned} \tag{316}$$

which implies

$$\dot{r} = r\left[1 - r^2\left(1 - \frac{\sin^2(2\theta)}{2}\right)\right]. \tag{317}$$

Thus, the null-clines for $\dot{r} = 0$ are $r = 0$ and

$$r = \frac{2}{2 - \sin^2(2\theta)}. \tag{318}$$

Additionally,

$$\begin{aligned} \dot{\theta} &= \frac{x\dot{y} - \dot{x}y}{r^2} \\ &= \frac{\left[x^2 + xy - xy^3\right] - \left[xy - y^2 - x^3y\right]}{r^2} \\ &= \frac{x^2 + y^2 + xy^3 + x^3y}{r^2} \\ &= 1 + r^2\sin\theta\cos\theta\left[\sin^2\theta + \cos^2\theta\right] \\ &= 1 + \frac{r^2\sin(2\theta)}{2}. \end{aligned} \tag{319}$$

Along the null-cline in (318),

$$\dot{\theta} = 1 + \frac{\sin(2\theta)}{2} \cdot \frac{2}{2 - \sin^2(2\theta)} = 1 + \frac{\sin(2\theta)}{2 - \sin^2(2\theta)} \geq 1 + \frac{-1}{2-1} = \frac{1}{2} > 0, \tag{320}$$

where we note the max value obtained by $\sin(2\theta)$ is 1, and the expression for $\dot{\theta}$ is strictly increasing as a function of $\sin(2\theta)$ since

$$f(x) := 1 + x/(2 - x^2) \quad \Longrightarrow \quad f'(x) = \frac{(2 - x^2) - x(-2x)}{(2 - x^2)^2} = \frac{2 + x^2}{(2 - x^2)^2} > 0. \tag{321}$$

Because $\dot{\theta} \neq 0$ along the nontrivial null-cline for $\dot{r} = 0$, the only fixed point of the system occurs at the origin. Furthermore, since

$$\inf_{\theta} \frac{2}{2 - \sin^2(2\theta)} = \frac{2}{2 - 0} = 1 \quad \text{and} \quad \sup_{\theta} \frac{2}{2 - \sin^2(2\theta)} = \frac{2}{2 - 1} = 2, \tag{322}$$

it follows that $\dot{r} \geq 0$ for $r \leq 1$ and $\dot{r} \leq 0$ for $r \geq 2$. Therefore, the region R defined by $\{(r, \theta) : 1 \leq r \leq 2\}$ is closed. This region also does not contain any fixed points. And, it contains at least one trajectory; namely, the null-cline (318). Hence the Poincaré-Bendixson theorem asserts there is a closed orbit in R. This completes the proof. □

Remark: A phase plane plot for the previous problem is given below.

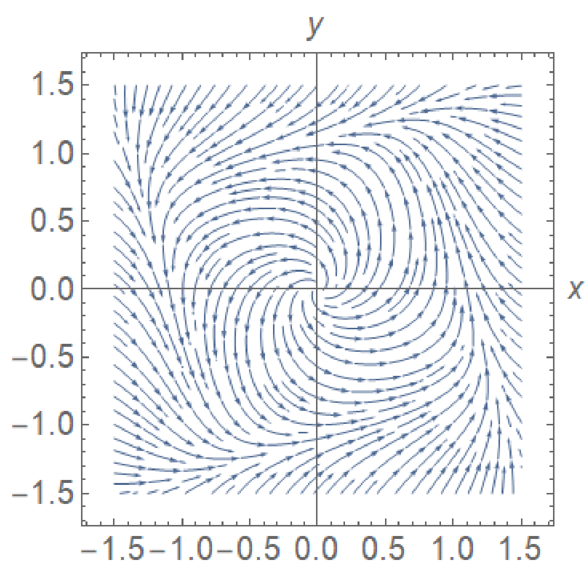

Figure 10: Phase plane for Strogatz Problem 7.3.3.

△

Problem 7.3.4. Consider the ODE system:

$$\dot{x} = x(1 - 4x^2 - y^2) - \frac{y(1+x)}{2}, \quad \dot{y} = y(1 - 4x^2 - y^2) + 2x(1+x). \tag{323}$$

a) Show the origin forms an unstable fixed point.

b) Show all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

Solution:

a) The Jacobian matrix $J(x, y)$ for this system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 1 - 12x^2 - y^2 - y/2 & -2xy - 1/2 \\ -8xy + 2(1+x) + 2x & 1 - 4x^2 - 3y^2 \end{pmatrix}. \tag{324}$$

This implies

$$J(0,0) = \begin{pmatrix} 1 & -1/2 \\ 2 & 1 \end{pmatrix}, \tag{325}$$

which has eigenvalues λ satisfying

$$0 = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2 \quad \Longrightarrow \quad \lambda = \frac{2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = 1 \pm i, \tag{326}$$

and so the origin forms an unstable spiral.

b) Differentiating in time reveals

$$\begin{aligned} \dot{V} &= 2(1 - 4x^2 - y^2)\left[-8x\dot{x} - 2y\dot{y}\right] \\ &= -2V^{1/2}\left[8x^2 - 4xy - 4x^2 + 2y^2V^{1/2} + 4xy + 4x^2\right] \\ &= -2V[8x^2 + 2y^2] \\ &\leq 0, \end{aligned} \tag{327}$$

with the inequality strict whenever (x, y) is neither along the ellipse $V = 0$ nor at the origin. Consider any trajectory originating at a location other than the origin. Along the trajectory, $\dot{V} < 0$ and $V \geq 0$, from which the monotone convergence theorem asserts V converges to a limit $V^\star$. This implies

$$0 = \dot{V}^\star = \lim_{t\to\infty} \dot{V} = \lim_{t\to\infty} -2V[8x^2 + y^2] \quad \Longrightarrow \quad \lim_{t\to\infty} V = V^\star = 0, \tag{328}$$

noting $8x^2 + y^2 > 0$ along the trajectory. This shows $V \to 0$ as $t \to \infty$, i.e., the trajectory approaches the ellipse as $t \to \infty$.

□

Remark: A phase plane plot for the previous problem is given below.

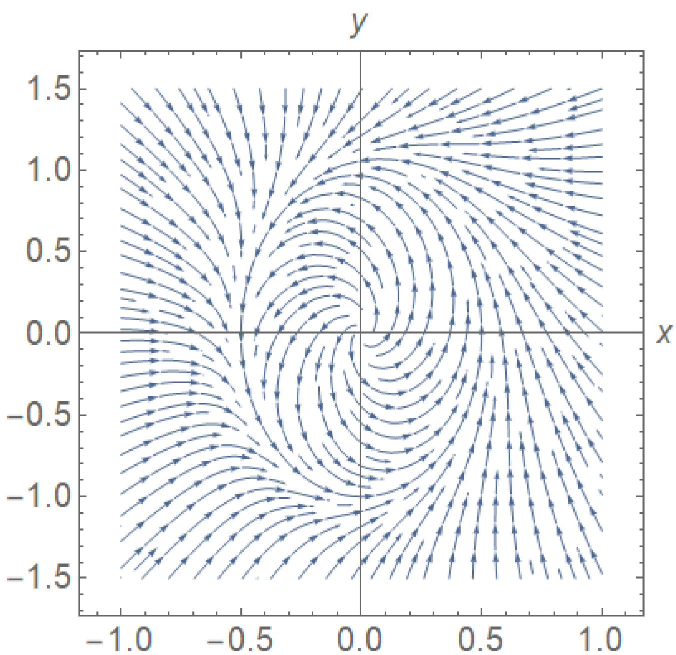

Figure 11: Phase plane for Strogatz Problem 7.3.4.

△

Problem 7.3.5. Show the following ODE system has at least one periodic solution:

$$\dot{x} = -x - y + x(x^2 + 2y^2), \quad \dot{y} = x - y + y(x^2 + 2y^2). \tag{329}$$

Solution:

We proceed by converting to polar coordinates and find a nontrivial periodic solution. Differentiating $r^2/2$ in time reveals

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= \left[-x^2 - xy + x^4 + 2x^2y^2\right] + \left[xy - y^2 + x^2y^2 + y^4\right] \\ &= -(x^2 + y^2) + (x^2 + y^2)^2 + x^2y^2 \\ &= -r^2 + r^4 + r^4\cos^2\theta\sin^2\theta \\ &= r^2\left[-1 + r^2\left(1 + \frac{\sin^2(2\theta)}{2}\right)\right]. \end{aligned} \tag{330}$$

Therefore the null-clines for $\dot{r} = 0$ are given by $r = 0$ and

$$r = \sqrt{\frac{2}{2 + \sin^2(2\theta)}}. \tag{331}$$

Additionally,

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{\left[x^2 - xy + x^3y + 2xy^3\right] - \left[-xy - y^2 + x^3y + 2xy^3\right]}{r^2} = \frac{x^2 + y^2}{r^2} = 1. \tag{332}$$

The fact $\dot{\theta} = 1$ everywhere implies $\theta = t + \theta_0$ and the only fixed point of this system is the origin. Thus, the ODE system admits the periodic solution

$$(r, \theta) = \left(\sqrt{\frac{2}{2 + \sin^2(2t)}}, t\right), \tag{333}$$

and we are done.[11] □

[11] Note we cannot use the Poincaré-Bendixson theorem for this problem since the limit cycle is unstable.

Remark: A phase plane plot for the previous problem is given below.

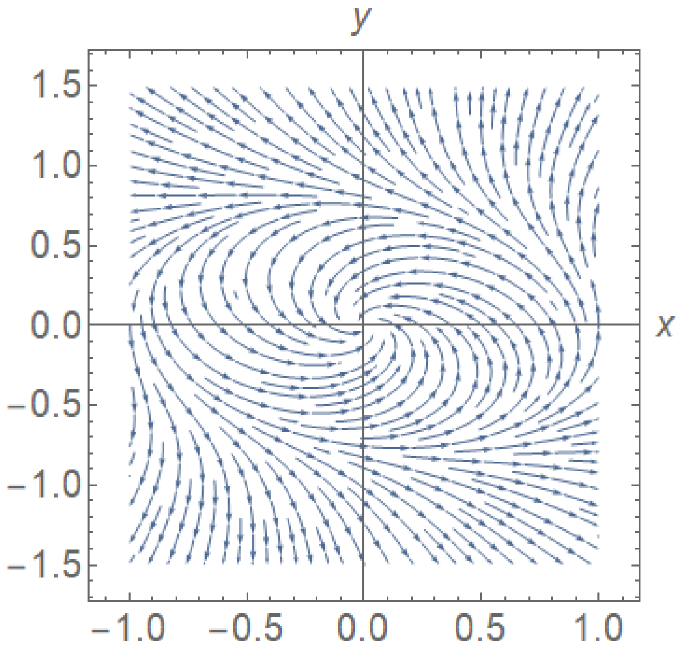

Figure 12: Phase plane for Strogatz Problem 7.3.5.

△

Bender and Orszag

The following are problems taken from the text of Bender and Orszag, reflecting types of problems that have begun to show up on the ADE qual in recent years and in MATH 266A course materials. Contrary to the authors of this textbook, I do not find much of the material "easy" (as they indicate by labels next to problems and sections).

Unfortunately, examples from this text are not provided in the online version. Please contact me via email if you wish to inquire about the solutions to these problems.

Past Homework Solutions

Problem 1. Solve the PDE

$$\begin{cases} u_t + \frac{1}{2} \cdot (u_x)^2 - u_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{393}$$

where a) $g = |\cdot|$ and b) where $g = -|\cdot|$.

Solution:

a) This is a Hamilton Jacobi equation with Hamiltonian $H(p) := p^2/2 - p$. We compute the Lagrangian $\mathcal{L}$ as the dual of H, i.e.,

$$\mathcal{L}(v) := H^*(v) := \sup_{p \in \mathbb{R}} \{p \cdot v - H(p)\} = \sup_{p \in \mathbb{R}} \left\{ p \cdot v - \frac{1}{2} \cdot p^2 + p \right\}. \tag{394}$$

We assume the supremum is taken on at the critical point of the braced expression (i.e., that it can be replaced by a max). At this critical point, we find

$$0 = D_p \left[p \cdot v - \frac{1}{2} \cdot p^2 + p \right] = v - p + 1 \quad \Longrightarrow \quad p = v + 1. \tag{395}$$

Hence

$$\mathcal{L}(v) = (v+1) \cdot v - \frac{1}{2}(v+1)^2 + (v+1) = \frac{1}{2}(v+1)^2. \tag{396}$$

Using the Hop-Lax formula, we find

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ t \cdot \mathcal{L}\left(\frac{x-y}{t}\right) + g(y) \right\} = \min_{y \in \mathbb{R}} \left\{ \frac{1}{2t} \cdot (x - y + 1)^2 + g(y) \right\}. \tag{397}$$

Note the expression to be minimized is convex. Thus, at a minimizer $y^\star$, the optimality condition with the subgradient is

$$0 \in \frac{\partial}{\partial y}\left[\frac{1}{2t}\cdot(x-y+1)^2+|y|\right]_{y=y^\star}=\frac{1}{t}\left(x-y^\star+1\right)+\operatorname{sgn}(y^\star) \qquad \Longrightarrow \qquad y^\star \in x+1+t\phi(y^\star), \quad (398)$$

where $\phi(y)$ equals the sign of y when $y \neq 0$ and $[-1,1]$ otherwise. (Return and complete.)

□

Qual Solutions

Reflections: We use a reflection to extend the domain of a function. Odd extensions are used to enforce a Direchlet condition (e.g., that the function is zero at the origin) while even reflections enforce a Neumann condition (e.g., the derivative is zero at the origin).

Entropy solutions: The entropy satisfying weak solution of a PDE is a function u for which[15]

1. u_ℓ and u_r are solutions to the PDE to the left and right of each shock curve C;
2. the Rankine-Hugoniot condition is satisfied along each shock C;
3. $F'(u_\ell) > \dot{s} > F'(u_r)$ along each shock C.

[15] There is also a more general condition than 3 given in the PDE text by Evans.

2018 Fall

F18.1.

a) Consider the dynamical system

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy, \tag{399}$$

where $a, b, c, d \in \mathbb{R}$ (with $ad - bc \geq 0$) are constants. Classify the equilibrium at $(0, 0)$ for all possible choices of the four constants. Indicate clearly all bifurcations that occur, and designate when you get closed orbits.

b) Consider the dynamical system

$$\dot{x} = -y + \alpha x(x^2 + y^2), \quad \dot{y} = x + \alpha y(x^2 + y^2), \tag{400}$$

where $\alpha \in \mathbb{R}$ is a constant. Determine, with appropriate arguments, the stability of the equilibrium point at the origin. Also draw the phase portraits for this system for all qualitatively different values of α.

Solution:

a) The Jacobian for the system is given by[16]

$$J(x, y) = \begin{pmatrix} \partial \dot{x}/\partial x & \partial \dot{x}/\partial y \\ \partial \dot{y}/\partial x & \partial \dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{401}$$

which has eigenvalues satisfying the characteristic equation

$$0 = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \tau\lambda + \Delta, \tag{402}$$

where $\tau := a + d$ is the trace and $\Delta := ad - bc$ is the discriminant. Thus,

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}. \tag{403}$$

Then we have the following cases.

[16] This problem is taken straight out of the explanations in Chapter 5 of Strogatz's text.

If $\Delta < 0$, then the eigenvalues are real and have opposite signs and, thus, form saddles points.

If $\Delta > 0$, the eigenvalues are either real with the same sign (nodes) or complex conjugate (spirals or centers). Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$. The parabola $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals, upon which we obtain degenerate and star nodes. The stability in these cases is determined by τ. If $\tau < 0$, both eigenvalues have negative real parts, making the fixed point stable. Unstable spirals occur when $\tau > 0$. Neutrally stable centers occur when $\tau = 0$.

If $\Delta = 0$, then at least one eigenvalue is zero. This means the origin is not an isolated fixed point.

We illustrate these results in Figure 13. From this, we see bifurcations occur along $\Delta = 0$, along $\tau = 0$, and along $4\Delta = \tau^2$. Centers (when $\tau = 0$) are when closed orbits occur.

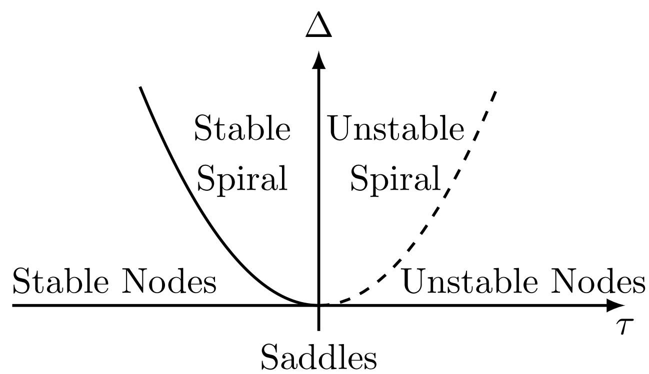

Figure 13: Classifications of fixed point determined by τ and Δ.

b) We proceed[17] by using standard polar coordinates (r, θ). Observe

$$\dot{r} = \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y} = \frac{1}{r}\left[r\dot{x} + y\dot{y}\right] = \frac{1}{r}\left[x\left(-y + \alpha x r^2\right) + y\left(x + \alpha y r^2\right)\right] = \alpha r^3. \tag{404}$$

This implies

$$-r\sin\theta + \alpha r^3\cos\theta = -y + \alpha x(x^2+y^2) = \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad\Longrightarrow\quad r\sin\theta[\dot{\theta}-1] = \underbrace{[\dot{r} - \alpha r^3]}_{=0}\cos\theta = 0. \tag{405}$$

[17]Note this is Example 6.3.2 in Strogatz's text.

Since this holds for all $y = r\sin\theta$, it follows that $\dot{\theta} = 1$, i.e., the trajectories are proceed counterclockwise with constant angular speed unity. Consequently, if $\alpha = 0$, then r is constant in time, which implies the origin forms a stable equilibrium point. If $\alpha > 0$, then r is increasing in time (strictly for $r \neq 0$), in which case the origin forms an unstable fixed point. If $\alpha < 0$, then r is decreasing in time (strictly for $r \neq 0$), making the origin asymptotically stable.

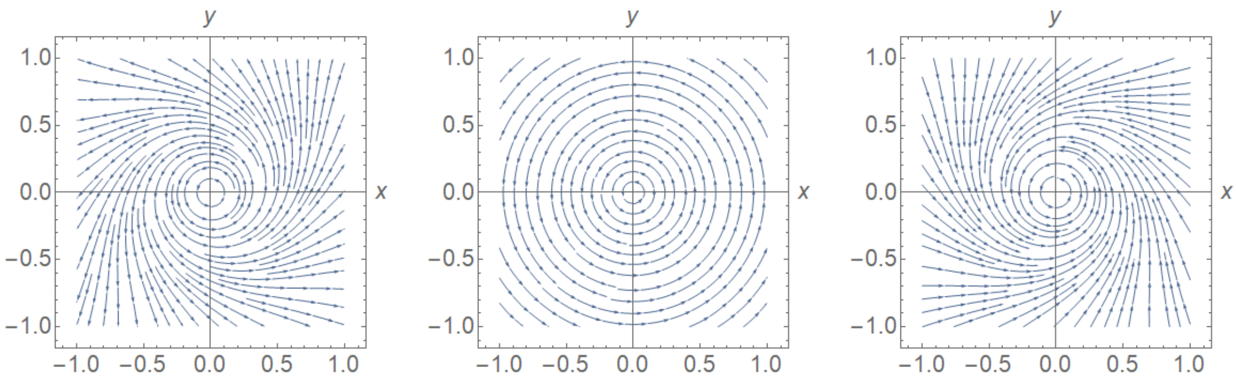

Figure 14: Plots of the ODE system for $\alpha = 1$, $\alpha = 0$, and $\alpha = -1$, from left to right.

□

F18.2. Consider the equation

$$xy'' + (2x-1)y' + \frac{1}{x}y = 0. \tag{406}$$

a) Classify the points $x = 0$ and $x = \infty$ as ordinary, regular, or irregular singular points.

b) For $x = 0$, determine the indicial equation and indicial exponents. Find the series expansion about $x = 0$ for the solution of (406) that satisfies $y'(0) = 1$, and from it obtain the solution in closed form. Why is one initial condition sufficient to determine this solution uniquely?

Solution:

a) The ODE may be rewritten as

$$y'' + py' + qy = 0, \tag{407}$$

where $p(x) = (2 - 1/x)$ and $q(x) = 1/x^2$. Since both p and q blow up as $x \longrightarrow 0$, the point $x = 0$ is not ordinary. However, since $xp = 2x - 1$ and $x^2 q = 1$ are analytic in a neighborhood of $x = 0$, the point $x = 0$ is a regular singular point.

The point $x = \infty$ will be classified according to how $x = 1/t$ is classified at $t = 0$. Observe

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = -x^{-2}\frac{\mathrm{d}y}{\mathrm{d}t}, \tag{408}$$

and so

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[-x^{-2}\frac{\mathrm{d}y}{\mathrm{d}t}\right] = 2x^{-3}\frac{\mathrm{d}y}{\mathrm{d}t} - x^{-2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}t} = 2t^3\frac{\mathrm{d}y}{\mathrm{d}t} - t^4\frac{\mathrm{d}^2y}{\mathrm{d}t^2}. \tag{409}$$

Thus, the ODE becomes

$$0 = \left(2t^3\frac{\mathrm{d}y}{\mathrm{d}t} - t^4\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) + (2-t)\left(-t^2\frac{\mathrm{d}y}{\mathrm{d}t}\right) + t^2 y \quad \Longrightarrow \quad 0 = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (2t^{-2} - 3t^{-1})\frac{\mathrm{d}y}{\mathrm{d}t} - t^{-2}y. \tag{410}$$

Since $(2t^{-2} - 3t^{-1})$ and $-t^{-2}$ are not analytic in a neighbhorhood of $t = 0$ while $t^2(2t^{-2} - 3t^{-1}) = 2 - 3t$ and $t^2(-t^{-2}) = -1$ are, we deduce $t = 0$ is a regular singular point. Whence $x = \infty$ is a regular singular point.

b) Since $x = 0$ is a regular singular point, Fuch's result states a solution $y_1(x)$ may be expressed as

$y_1(x) = x^\alpha A(x)$, where A is analytic, i.e., there are scalars $\{c_n\}_{n=0}^\infty$ such that

$$y_1(x) = \sum_{n=0}^\infty c_n x^{n+\alpha}, \tag{411}$$

where we assume α is chosen such that $c_0 \neq 0$. Plugging this into our differential equation reveals

$$\begin{aligned} 0 &= x^2 y_1'' + (2x^2 - x)y_1' + y_1 \\ &= \sum_{n=0}^\infty c_n \left[(\alpha+n)(\alpha+n-1)x^{n+\alpha} + 2(\alpha+n)x^{n+\alpha+1} - (\alpha+n)x^{n+\alpha} + x^{n+\alpha}\right] \\ &= c_0(\alpha-1)^2 x^\alpha + \sum_{n=1}^\infty c_n(\alpha+n-1)^2 x^{n+\alpha} + + \sum_{k=1}^\infty 2c_{k-1}(\alpha+k-1)x^{k+\alpha} \\ &= c_0(\alpha-1)^2 x^\alpha + \sum_{n=1}^\infty (\alpha+n-1)\left[(\alpha+n-1)c_n + 2c_{n-1}\right] x^{n+\alpha}. \end{aligned} \tag{412}$$

Since this holds for all x in a neighbhorhood of the origin and $c_0 \neq 0$, the first term reveals the indicial exponents are $\alpha_1 = \alpha_2 = 1$. The coefficients in the series must also be identically zero, which yields the recurrence relation, upon plugging in $\alpha = 1$,

$$c_n = -\frac{2}{n} \cdot c_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{413}$$

Taking $c_0 = 1$, we see

$$y_1(x) = \sum_{n=0}^\infty \frac{(-2)^n}{n!} x^{n+1} = x \sum_{n=0}^\infty \frac{(-2x)^n}{n!} = x\exp(-2x). \tag{414}$$

Since the indicial exponent is repeated, the second linearly independent solution $y_2(x)$ of the ODE is of the form

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=0}^\infty b_n x^{n+1}, \tag{415}$$

where

$$b_n = \frac{\partial}{\partial \alpha}\left[c_n(\alpha)\right]_{\alpha=1}. \tag{416}$$

Therefore, the general solution to the ODE is of the form

$$y(x) = d_1 y_1(x) + d_2 y_2(x), \tag{417}$$

for some scalars $d_1, d_2 \in \mathbb{R}$. However, since

$$y_2'(x) = y_1'(x)\ln(x) + \frac{y_1(x)}{x} + o\left(1\right) \quad \text{as } x \longrightarrow 0, \tag{418}$$

and

$$y_1'(0) = \left[(1-2x)\exp(-2x)\right]_{x=0} = 1, \tag{419}$$

and $y_1(0) = 0$, we see $y_2'(0)$ is undefined. Therefore, initial condition then reveals $d_2 = 0$ and

$$1 = y'(0) = d_1 y_1'(0) = d_1 \quad \Longrightarrow \quad y(x) = y_1(x) = x\exp(-2x). \tag{420}$$

□

F18.3. Consider the Chebyshev equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{1/2}\frac{\mathrm{d}y}{\mathrm{d}x}\right] + n^2(1-x^2)^{-1/2}y = 0 \quad \text{for } x \in (-1,1), \tag{421}$$

with integers $n \geq 0$.

a) Find the general solution to (421).

b) Denote by $T_n(x) = \cos(n \arccos(x))$ the degree-n polynomial solution of (421). Show that the $T_n(x)$ satisfy the orthogonality relation

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}\ \mathrm{d}x = 0, \quad \text{for } m \neq n. \tag{422}$$

Determine the expansion of the function $g(x) = (1-x^2)^{1/2}$ in terms of the $T_n(x)$.

Solution:

a) Consider the change of variables $x = \cos\theta$ where $0 < \theta < \pi$. In this case,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{1}{\sin\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta}, \tag{423}$$

where we note

$$x = \cos\theta \quad \Longrightarrow \quad 1 = -\sin\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \quad \Longrightarrow \quad \frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{1}{\sin\theta}. \tag{424}$$

Thus, for $\theta \in (0,\pi)$,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{1/2}\frac{\mathrm{d}y}{\mathrm{d}x}\right] + n^2(1-x^2)^{-1/2}y \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin\theta \cdot -\frac{1}{\sin\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta}\right] + \frac{n^2 y}{\sin\theta} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{\mathrm{d}y}{\mathrm{d}\theta}\right] + \frac{n^2 y}{\sin\theta} \\ &= \frac{1}{\sin\theta}\left[\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2} + n^2 y\right]. \end{aligned} \tag{425}$$

Since $\theta \in (0,\pi)$, the term $\sin\theta$ is nonzero, thereby implying the expression in brackets is zero. The general solution is, thus,

$$y = c_1\sin(n\theta) + c_2\cos(n\theta) = c_1\sin(n\arccos(x)) + c_2\cos(n\arccos(x)), \tag{426}$$

for some scalars $c_1, c_2 \in \mathbb{R}$.

b) Using the same change of variables as before, for $m \neq n$,

$$\begin{aligned} 0 &= \int_{-1}^{1} T_n(x) T_m(x) (1-x^2)^{-1/2} \, \mathrm{d}x \\ &= -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{0}^{\pi} \cos((m+n)\theta) + \cos((m-n)\theta) \, \mathrm{d}\theta \\ &= \frac{1}{2}\left[\frac{\sin((m+n)\theta)}{m+n} + \frac{\sin((m-n)\theta)}{m-n}\right]_{\theta=0}^{\pi} \\ &= 0, \end{aligned} \tag{427}$$

where we note

$$x = \cos\theta \quad \Longrightarrow \quad \mathrm{d}x = -\sin\theta \mathrm{d}\theta \quad \Longrightarrow \quad -\mathrm{d}\theta = \frac{\mathrm{d}x}{\sin\theta} = (1-x^2)^{-1/2}\mathrm{d}x. \tag{428}$$

Let us now consider the expansion of $g(x)$. We seek to show identify scalars $\{c_n\}_{n=0}^{\infty}$ such that

$$g(x) = \sum_{n=0}^{\infty} c_n T_n(x). \tag{429}$$

We proceed by working with the variable θ, considering the Sturm-Liouville problem

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\theta^2} + n^2 z = 0, \quad z'(0) = z'(\pi) = 0. \tag{430}$$

For each n, the general solution is of the form

$$z = \alpha_1 \sin(n\theta) + \alpha_2 \cos(n\theta), \tag{431}$$

and the boundary conditions imply $\alpha_1 = 0$ and n is an integer. By Sturm-Liouville theory, the resulting eigenfunctions $\{\cos(n\theta)\}_{n=0}^{\infty}$ form an eigenbasis and are orthogonal with respect to the scalar product

$$\langle f_1, f_2 \rangle := \int_0^{\pi} f_1(\theta) f_2(\theta) \, \mathrm{d}\theta. \tag{432}$$

Therefore, there exists scalars $\{d_n\}_{n=0}^{\infty}$ such that $\sin\theta$ may be expressed on $[0,\pi]$ via

$$\sin\theta = \sum_{n=0}^{\infty} d_n \cos(n\theta). \tag{433}$$

By the orthogonality of the eigenfunctions, we have the standard result for the coefficients

$$d_n = \frac{\langle \sin\theta, \cos(n\theta)\rangle}{\langle \cos(n\theta), \cos(n\theta)\rangle} = \frac{\int_0^{\pi} \sin(\theta)\cos(n\theta)\ \mathrm{d}\theta}{\int_0^{\pi} \cos^2(n\theta)\ \mathrm{d}\theta}, \quad \text{for all } n \geq 0. \tag{434}$$

However, for $x \in (-1,1)$ and the change of variables $x = \cos\theta$, we see $\theta \in (0,\pi)$, and so

$$g(x) = \sqrt{1-\cos^2\theta} = |\sin\theta| = \sin\theta = \sum_{n=0}^{\infty} d_n \cos(n\theta) = \sum_{n=0}^{\infty} d_n T_n(x). \tag{435}$$

This verifies (429), taking $c_n = d_n$, and the proof is complete.

$\square$

F18.4. For a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, consider the parabolic PDE

$$\begin{cases} u_t - \Delta u = (1-u)_+ & \text{in } \Omega \times (0,\infty), \\ u = \ell & \text{on } \partial\Omega \times (0,\infty), \\ u = g & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{436}$$

where g is a smooth function that vanishes on $\partial\Omega$.

a) Show that if $\ell(x), g(x) \leq 1$, then $u(x,t) \leq 1$ for all $t > 0$.

b) Supposing[18] instead that g does not vanish along $\partial\Omega$ and that $g(x) > 1$ and $\ell(x) > 1$, show that $u(x,t) > 1$ for all $t > 0$.

Solution:

a) Fix $T > 0$ and take $\Omega_T := \Omega \times (0,T]$ and Γ_T to be the parabolic boundary of Ω_T. Let $\varepsilon > 0$ and set $v := u - \varepsilon e^t$. Since $\overline{\Omega}_T$ is compact and v is continuous, v attains its supremum over $\overline{\Omega}_T$. By way of contradiction, suppose this supremum is at least unity. By the continuity of v, it follows that there exists a point in $\overline{\Omega}_T$ at which $v = 1$. Let $(x^\star, t^\star) \in \overline{\Omega}_T$ be such that $v(x^\star, t^\star) = 1$, with $t^\star$ the first time at which this occurs. Note $t^\star > 0$ since

$$v(x,t) = g(x) - \varepsilon e^0 \leq 1 - \varepsilon < 1 \quad \text{on } \Omega \times \{t = 0\}. \tag{437}$$

And, $x^\star \in \Omega$ since

$$v(x,t) = \ell(x) {}_- \varepsilon e^{t^\star} \leq 1 - \varepsilon e^{t^\star} \leq 1 - \varepsilon < 1 \quad \text{on } \partial\Omega \times (0,\infty). \tag{438}$$

Therefore, $(x^\star, t^\star) \in \overline{\Omega}_T - \Gamma_T = \Omega_T$. By the fact $t^\star$ is the first time at which $v = 0$, we see $v_t(x^\star, t^\star) \geq 0$. Since $x^\star$ is a local maximizer of $v(\cdot, t^\star)$, it follows that $\Delta v(x^\star, t^\star) = 0$. Whence, at $(x^\star, t^\star)$,

$$0 \leq v_t - \Delta v = u_t - \Delta v - \varepsilon e^{t^\star} = (1-u)_+ - \varepsilon e^{t^\star} = \left(-\varepsilon e^{t^\star}\right)_+ - \varepsilon e^{t^\star} = -\varepsilon e^{t^\star} < 0, \tag{439}$$

where the third equality holds since at the indicated point $1 = v = u - \varepsilon e^{t^\star}$. This inequality implies

[18] We believe the original prompt was in error here. For g could not be smooth and vanish at the boundary $\partial\Omega$ while simultaneously be greater than unity everywhere in Ω. We have rewritten what we think it should have stated.

$0 < 0$, a contradiction. Therefore,

$$\sup_{\overline{\Omega}_T} v < 1, \tag{440}$$

and so

$$u < 1 + \varepsilon e^t \leq 1 + \varepsilon e^T \quad \text{in } \overline{\Omega}_T. \tag{441}$$

Since $\varepsilon > 0$ was arbitrarily chosen, we may let $\varepsilon \longrightarrow 0^+$ to deduce

$$u \leq 1 \quad \text{in } \overline{\Omega}_T. \tag{442}$$

And, because this result holds for arbitrary $T > 0$, we may let $T \longrightarrow \infty$ to deduce $u \leq 1$ for all times, as desired.

b) Let $T > 0$ be given and again set $\Omega_T := \Omega \times (0, T]$ and Γ_T to be the parabolic boundary. Since Γ_T is compact and u is continuous, it follows from the hypotheses given that

$$\inf_{\Gamma_T} u > 1. \tag{443}$$

Choose ε such that $0 < \varepsilon < \inf_{\Gamma_T} u - 1$, and then define $v := u - \varepsilon e^{-t}$. It then follows that

$$v = u - \varepsilon e^{-t} \geq u - \varepsilon > 1 \quad \text{on } \Gamma_T. \tag{444}$$

Since v is continuous and $\overline{\Omega}_T$ is compact, v attains its infimum over $\overline{\Omega}_T$. By way of contradiction, suppose the infimum is less than or equal to unity. By (444), it follows that any minimizer of v over $\overline{\Omega}_T$ is contained in Ω_T. By the continuity of v, it follows that there exists a point in Ω_T at which $v = 1$. Let $(\tilde{x}, \tilde{t})$ be such a point with $\tilde{t}$ the first time at which this occurs. Then $v_t(\tilde{x}, \tilde{t}) \leq 0$ and, because $\tilde{x}$ is a local minimizer of $v(\cdot, \tilde{t})$, we deduce $\Delta v(\tilde{x}, \tilde{t}) \geq 0$. Therefore, at the minimizer,

$$0 \geq v_t - \Delta v = u_t - \Delta u + \varepsilon e^{-\tilde{t}} = (1-u)_+ + \varepsilon e^{-\tilde{t}} = \underbrace{\left(-\varepsilon e^{-\tilde{t}}\right)_+}_{=0} + \varepsilon e^{-\tilde{t}} > 0, \tag{445}$$

a contradiction. Thus, the minimizer of v over $\overline{\Omega}_T$ exceeds unity, and so

$$\inf_{\overline{\Omega}_T} u = \inf_{\overline{\Omega}_T} v + \varepsilon e^{-t} \geq \left(\inf_{\overline{\Omega}_T} v\right) + \varepsilon e^{-T} > \inf_{\overline{\Omega}_T} v > 1 \quad \text{in } \overline{\Omega}_T. \tag{446}$$

Because this holds for arbitrary $T > 0$, it follows that, for all $(x, T) \in \overline{\Omega} \times (0, \infty)$, $u(x, T) > 1$, as desired.

□

F18.5. Consider the following initial-boundary problem for $u = u(x,t)$ in the domain $\{x > 0\} \times \{t > 0\}$:

$$\begin{cases} u_t - u_{xx} + au = 0 & \text{in } \{x > 0\} \times \{t > 0\}, \\ u = f & \text{on } \{x > 0\} \times \{t = 0\}, \\ u = g & \text{on } \{x = 0\} \times \{t > 0\}, \end{cases} \tag{447}$$

where $f(x)$ and $g(t)$ are continuous functions with compact support and a is a constant. Find an explicit solution of this problem.

Solution:

We proceed[19] by defining a new function and then performing an odd reflection. Set $v(x,t) := u(x,t)e^{at}$. Then

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } \{x > 0\} \times \{t > 0\}, \\ v = f & \text{on } \{x > 0\} \times \{t = 0\}, \\ v = ge^{at} & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{448}$$

Then define $w(x,t) := v(x,t) - f(x) - g(t)$. Because $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to \mathbb{R}$ are compactly supported, $f(0) = g(0) = 0$. This implies

$$w(x,0) = v(x,0) - f(x) - g(0) = f(x) - f(x) - 0 = 0 \quad \text{and} \quad w(0,t) = v(0,t) - f(0) - g(t) = g(t) - 0 - g(t) = 0. \tag{449}$$

Now define the odd reflection $\tilde{w}(x,t)$ by

$$\tilde{w}(x,t) := \begin{cases} w(x,t) & \text{if } x > 0, \\ -w(-x,t) & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{450}$$

Differentiating reveals

$$\tilde{w}_t(x,t) - \tilde{w}_{xx}(x,t) = w_t(x,t) - w_{xx}(x,t) = \phi(x,t) \quad \text{in } \{x > 0\} \times \{t > 0\}, \tag{451}$$

and

$$\tilde{w}_t(x,t) - \tilde{w}_{xx}(x,t) = -w_t(-x,t) + w_{xx}(-x,t) = \phi(x,t) \quad \text{in } \{x < 0\} \times \{t > 0\}, \tag{452}$$

[19] See Problem 2.15 in Evans' text on page 16 and also the solution to S14.1.

where

$$\phi(x,t) := \begin{cases} -(ge^{at})' + f''(x) & \text{if } x > 0, \\ (ge^{at})' - f''(-x) & \text{if } x < 0, \end{cases} \tag{453}$$

and we momentarily assume f and g are smooth. This assumption may be relaxed upon arriving at our final expression for $u(x,t)$. Compiling our results, we may write

$$\begin{cases} \tilde{w}_t - \tilde{w}_{xx} = \phi & \text{in } \mathbb{R} \times (0,\infty), \\ \tilde{w} = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \\ \tilde{w} = 0 & \text{on } \{x = 0\} \times (0,\infty). \end{cases} \tag{454}$$

For each $t \in (0,\infty)$, observe, for fixed $s \in (0,t)$,

$$\tilde{w}(x,t;s) := \int_{-\infty}^{\infty} \Phi(x-\xi, t-s)\phi(\xi,s)\ \mathrm{d}\xi \tag{455}$$

forms a solution to

$$\begin{cases} \tilde{w}_t(\cdot;s) - \tilde{w}_{xx}(\cdot;s) = 0 & \text{in } \mathbb{R} \times (s,\infty), \\ \tilde{w}(\cdot;s) = 0 & \text{on } \mathbb{R} \times \{t = s\}, \\ \tilde{w}(\cdot;s) = \phi(\cdot,s) & \text{on } \{x = 0\} \times (s,\infty), \end{cases} \tag{456}$$

where Φ is the fundamental solution to the heat equation:

$$\Phi(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right). \tag{457}$$

Duhamel's principle asserts

$$\tilde{w}(x,t) = \int_0^t \tilde{w}(x,t;s)\ \mathrm{d}s = \int_0^t \int_{-\infty}^{\infty} \Phi(x-\xi,t-s)\phi(\xi,s)\ \mathrm{d}\xi \mathrm{d}s \tag{458}$$

Substituting in the definition of ϕ, we see

$$\tilde{w}(x,t) = \int_0^t \left[\int_0^{\infty} \Phi(x-\xi,t-s)\left[f''(\xi) - (g(s)e^{as})'\right] \mathrm{d}\xi + \int_{-\infty}^{0} \Phi(x-\xi,t-s)\left[-f''(-\xi) + (g(s)e^{as})'\right] \mathrm{d}\xi \right] \mathrm{d}s \tag{459}$$

An explicit solution (which we currently neglect to fully write down) is then given by integrating by parts to take the derivatives off of f and g and place them onto Φ, which is smooth. Then, in $\{x > 0\} \times \{t > 0\}$,

$$u(x,t) = e^{-at} v(x,t) = e^{-at} \left(w(x,t) + f(x) + g(t) \right) = e^{-at} \left(\tilde{w}(x,t) + f(x) + g(t) \right). \tag{460}$$

□

F18.6. For a bounded domain Ω in $\mathbb{R}^n$ and for

$$u \in \mathcal{A} := \{w \in C^1(\Omega) : w|_{\partial\Omega} = 0, \quad \int_\Omega w = 1\}, \tag{461}$$

consider the energy

$$E(u) := \int_\Omega \sqrt{1 + |Du|^2}\ \mathrm{d}x. \tag{462}$$

a) Show that $E(u)$ has at most one minimizer among $u \in \mathcal{A}$.

b) Let $\Omega := \{|x| < 1\}$ and suppose that $u^\star$ minimizes $E(u)$ over $\mathcal{A}$. Show that $u^\star$ is a radial function.

Solution:

a) Let $u, v \in \mathcal{A}$. Assume u and v are distinct minimizers of E over $\mathcal{A}$. It suffices to show $u = v$. Note $\frac{1}{2}(u+v) \in \mathcal{A}$. Define $f(\alpha) := (1+\alpha^2)^{1/2}$ and observe

$$f'(\alpha) = \alpha(1+\alpha^2)^{-1/2} \quad \Longrightarrow \quad f''(\alpha) = (1+\alpha^2)^{-1/2} - \alpha^2(1+\alpha^2)^{-3/2} = (1+\alpha^2)^{-3/2} > 0, \tag{463}$$

which shows f is strictly convex and strictly increasing for positive arguments. Therefore,

$$f\left(\frac{1}{2}|Du + Dv|\right) \leq f\left(\frac{1}{2}|Du| + \frac{1}{2}|Dv|\right) \leq \frac{1}{2}f(|Du|) + \frac{1}{2}f(|Dv|), \tag{464}$$

where the final inequality is strict whenever $|Du| \neq |Dv|$. Consequently,

$$E\left(\frac{1}{2}(u+v)\right) = \int_\Omega f\left(\frac{1}{2}|Du + Dv|\right)\ \mathrm{d}x \leq \frac{1}{2}E(u) + \frac{1}{2}E(v) = E(u). \tag{465}$$

Because u and v are minimizers, the left hand side of (465) is bounded below by $E(u)$. This implies

$$\int_\Omega f\left(\frac{1}{2}|Du + Dv|\right)\ \mathrm{d}x = \frac{1}{2}\int_\Omega f(|Du|) + f(|Dv|)\ \mathrm{d}x, \tag{466}$$

and so

$$0 = \int_\Omega \underbrace{\frac{1}{2}\left[f(|Du|) + f(|Dv|)\right] - f\left(\frac{1}{2}|Du + Dv|\right)}_{\geq 0}\ \mathrm{d}x. \tag{467}$$

It follows from undergraduate analysis that the integrand is identically zero, from which (464) implies

$|Du| = |Dv|$ in Ω, and so

$$f\left(\frac{1}{2}|Du + Dv|\right) = f(|Du|) \quad \text{in } \Omega. \tag{468}$$

However,

$$\begin{aligned} f\left(\frac{1}{2}|Du + Dv|\right) &= \left(1 + \frac{1}{4}|Du + Dv|^2\right)^{1/2} \\ &= \left(1 + \frac{1}{4}|Du|^2 + \frac{1}{4}|Dv|^2 + \frac{1}{2}|Du||Dv|\cos\theta\right)^{1/2} \\ &= \left(1 + \frac{1+\cos\theta}{2}|Du|^2\right)^{1/2} \\ &= f\left(\sqrt{\frac{1+\cos\theta}{2}}|Du|\right), \end{aligned} \tag{469}$$

where $\theta \in [0, \pi]$ denotes the angle between Du and Dv and the third equality holds since $|Du| = |Dv|$ in Ω. Thus,

$$f(|Du|) = f\left(\sqrt{\frac{1+\cos\theta}{2}}|Du|\right) \quad \text{in } \Omega, \tag{470}$$

which, by the fact f is strictly increasing for positive arguments, can only be the case when $\cos\theta = 1$ everywhere in Ω, thereby implying Du and Dv are parallel. Since Du and Dv are parallel and with equal magnitude in Ω, we see $Du = Dv$ in Ω. This shows $u = v + c$ in Ω for some constant $c \in \mathbb{R}$. However,

$$1 = \int_\Omega u \ \mathrm{d}x = \int_\Omega v + c \ \mathrm{d}x = 1 + \int_\Omega c \ \mathrm{d}x = 1 + c|\Omega| \quad \Longrightarrow \quad c = 0. \tag{471}$$

Therefore, $u = v$ in Ω, and the proof is complete.

b) Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Define $v(x) := u^\star(Qx)$. Using the tensor notation for derivatives and the summation convention, we see

$$v_i = u_j^\star Q_{ji} \quad \Longrightarrow \quad |Dv|^2 = v_i v_i = (u_j^\star Q_{ji})(u_k^\star Q_{ki}) = u_j^\star u_k^\star Q_{ji} Q_{ik}^T = u_j^\star u_k^\star \delta_{jk} = u_j^\star u_j^\star = |Du^\star|^2. \tag{472}$$

And,

$$\int_\Omega v(x) \ \mathrm{d}x = \int_\Omega u^\star(Rx) \ \mathrm{d}x = \int_\Omega u^\star(y) \ \mathrm{d}y = 1, \tag{473}$$

where we use change of variables $y = Rx$ and note the orthogonality of R implies $|R| = 1$, and so $\mathrm{d}y = \mathrm{d}x$, and $R\Omega = \Omega$ since Ω is rotationally invariant. We additionally see $v(x) = u(Rx) = 0$ on $\partial\Omega$

since $x \in \partial\Omega$ implies $Rx \in \partial\Omega$, by the symmetry of Ω. Thus, $v \in \mathcal{A}$. Furthermore, (472) implies

$$E(v) = \int_\Omega f(|Dv|)\ \mathrm{d}x = \int_\Omega f(|Du^\star|)\ \mathrm{d}x = E(u^\star). \tag{474}$$

By our result in a), the minimizer of E over $\mathcal{A}$ is unique. Therefore, $v = u^\star$ and, as Q was an arbitrary orthogonal matrix, we conclude $u^\star$ is radial.

□

Remark: Now suppose $u^\star$ minimizes E over $\mathcal{A}$ and v is a perturbation function, i.e., $v \in C^1(\Omega)$ and $v|_{\partial\Omega} = 0$. Then

$$\delta E(u^\star, v) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[E(u+\varepsilon v)\right]_{\varepsilon=0} = \left[\int_\Omega f'(|Du+\varepsilon v|)\cdot \frac{Du+\varepsilon Dv}{|Du+\varepsilon Dv|}\cdot Dv \ \mathrm{d}x\right]_{\varepsilon=0} = \int_\Omega f'(|Du|)\cdot\frac{Du}{|Du|}\cdot Dv \ \mathrm{d}x, \quad (475)$$

which may be simplified as

$$\delta E(u^\star, v) = \int_\Omega \frac{Du}{\sqrt{1+|Du|^2}}\cdot Dv \ \mathrm{d}x. \quad (476)$$

Define

$$J(u) := \int_\Omega u \ \mathrm{d}x, \quad (477)$$

and so

$$\delta J(u^\star, v) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[J(u^\star+\varepsilon v)\right]_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_\Omega u^\star + \varepsilon v \ \mathrm{d}x\right]_{\varepsilon=0} = \left[\int_\Omega v \ \mathrm{d}x\right]_{\varepsilon=0} = \int_\Omega v \ \mathrm{d}x. \quad (478)$$

By Lagrange's theorem of multipliers, there exists $\lambda \in \mathbb{R}$ such that, for all test functions v,

$$\delta E(u^\star, v) = \lambda \delta J(u^\star, v), \quad (479)$$

which implies

$$0 = \int_\Omega \frac{Du}{\sqrt{1+|Du|^2}}\cdot Dv - \lambda v \ \mathrm{d}x = \int_\Omega \left(\frac{Du}{\sqrt{1+|Du|^2}} - \lambda x\right)\cdot Dv \ \mathrm{d}x. \quad (480)$$

Because this holds for an arbitrary test function v, we see

$$Du(x) = \lambda\sqrt{1+|Du|^2}x \ \text{ in } \Omega. \quad (481)$$

This could be used to show u is radial. Also, this implies

$$|Du|^2 = (1+|Du|^2)|x|^2 \quad \Longrightarrow \quad |Du|^2 = \frac{|x|^2}{1-|x|^2} \quad \Longrightarrow \quad f(|Du|) = \left(1+\frac{|x|^2}{1-|x|^2}\right)^{1/2} = \frac{1}{\sqrt{1-|x|^2}}, \quad (482)$$

and so

$$E(u) = \int_{B(0,1)} \frac{1}{\sqrt{1-|x|^2}} \ \mathrm{d}x. \quad (483)$$

△

F18.7.

a) Consider the linear equation

$$u_t + a u_x = 0, \tag{484}$$

where $a > 0$. Solve the initial-boundary value problem for this PDE in the domain $\{x > 0\} \times \{t > 0\}$ with boundary conditions $u(x, 0) = 0$, $u(0, t) = 1$. Draw a characteristic diagram for this problem (a graph of the solution in the xt plane).

b) Consider the nonlinear equation

$$u_t + (u^3)_x = 0, \tag{485}$$

for viscous flow down an inclined plane. Solve the initial-boundary value problem for this PDE in the domain $\{x > 0\} \times \{t > 0\}$ with boundary conditions $u(x, 0) = 0$ and $u(0, t) = 1$. Here $x = 0$ corresponds to a gate that release a fluid with height $u(0, t)$. Draw a characteristics diagram for this problem.

c) Consider the same problem as in b), but now with the boundary conditions $u(x, 0) = 1$ and $u(0, t) = 0$, corresponding to a uniform flow with the gate closing at time $t = 0$. Find a solution that is continuous in the domain $\{x \geq 0\} \times \{t > 0\}$. Draw a characteristics diagram for this problem. Is the solution uniformly continuous? Explain your answer.

Solution:

a) We proceed by using the method of characteristics. Define $F(p, q, z, x, t) = q + ap$. Taking $q = u_t$, $z = u$, and $p = u_x$, we see $F = 0$ and obtain the system of characteristic ODE:

$$\begin{cases} \dot{x}(s) = F_p = a, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = t_0, \\ \dot{z}(s) = F_p p + F_q q = ap + q = 0, \quad z(0) = z^0, \end{cases} \tag{486}$$

where $z^0 = 0$ if the characteristic originates along the x axis and $z^0 = 1$ if the characteristic originates along the t axis. This implies $t = t_0 + s$, z is constant along characteristics, and

$$x(s) = x_0 + \int_0^s \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^s a\ \mathrm{d}\tau = x_0 + as = x_0 + a(t - t_0). \tag{487}$$

This shows the characteristics are linear and with identical slope, proceeding to the right in time. For a point (x,t) in the first quadrant, we take (x_0, t_0) to be the boundary point along the first quadrant which is connected to (x,t) with the line segment connecting the two having slope matching the characteristics. If $x > at$, then $t_0 = 0$ and the initial condition implies $u(x,t) = 0$. If $x < at$, then $x_0 = 0$ and the boundary condition implies $u(x,t) = 1$. In summary, in $\{x > 0\} \times \{t > 0\}$ the solution to the PDE is

$$u(x,t) = \begin{cases} 1 & \text{if } x < at, \\ 0 & \text{if } x > at. \end{cases} \tag{488}$$

The desired characteristic diagram is in Figure 15.

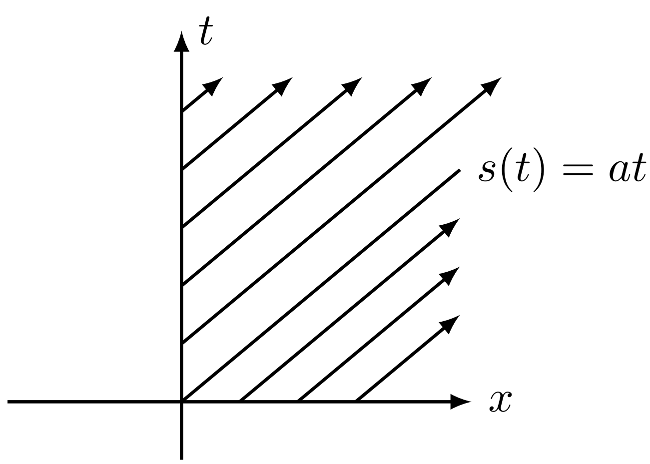

Figure 15: Characteristic Diagram for F18.7a.

b) Again we proceed by the method of characteristics. Define $F(p,q,z,x,t) = q + 3z^2p$. Taking $q = u_t$, $z = u$, and $p = u_x$ yields $F = 0$ and gives rise to the system of characteristic ODE:

$$\begin{cases} \dot{x}(s) = F_p = 3z^2, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = t_0, \\ \dot{z}(s) = F_p p + F_q q = 3z^2 p + q = 0, \quad z(0) = z^0, \end{cases} \tag{489}$$

where z^0 is as in a). This implies $t = t_0 + s$ and z is constant along characteristics, from which we deduce

$$x(s) = x_0 + \int_0^s \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^s 3z^2(\tau)\ \mathrm{d}\tau = x_0 + 3sz^0 = \begin{cases} x_0 + 3s & \text{if } x_0 = 0, \\ x_0 & \text{if } x_0 > 0. \end{cases} \tag{490}$$

Consequently, the characteristics crash and a shock occurs at $(x,t) = (0,0)$. The Rankine-Hugoniot condition implies that if the shock is parameterized by $(\tilde{x}(t), t)$ then the velocity $\sigma = \dot{\tilde{x}}$ of the shock curve satisfies

$$\sigma = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{u_\ell^3 - u_r^3}{u_\ell - u_r} = \frac{1^3 - 0}{1 - 0} = 1, \tag{491}$$

where u_ℓ and u_r are the limiting functions values approaching the shock from the left and right, respectively. With the fact $\tilde{x}(0) = 0$, this implies $\tilde{x}(t) = t$. Consequently, we deduce, in $\{x > 0\} \times \{t > 0\}$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ 0 & \text{if } x > t. \end{cases} \tag{492}$$

The characeristic diagram is given in Figure 16.

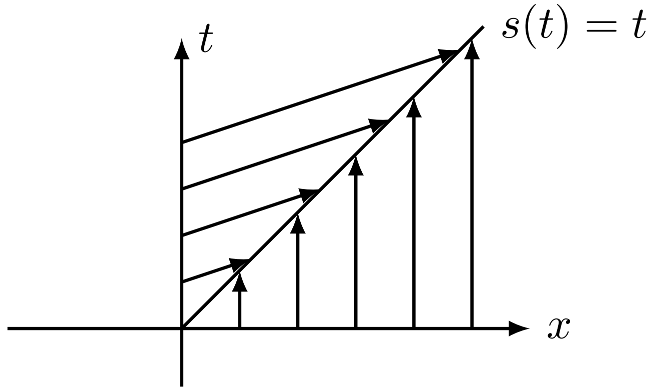

Figure 16: Characteristic Diagram for F18.7b.

c) In this case, because $u_\ell < u_r$, a rarefaction wave occurs. Note we have the same system of characteristic ODE as in b). This implies $t = t_0 + s$ and z is constant along characteristics, from which we deduce, with the new initial/boundary conditions,

$$x(s) = x_0 + \int_0^s \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^s 3z^2(\tau)\ \mathrm{d}\tau = x_0 + 3sz^0 = \begin{cases} x_0 & \text{if } x_0 = 0, \\ x_0 + 3s & \text{if } x_0 > 0. \end{cases} \tag{493}$$

This tells us the solution along the t axis and for $x > 3t$, but not in the remaining portion of the first quadrant. We look for a solution of the form $u(x,t) = v(x/t)$. Plugging this into the PDE implies (with $v = v(x/t)$)

$$0 = u_t + 3u^2 u_x = v' \cdot -\frac{x}{t^2} + 3v^2 v' \cdot \frac{1}{t} = \frac{v'}{t}\left[3v^2 - \frac{x}{t}\right] \quad \Longrightarrow \quad v\left(\frac{x}{t}\right) = \left(\frac{x}{3t}\right)^{1/2}, \tag{494}$$

where the implication holds, assuming $v' \neq 0$, and the square root is positive since $u = 1$ along $x = 3t$. Indeed, for our choice of v, this assumption holds. Therefore, we conclude, in $\{x \geq 0\} \times \{t > 0\}$,

$$u(x,t) = \begin{cases} \sqrt{x/3t} & \text{if } 0 < x \leq 3t, \\ 1 & \text{if } x > 3t. \end{cases} \tag{495}$$

The characteristic diagram is provided in Figure 17.

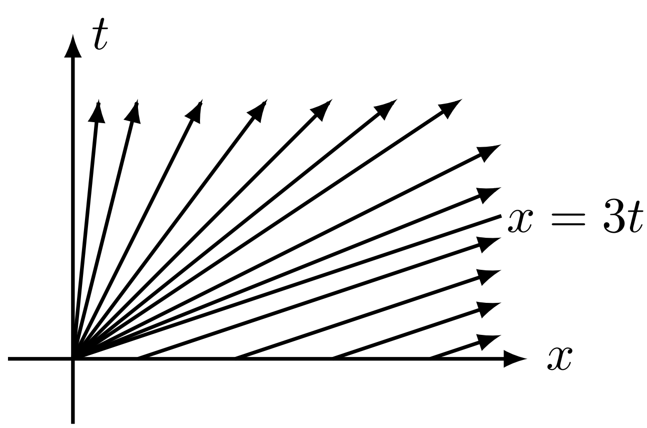

Figure 17: Characteristic Diagram for F18.7c.

Lastly, note u is continuous in the domain $\{x \geq 0\} \times \{t > 0\}$. By way of contradiction, suppose u is also uniformly continuous and let $\varepsilon = 1/2$. Then there exists $\delta > 0$ such that

$$\|(x_1, t_1) - (x_2, t_2)\| < \delta \quad \Longrightarrow \quad |u(x_1, t_1) - u(x_2, t_2) < \varepsilon = \frac{1}{2}. \tag{496}$$

However, consider the point $(x_1, t_1) = (\delta/2, \delta/6)$ at which $u = 1$ and $(x_2, t_2) = (0, \delta/6)$ at which $u = 0$. Then

$$\|(x_1, t_1) - (x_2, t_2)\| = \frac{\delta}{2} < \delta \quad \Longrightarrow \quad 1 = |1 - 0| = |u(x_1, t_1) - u(x_2, t_2)| < \frac{1}{2}, \tag{497}$$

a contradiction. Thus, u is not uniformly continuous.

□

F18.8. The equation of motion of a vibrating beam is

$$-c_m u_{tt} = EIu_{xxxx}, \tag{498}$$

where u is the displacement of the beam as a function of its position along its axis, the constant $c_m = \rho A$ is the linear mass density of the beam, E is the elastic modulus, and I is the moment of inertia. If the beam is simply supported at its ends, it satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ (no displacement at its ends) and $u_{xx}(0,t) = u_{xx}(L,t) = 0$ (zero bending moments).

a) Compute the solution of this problem, given the initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$.

b) Find the solution of the vibrating-string equation

$$u_{tt} = c^2 u_{xx} \tag{499}$$

with fixed boundary conditions $u(0,t) = u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. Compare how the spectrum of the normal modes scales with the length of the string versus the length of the beam.

Solution:

a) We proceed by using separation of variables. Assume $u(x,t) = f(x)\phi(t)$. The provided PDE implies

$$-\alpha f\phi'' = f''''\phi \quad \implies \quad \frac{\phi}{\phi''} = -\alpha\frac{f}{f''''}, \tag{500}$$

where $\alpha := c_m/EI > 0$. Since the left and right hand sides are functions of independent variables, there must exist $\mu \in \mathbb{R}$ such that

$$\mu = -\alpha\frac{f}{f''''} \quad \implies \quad -\frac{\mu}{\alpha}f'''' = f. \tag{501}$$

Then observe

$$\int_0^L ff''''\ \mathrm{d}x = \int_0^L -f'f'''\ \mathrm{d}x + \left[ff'''\right]_0^L = \int_0^L (f'')^2\ \mathrm{d}x + \underbrace{\left[ff''' - f'f''\right]_0^L}_{=0}, \tag{502}$$

and so

$$\int_0^L f^2 \, \mathrm{d}x = -\frac{\mu}{\alpha}\int_0^L f f'''' \, \mathrm{d}x = -\frac{\mu}{\alpha}\int_0^L (f'')^2 \, \mathrm{d}x. \tag{503}$$

If $f'' = 0$ in $(0, L)$, then f is linear, from which the boundary conditions imply f is identically zero. This results in obtaining the trivial solution $u \equiv 0$. Assuming u is not the trivial solution, it follows that f is nonzero and $\mu < 0$ so that $f - |\mu/\alpha| f'''' = 0$. Letting $\gamma := (-\mu/\alpha)^{1/4}$, each linearly independent solution is of the form $f = \exp(\omega\gamma x)$, where $\omega \in \{\pm 1, \pm i\}$ is a fourth root of unity. Using Euler's formula for sines and cosines and using hyperbolic sine and cosine also (in place of $\exp(\gamma x)$ and $\exp(-\gamma x)$), we may write that f satisfies

$$f_k(x) = c_1 \sin(\gamma x) + c_2 \cos(\gamma x) + c_3 \sinh(\gamma x) + c_4 \cosh(\gamma x), \tag{504}$$

for some scalars $c_1, c_2, c_3, c_4 \in \mathbb{R}$. The condition $u(0,t) = 0$ implies $c_2 + c_4 = 0$. The condition $u_{xx}(0,t) = 0$ implies $\gamma^2(-c_2 + c_4) = 0$. Thus, $c_2 = c_4 = 0$. The conditions $u(L,t) = 0$ and $u_{xx}(L,t) = 0$, respectively, imply

$$0 = c_1 \sin(\gamma L) + c_3 \sinh(\gamma L) \quad \text{and} \quad 0 = \gamma^2 \left[-c_1 \sin(\gamma L) + c_3 \sinh(\gamma L)\right]. \tag{505}$$

This implies either that $c_1 = c_3 = 0$ or that $\sin(\gamma L) = 0$ and $c_3 = 0$. The first case cannot hold since this would yield the trivial solution. Consequently, we see

$$f(x) = c_1 \sin(\gamma x), \tag{506}$$

where $\gamma = k\pi/L$ for some nonnegative integer k, which implies

$$\frac{k\pi}{L} = \gamma = \left(-\frac{\mu_k}{\alpha}\right)^{1/4} \quad \Longrightarrow \quad \mu_k = -\alpha\left(\frac{k\pi}{L}\right)^{1/4}. \tag{507}$$

Because this holds for an arbitrary k, we may expand a general function f on $[0, L]$ via an odd periodic extension of f onto $[-L, L]$ with sine functions. That is,

$$f(x) = \sum_{k=0}^{\infty} d_k f_k(x) = \sum_{k=0}^{\infty} d_k \sin\left(\frac{k\pi x}{L}\right), \tag{508}$$

where each coefficient is given by

$$d_k := \frac{\int_0^L f(x)\sin(k\pi x/L)\ \mathrm{d}x}{\int_0^L \sin(k\pi x/L)^2\ \mathrm{d}x}. \tag{509}$$

And, the associated ϕ_k satisfy

$$\frac{\phi_k}{\phi_k''} = \mu_k \quad \Longrightarrow \quad \phi_k(t) = b_{1,k}\sin(\sqrt{-\mu_k}t) + b_{2,k}\cos(\sqrt{-\mu_k}t), \tag{510}$$

for some scalars $b_{1,k}, b_{2,k} \in \mathbb{R}$. Our assumption that $u(x,0) = f(x)$ implies $\phi_k(0) = 1$, and so $b_{2,k} = 1$. And, from the condition $u_t(x,0) = g(x)$ we take

$$\beta_k = \phi_k'(0) \tag{511}$$

where

$$g(x) = \sum_{k=0}^{\infty} \beta_k f_k(x), \quad \beta_k := \frac{\int_0^L g(x)\sin(k\pi x/L)\ \mathrm{d}x}{\int_0^L \sin(k\pi x/L)^2\ \mathrm{d}x} \tag{512}$$

This can be used to determine $b_{1,k}$. Having this, we may write $\phi_k(t)$. Compiling our results, we write

$$u(x,t) = \sum_{k=0}^{\infty} d_k f_k(x)\phi_k(t). \tag{513}$$

b)

□

2018 Spring

S18.1. Consider the following non-dimensionalized model for glycolysis:

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y, \\ \dot{y} &= b - ay - x^2 y, \end{aligned} \tag{514}$$

where $x \geq 0$ is the concentration of ADP, $y \geq 0$ is the concentration of F6P, and $a, b > 0$ are kinetic parameters. Determine the equilibrium points and their linear stability, and show that a periodic orbit exists if and only if a and b satisfy an appropriate condition (which you should determine). Draw the phase portrait in this case.

Solution:

First[20] we find the nullclines. The first equation reveals $\dot{x} = 0$ on the curve $y = x/(a + x^2)$ and the second equation reveals $\dot{y} = 0$ on the curve $y = b/(a + x^2)$. These null clines are shown in the following figure. Note the direction of the flow shown is given by the sign of $\dot{x}$ and $\dot{y}$ in the different regions.

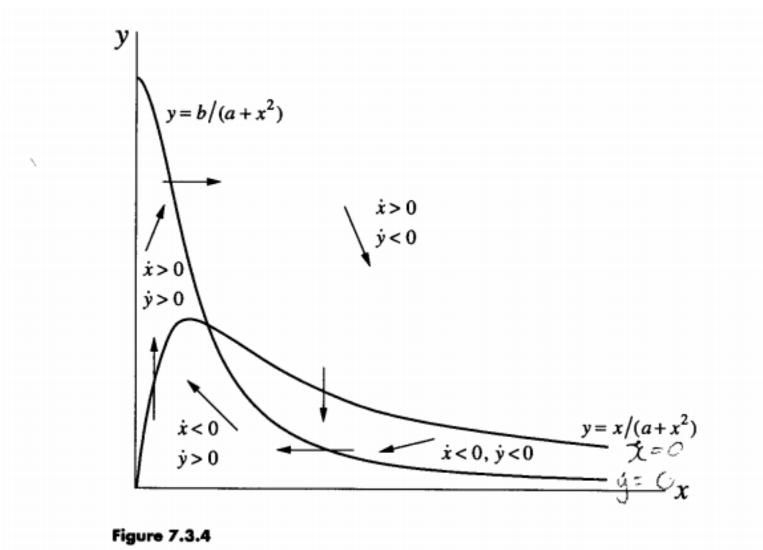

Figure 18: Snippet from Strogatz's text (p. 206)

We claim the region enclosed by $(0, b/a)$, $(b, b/a)$, straight with slope -1 to the to the null cline $y = x/(a+x)^2$,

[20]This solution follows directly from Strogatz's *Nonlinear Dynamics and Chaos* (pp. 205–208).

and then straight down to the x-axis, back to the origin, and up to $(0, b/a)$. Observe

$$\dot{x} - (-\dot{y}) = -x + ay + x^2y + (b - ay - x^2y) = b - x \quad \Longrightarrow \quad -\dot{y} > \dot{x} \text{ if } x > b. \tag{515}$$

This implies the vector fields points inward on the diagonal line, because $\mathrm{d}y/\mathrm{d}x < -1$, and therefore the vectors are steeper than the diagonal line. Thus the region is a trapping region, as claimed. Now, we must find conditions under which the fixed point in this region is a repeller. Then we may consider the trapping region that is punctured by removing this point since such a point drive all neighboring trajectories into the trapping region. Since this region is free of fixed points, we may then apply the Poincaré-Bendixson theorem.

The Jacobian $J(x, y)$ for the system is

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 2xy - 1 & a + x^2 \\ -2xy & -(a + x^2) \end{pmatrix}. \tag{516}$$

At a nonzero fixed point, adding our two equations yields

$$0 = 0 + 0 = \left(-x + ay + x^2y\right) + \left(b - ay - x^2y\right) = b - x \quad \Longrightarrow \quad x = b. \tag{517}$$

This, in turn, implies

$$0 = -b + ay + b^2y \quad \Longrightarrow \quad y = \frac{b}{a + b^2}. \tag{518}$$

Thus, the single fixed point of the system is at $(x^*, y^*) := (b, b/(a + b^2))$. Then

$$|J(x,y)| = -(2xy - 1)\left(a + x^2\right) - (-2xy)\left(a + x^2\right) = \left(a + x^2\right)(2xy - 2xy + 1) = a + x^2, \tag{519}$$

which implies $|J(x^*, y^*)| = a + b^2 > 0$. Furthermore, the trace τ of $J(x^*, y^*)$ is

$$\tau = -\frac{b^4 + (2a - 1)b^2 + (a + a^2)}{a + b^2}, \tag{520}$$

which implies

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4(a + b^2)}}{2}. \tag{521}$$

Consequently, the fixed point is unstable for $\tau > 0$ (i.e., a repeller) and stable for $\tau < 0$. We see $\tau = 0$ precisely when

$$b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right), \tag{522}$$

which defines a curve in (a, b) space yielding regions of parameters corresponding to a stable limit cycle existing or not. The Poincaré-Bendixson theorem of nonlinear dynamics tells us if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Whence a periodic orbit exists precisely when a and b are chosen such that $\tau > 0$.

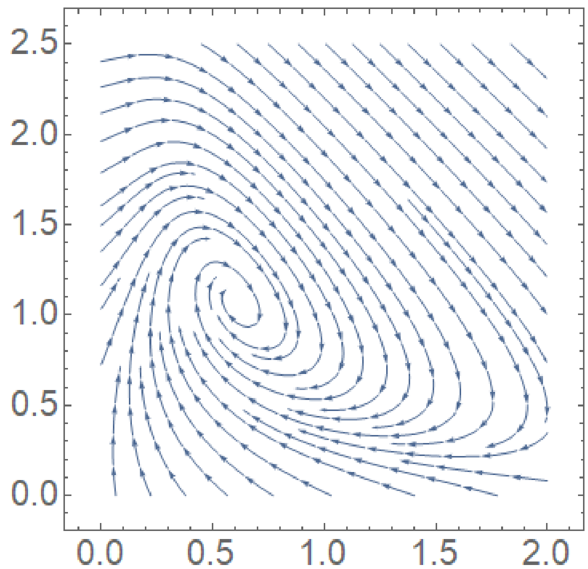

Figure 19: Plot of the phase plane with $a = 0.2$ and $b = 0.6$, which implies $\tau < 0$.

□

S18.3. Consider the ordinary differential equation

$$x^3y'' + y = 0. \tag{523}$$

a) Show that the ODE has a regular singular point at $x_0 = \infty$ and determine its indicial components.

b) The leading behavior of a particular solution to (523) is $t(x) \sim x$ as $x \to \infty$. By considering the largest terms in a singular series solution, determine the next largest term in the expansion of $y(x)$ for large positive x.

Solution:

a) To classify the point $x_0 = \infty$, we analytically map the point at infinity into the origin using the inversion transformation $x = 1/t$. We must show 0 is a regular singular point of the transformed ODE. Observe $\mathrm{d}x = -t^{-2}\mathrm{d}t$ so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -t^2\frac{\mathrm{d}y}{\mathrm{d}t} \quad \Longrightarrow \quad \frac{\mathrm{d}y^2}{\mathrm{d}x^2} = t^4\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2t^3\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{524}$$

This transforms the given ODE to

$$0 = \frac{1}{t^3}\cdot\left[t^4\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2t^3\frac{\mathrm{d}y}{\mathrm{d}t}\right] + y = t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + y. \tag{525}$$

Dividing by t on each side yields

$$0 = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{2}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y}{t} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + p_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + p_0(t)y, \tag{526}$$

where $p_1(t) = 2/t$ and $p_0(t) = 1/t$. Note p_1 and p_0 are not analytic in a neighborhood of 0; however, $(t-0)p_1(t) = 1$ and $(t-0)^2p_0(t) = t$ are analytic in a neighborhood of 0. Thus 0 is a regular singular point of (526). Therefore $x_0 = \infty$ is a regular singular point of (523).

Since $t = 0$ is a regular singular point, Fuch's theorem asserts there exists a Frobenius series solution y, i.e., y is of the form

$$y = \sum_{k=0}^{\infty} c_k t^{k+\alpha} = \sum_{k=0}^{\infty} c_k x^{-(k+\alpha)}, \tag{527}$$

where α is chosen such that $c_0 \neq 0$. This implies

$$\begin{aligned} 0 &= y + 2\frac{\mathrm{d}y}{\mathrm{d}t} + t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \\ &= \sum_{k=0}^{\infty} c_k\left[t^{k+\alpha} + 2(k+\alpha)t^{k+\alpha-1} + (k+\alpha)(k+\alpha-1)t^{k+\alpha-1}\right] \\ &= \sum_{k=0}^{\infty} c_k t^{k+\alpha} + c_0\alpha(\alpha+1)t^{\alpha-1} + \sum_{n=0}^{\infty} c_{n+1}\left[2(n+1+\alpha) + (n+1+\alpha)(n+\alpha)\right]t^{n+\alpha} \\ &= c_0\alpha(\alpha+1)t^{\alpha-1} + \sum_{k=0}^{\infty}\left[c_k + (k+1+\alpha)(k+2+\alpha)c_{k+1}\right]t^{k+\alpha}, \end{aligned} \tag{528}$$

where the equalities come from reindexing the series with $n = k-1$. Equating coefficients reveals

$$\alpha(\alpha+1) \quad \text{and} \quad 0 = c_k + (k+1+\alpha)(k+2+\alpha)c_{k+1}, \quad \text{for all } k \geq 0, \tag{529}$$

where we note we assumed $c_0 \neq 0$. This implies the indicial components[21] are $\alpha_1 = 0$ and $\alpha_2 = -1$.

b) Observe[22] our hypothesis implies $y(x) \sim x + \delta(x)$ as $x \to \infty$, where $|\delta| \ll x$ as $x \to \infty$. Consequently,

$$x + \delta \sim y = -x^3y'' = -x^3\delta'' \quad \Longrightarrow \quad x \sim -x^3\delta'' \quad \text{as } x \to \infty. \tag{530}$$

This implies

$$\delta'' \sim -x^{-2} \quad \Longrightarrow \quad \delta \sim \ln(x) + c_1x + c_2 \quad \text{as } x \to \infty. \tag{531}$$

Because $|\delta| \ll x$ as $x \to \infty$, it follows that $c_1 = 0$. Note constant functions are solutions to the ODE, and so we may subtract such solutions from our particular solution and still obtain a solution, i.e., we may take $c_2 = 0$. All that remains is to verify taking $\delta(x) = \ln(x)$ forms a consistent dominant balance. Indeed,

$$\lim_{x\to\infty}\frac{x}{\ln(x)} = \lim_{x\to\infty}\frac{1}{1/x} = \lim_{x\to\infty} x = \infty. \quad \Longrightarrow \quad |\ln(x)| \ll x \quad \text{as } x \to \infty. \tag{532}$$

From this, we conclude the next largest term is $\delta(x) = \ln(x)$.

□

[21] We presume the prompt meant with the phrase "indicial components" what Bender and Orzsag meant by "indicial exponents". See, e.g., §3.3 of their text.

[22] This question is strikingly similar to Problem 3.22b in Bender and Orszag's text.

Remark: We wish to draw further insights regarding the previous problem. A solution to (523) given by

$$y_1 = \sum_{k=0}^{\infty} c_k t^k, \tag{533}$$

where we take $\alpha = \alpha_1 = 0$ in the recurrence relation in (529). However, note $N = \alpha_1 - \alpha_2 = 1$ and using the recurrence relation in (529) with $\alpha = \alpha_2$ reveals

$$0 = 0c_1 = \left[\underbrace{(N + 1 + \alpha_2)}_{=0}(N + 2 + \alpha_2)c_N \right]_{N=1} = -c_0, \tag{534}$$

which cannot be the case for a Frobenius series since we assume $c_0 \neq 0$. Thus, there is only one solution to the ODE in Frobenius form. Following Bender and Orszag's text, we see the second solution is of the form

$$\begin{aligned} y_2 &= \sum_{k=0}^{\infty} d_k t^{k+\alpha_2} - \frac{\partial}{\partial \alpha} \left[y(x,\alpha) \right]_{\alpha=\alpha_1} \\ &= \sum_{k=0}^{\infty} d_k t^{k+\alpha_2} - y(t,\alpha_1)\ln(t) - \sum_{k=0}^{\infty} b_k t^{t+\alpha_1} \\ &= y_1(t)\ln(t) + \sum_{k=0}^{\infty} q_k t^{k-1}, \end{aligned} \tag{535}$$

where

$$y(t,\alpha) = \sum_{k=0}^{\infty} c_k(\alpha) t^{k+\alpha} \quad \text{and} \quad b_k = \frac{\partial}{\partial \alpha} \left[c_k(\alpha) \right]_{\alpha=\alpha_1}, \tag{536}$$

and

$$\sum_{k=0}^{\infty} q_k t^{k-1} = \sum_{k=0}^{\infty} d_k t^{k+\alpha_2} - \sum_{k=0}^{\infty} b_k t^{t+\alpha_1}. \tag{537}$$

We may take $c_0 = d_0 = 1$. Thus, the general solution is of the form

$$y(t) = \beta_1 y_1(t) + \beta_2 y_2(t), \tag{538}$$

for scalars $\beta_1, \beta_2 \in \mathbb{R}$. Assume $y \sim 1/t$ as $t \to 0^+$ and observe, as $t \to 0^+$,

$$\begin{aligned} t^{-1} \sim y &\sim \beta_1 \left[c_0 + c_1 t\right] + \beta_2 \left[(c_0 + c_1 t)\ln(t) + \left(q_0 t^{-1} + q_1\right)\right] \\ &= \beta_2 q_0 t^{-1} + \beta_2 c_0 \ln(t) + (\beta_1 c_0 + \beta_1 c_1 + \beta_2 c_1 t \ln(t) + \beta_2 q_1) \\ &\sim \beta_2 q_0 t^{-1} + \beta_2 c_0 \ln(t) \\ &= \beta_2 \left(t^{-1} + \ln(t)\right), \end{aligned} \tag{539}$$

where we may ignore the omitted terms (asymptotically) since our assumption implies $\beta_2 = 1$. Consequently, we see

$$y \sim t^{-1} + \ln(t) \quad \text{as } t \to 0^+ \qquad \Longrightarrow \qquad y \sim x + \ln(x^{-1}) = x - \ln(x) \quad \text{as } x \to \infty. \tag{540}$$

This proves the next largest term in the expansion of $y(x)$ for large positive x is $-\ln(x)$. In fact, upon consulting (539), we see the succeeding largest term in the expansion will be a constant since

$$\beta_2 c_1 t \ln(t) = 1 \cdot -\frac{c_0}{2} t \ln(t) = -\frac{1}{2} t \ln(t) = t \ln(t^{-1/2}) \to 0 \ \text{ as } t \to 0^+. \tag{541}$$

△

S18.4. We seek a solution $u : \Omega \times [0, \infty) \to \mathbb{R}$ of the PDE

$$\begin{cases} u_t - \Delta u + u\|Du\| = 0 & \text{in } \Omega \times (0,\infty), \\ u = f & \text{on } \partial\Omega \times [0,\infty), \\ u = g & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{542}$$

where $\Omega \subset \mathbb{R}^d$ is the interior of a connected compact set, and $\|\cdot\|$ is the usual Euclidean norm. Assume the boundary and initial conditions are smooth. Show that there is at most one $C^{1,2}(\Omega \times \mathbb{R})$ solution of this PDE.[23]

Solution:

Let u and v be $C^{1,2}(\Omega \times (0, \infty))$ solutions of the given PDE, and set $w := u - v$. Fix a time $T > 0$. Then take $\Omega_T := \Omega \times (0, T]$ and let Γ_T be the parabolic boundary of Ω_T so that

$$\begin{cases} w_t - \Delta w = v\|Dv\| - u\|Du\| & \text{in } \Omega_T, \\ w = 0 & \text{on } \Gamma_T. \end{cases} \tag{543}$$

Now fix $\varepsilon > 0$ and define $\tilde{w} := w + \varepsilon e^t$. Since $\overline{\Omega}_T$ is compact, the continuous function $\tilde{w}$ attains its infimum over $\overline{\Omega}_T$. By way of contradiction, suppose the infimum of $\tilde{w}$ is nonpositive. By the continuity of $\tilde{w}$, it follows that there exists a first time $t^\star$ and and a point $x^\star \in \overline{\Omega}$ such that $\tilde{w}(x^\star, t^\star) = 0$. Note $t^\star > 0$ and $x^\star \in \Omega$ since

$$\tilde{w} = w + \varepsilon e^t \geq w + \varepsilon e^t = \varepsilon e^t \geq \varepsilon > 0 \quad \text{on } \Gamma_T. \tag{544}$$

Thus, $(x^\star, t^\star) \in \Omega_T$. Consequently, $w_t(x^\star, t^\star) \geq 0$ and, because $x^\star$ is a local minimizer of $w(\cdot, t^\star)$, $\Delta w(x^\star, t^\star) \geq 0$ and, at $(x^\star, t^\star)$,

$$0 = D\tilde{w} = Dw = Du - Dv \quad \Longrightarrow \quad Du = Dv. \tag{545}$$

Combining these facts, we see, at $(x^\star, t^\star)$,

$$0 \geq \tilde{w}_t - \Delta\tilde{w} = w_t - \Delta w + \varepsilon e^{t^\star} = v\|Dv\| - u\|Du\| + \varepsilon e^{t^\star} = -w\|Du\| + \varepsilon e^{t^\star} = (\|Du\| + 1)\, \varepsilon e^{t^\star} > 0, \tag{546}$$

a contradiction. The final equality holds since, at the given point, $0 = \tilde{w} = w + \varepsilon e^{t^\star}$. The contradiction

[23] *We presume the authors meant to only include nonnegative times.*

proves $\tilde{w} > 0$ in $\overline{\Omega}_T$. Therefore,

$$w \geq -\varepsilon e^t \geq -\varepsilon e^T \quad \text{in } \overline{\Omega}_T. \tag{547}$$

Since $\varepsilon > 0$ was arbitrary, we may let $\varepsilon \to 0^+$ to deduce $w \geq 0$ in $\overline{\Omega}_T$. Because this holds for arbitrary $T > 0$, we may then let $T \to \infty$ to deduce

$$w \geq 0 \quad \Longrightarrow \quad u \geq v \quad \text{in } \overline{\Omega} \times [0, \infty). \tag{548}$$

We may repeat at analogous argument, swapping the roles of u and v to deduce $u \leq v$, from which it follows that $u = v$. Whence any solution to the PDE is necessarily unique. □

S18.5. Consider entropy solutions $u(x,t) : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ of the flux conservation equation

$$u_t + f(u)_x = 0 \tag{549}$$

with initial condition

$$u(x,0) = \begin{cases} x & \text{if } x \in (0,1), \\ 0 & \text{otherwise,} \end{cases} \tag{550}$$

and with flux function $f(u) = u^3/3$.

a) Derive the Rankine-Hugenoit condition for the propagation of discontinuous solutions of this PDE.

b) Find the long time solution of the PDE. You may assume $u \geq 0$, so $f(u)$ is convex, and also that at long times, the solution can be broken into three parts:

$$u(x,t) = \begin{cases} t^\alpha g(x/t^\beta) & \text{if } 0 < x < h(t), \\ 0 & \text{otherwise,} \end{cases} \tag{551}$$

for some exponents α and β and positive functions g and h, all of which you should determine.

Solution:

a) First suppose u is a smooth solution to the PDE and $v : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ is a test function, i.e., smooth with compact support. Then integrating the PDE yields

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty}\int_0^{\infty} [u_t + f(u)_x]v \ \mathrm{d}x\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t - \int_{-\infty}^{\infty} [uv]_{t=0} \ \mathrm{d}x, \end{aligned} \tag{552}$$

where we have used integration by parts with the compact support of v to see all the nonlisted boundary terms vanish. The right hand side of (552) makes sense even when u is not smooth. Consequently, we say u is an integral solution to the PDE when $u \in L^\infty(\mathbb{R} \times (0,\infty))$ satisfies

$$0 = \int_{-\infty}^{\infty}\int_0^{\infty} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t - \int_{-\infty}^{\infty} [uv]_{t=0} \ \mathrm{d}x, \tag{553}$$

for all test functions v.

Now suppose u is an integral solution of the PDE, and let C be a curve of discontinuity in u. We seek to derive a condition for the propagation of this discontinuity. Let $V \subset \mathbb{R} \times (0, \infty)$ be a bounded open subset that intersects with C and with V_ℓ and V_r the portions of V to the left and right of C, respectively. We assume u is smooth to the left and right of C, but not along C. If v is a test function with compact support in V_ℓ, then we see

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \int_0^{\infty} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t - \int_{-\infty}^{\infty} [uv]_{t=0} \ \mathrm{d}x \\ &= \iint_{V_\ell} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t \\ &= \iint_{V_\ell} (u_t + f(u)_x)\, v \ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{554}$$

By the arbitrariness of the test function v, it follows that

$$u_t + f(u)_x = 0 \ \text{ in } V_\ell. \tag{555}$$

Likewise, we deduce

$$u_t + f(u)_x = 0 \ \text{ in } V_r. \tag{556}$$

Now consider a test function v with compact support in V that does not necessarily vanish along C. Then

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \int_0^{\infty} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t - \int_{-\infty}^{\infty} [uv]_{t=0} \ \mathrm{d}x \\ &= \iint_{V_\ell} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t + \iint_{V_r} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{557}$$

However, integration by parts reveals

$$\iint_{V_\ell} -uv_t - f(u)v_x \ \mathrm{d}x\mathrm{d}t = \underbrace{\iint_{V_\ell} (u_t + f(u)_x)\, v \ \mathrm{d}x\mathrm{d}t}_{=0} + \int_C \left[-f(u_\ell)\nu^1 - u_\ell \nu^2 \right] v \ \mathrm{d}\ell, \tag{558}$$

where the underbraced term equals zero by (555), $\nu = (\nu^1, \nu^2)$ is the outward normal along ∂V_ℓ, pointing from V_ℓ into V_r, and u_ℓ is the limiting value of u approaching C from the left. Likewise,

$$\iint_{V_r} -uv_t - f(u)v_x \, \mathrm{d}x\mathrm{d}t = \underbrace{\iint_{V_r} (u_t + f(u)_x)\, v \, \mathrm{d}x\mathrm{d}t}_{=0} - \int_C \left[-f(u_r)\nu^1 - u_r\nu^2\right] v \, \mathrm{d}\ell, \tag{559}$$

where a minus since is used since the outward normal along ∂V_r is $-\nu$ and u_r is the limiting value of u approaching C from the right. Combining our results in (557), (558), and (559), we see

$$0 = \int_C \left((f(u_r) - f(u_\ell))\nu^1 + (u_r - u_\ell)\nu^2\right) v \, \mathrm{d}\ell. \tag{560}$$

Again by the arbitrariness of the test function v, this shows

$$0 = (f(u_r) - f(u_\ell))\nu^1 + (u_r - u_\ell)\nu^2 \ \text{ along } C. \tag{561}$$

Assuming C is sufficiently smooth, we can let $s(t)$ be the parameterization of x along C so that $(x,t) = (s(t),t)$ along C. Then $\nu = \frac{1}{\sqrt{1+\dot{s}^2}}(-1,\dot{s})$, which implies

$$0 = (f(u_r) - f(u_\ell))(-1) + (u_r - u_\ell)(\dot{s}) \ \text{ along } C, \tag{562}$$

and so

$$\dot{s}(u_\ell - u_r) = f(u_\ell) - f(u_r) \ \text{ along } C. \tag{563}$$

The condition (563) is the Rankine-Hugenoit condition for the propagation of discontinuities.

b) We proceed via the method of characteristics. Let $F(p,q,z,x,t) = q + pz^2$. Taking $p = u_x$, $q = u_t$, and $z = u$ gives $F = 0$ and rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z^2, \quad x(0) = x^0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = z^2 p + 1q = 0, \quad z(0) = q(x^0), \end{cases} \tag{564}$$

where $q(x) := u(x,0)$. This implies $t = s$ and z is constant along characteristics. Observe

$$x = x^0 + tz^2(0) = x^0 + tq^2(x^0) = \begin{cases} x^0 + t(x^0)^2 & \text{if } x^0 \in (0,1), \\ x^0 & \text{otherwise.} \end{cases} \tag{565}$$

Consequently, the characteristics are straight lines. For $x^0 \notin (0,1)$ we see they are vertical lines, and the characteristics immediately crash at $x^0 = 1$. Let $(s(t),t)$ give the parameterization of the resulting discontinuity. Then $s(0) = 1$ and applying the Rankine-Hugenoit condition reveals

$$\dot{s} = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{3}(1)^3 - \frac{1}{3}(0)^2}{1-0} = \frac{1}{3}. \tag{566}$$

Thus $s(t) = 1 + \dfrac{t}{3}$. By way of contradiction, suppose there is a time $t > 0$ at which another shock curve occurs. At such time, we have

$$x^0(1 + tx^0) = x = s(t) = 1 + \frac{t}{3} \quad \Longrightarrow \quad t = \frac{1 - x^0}{(x^0)^2 - 1/3}. \tag{567}$$

For all $(x,t) \in \mathbb{R} \times (0,1)$,

$$u(x,t) = \begin{cases} ?? & \text{if } 0 < x < 1 + \frac{1}{3}, \\ 0 & \text{otherwise} \end{cases} \tag{568}$$

□

S18.7. let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary $\partial\Omega$. Recall the notation $g \in C^1(\overline{\Omega})$ means there exists an open set O containing $\overline{\Omega}$ such that $g \in C^1(O)$. Let $f_1, \dots, f_d \in C^1(\overline{\Omega})$ be such that

$$\sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i} = 0 \ \text{ in } \Omega. \tag{569}$$

Suppose $u \in C^2(\overline{\Omega})$ and

$$\begin{cases} \Delta u + \sum_{i=1}^d f_i u_{x_i} - u^3 - u^5 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{570}$$

Show that u is identically zero in Ω.

Solution:

Set $q := (f_1, f_2, \dots, f_d)$ so that $q : \Omega \to \mathbb{R}^d$. Then $\nabla \cdot q = 0$ in Ω and

$$\begin{cases} \Delta u + q \cdot Du - u^3 - u^5 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{571}$$

Using integration by parts and the fact $\nabla \cdot q = 0$ in Ω, we see

$$\int_\Omega (\nabla \cdot (qu))\, u \ \mathrm{d}x = \int_\Omega (q \cdot Du) u \ \mathrm{d}x = \int_\Omega (uq) \cdot Du \ \mathrm{d}x = -\int_\Omega (\nabla \cdot (uq))\, u \ \mathrm{d}x + \int_{\partial\Omega} u^2 q \cdot \nu \ \mathrm{d}\sigma. \tag{572}$$

Since $u = 0$ on $\partial\Omega$, the integral on the boundary vanishes, which implies

$$\int_\Omega (\nabla \cdot (qu))\, u \ \mathrm{d}x = -\int_\Omega (\nabla \cdot (uq))\, u \ \mathrm{d}x \quad \Longrightarrow \quad \int_\Omega (qu) \cdot Du \ \mathrm{d}x = 0, \tag{573}$$

and so

$$0 \leq \int_\Omega |Du|^2 \ \mathrm{d}x = -\int_\Omega u \Delta u \ \mathrm{d}x + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \mathrm{d}\sigma = -\int_\Omega u \left(u^5 + u^3 - q \cdot Du\right) \ \mathrm{d}x, \tag{574}$$

where the final equality holds by using the PDE and the fact $u = 0$ on $\partial\Omega$. This implies

$$0 \leq -\int_\Omega u \left(u^5 + u^3 - q \cdot Du\right) \ \mathrm{d}x = \int_\Omega -u^6 - u^4 + (qu) \cdot Du \ \mathrm{d}x \leq \int_\Omega (qu) \cdot Du \ \mathrm{d}x = 0. \tag{575}$$

Therefore $Du = 0$ in Ω, which implies u is constant in each connected subset of Ω. Because $u = 0$ on $\partial\Omega$, it then follows that $u = 0$ in Ω. This completes the proof. $\square$

S18.8. Let $\Phi \in C^3(\mathbb{R}^d)$ be such that Φ and its first derivatives are bounded. We consider the Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$L(x,v) := \frac{1}{2}|v|^2 - \Phi(x). \tag{576}$$

Given $T \in (0,\infty)$ and $x, y \in \mathbb{R}^d$, we define the minimal action

$$C(x,y) := \inf_{\sigma:[0,T]\to\mathbb{R}^d} \left\{ \int_0^T L(\sigma(\tau), \dot{\sigma}(\tau))\ \mathrm{d}\tau : \sigma \in C^1([0,T]),\ \sigma(0) = x,\ \sigma(T) = y \right\}. \tag{577}$$

a) Show that if $\Phi \equiv 0$, then $\sigma_0(\tau) = (1 - \tau/T)x + (\tau/T)y$ is the unique path whose action is $C(x,y)$.

b) Show that if Φ is concave, then $C(x,y)$ has at most one minimizer.

Solution:

a) Fix $x, y \in \mathbb{R}^d$ and define the admissibility class

$$\mathcal{A} := \left\{ \sigma : [0,T] \to \mathbb{R} : \sigma \in C^1,\ \sigma(0) = x, \sigma(T) = y \right\}. \tag{578}$$

We claim $\mathcal{A}$ is convex. Indeed, if $u, v \in \mathcal{A}$ and if $\lambda \in (0,1)$, then $\lambda u + (1-\lambda)v \in \mathcal{A}$ since this function is in C^1 and

$$(\lambda u + (1-\lambda)v)(0) = \lambda x + (1-\lambda)x = x \quad \text{and} \quad \lambda u + (1-\lambda)v(T) = \lambda y + (1-\lambda)y = y. \tag{579}$$

Define the mapping $J : \mathcal{A} \to \mathbb{R}$ by

$$J(u) := \int_0^T L(u(\tau), \dot{u}(\tau))\ \mathrm{d}\tau. \tag{580}$$

We must show σ_0 is the unique minimizer of J over $\mathcal{A}$. By our work in b) below, we know since $\Phi \equiv 0$ is convex, J is strictly convex. This implies the only extremizer of J is its minimizer. So, it suffices to show σ_0 is the unique extremizer of J over $\mathcal{A}$. Suppose u is an extremizer of J over $\mathcal{A}$ and

$v : [0, T] \to \mathbb{R}$ is C^1 with $v(0) = v(T) = 0$. Then $u + \varepsilon v \in \mathcal{A}$ for all $\varepsilon \in \mathbb{R}$. For $\varepsilon \neq 0$ observe

$$\begin{aligned} \frac{J(u+\varepsilon v) - J(u)}{\varepsilon} &= \frac{1}{\varepsilon}\int_0^T L(u+\varepsilon v, \dot{u}+\varepsilon\dot{v}) - L(u,\dot{u})\ \mathrm{d}\tau \\ &= \frac{1}{\varepsilon}\int_0^T \frac{1}{2}|\dot{u}+\varepsilon\dot{v}|^2 - \Phi(u+\varepsilon v) - \frac{1}{2}|\dot{u}|^2 + \Phi(u)\ \mathrm{d}\tau \\ &= \int_0^T \dot{u}\cdot\dot{v} + \varepsilon|\dot{v}|^2 - \frac{\Phi(u+\varepsilon v)-\Phi(u)}{\varepsilon}\ \mathrm{d}\tau. \end{aligned} \tag{581}$$

Since $\dot{u}, \dot{v} \in C^1([0,T])$ and $[0,T]$ is compact, we know $\|\dot{u}\|, \|\dot{v}\| \in L^\infty([0,T])$. For each $t \in [0,T]$, let Γ_t be the line segment connecting $u(t)$ and $(u+\varepsilon v)(t)$. Then by the fundamental theorem of line integrals

$$\left|\frac{\Phi(u+\varepsilon v)-\Phi(u)}{\varepsilon}\right| = \left|\frac{1}{\varepsilon}\int_{\Gamma_t} D\Phi\cdot \mathrm{d}\vec{\ell}\right| \leq \frac{1}{\varepsilon}\int_{\Gamma_t}\|D\Phi\|_\infty\ \mathrm{d}\ell = \frac{1}{\varepsilon}\cdot\varepsilon\|v\|_\infty\|D\Phi\|_\infty = \|v\|_\infty\|D\Phi\|_\infty, \tag{582}$$

where we note $|\Gamma_t| = \varepsilon\|v\|_\infty$ and use the fact the first derivatives of Φ are bounded to write $D\Phi \in L^\infty(\mathbb{R}^d)$. This shows for each $t \in [0,T]$ and $|\varepsilon| \leq 1$

$$\left|\dot{u}\cdot\dot{v} + \varepsilon|\dot{v}|^2 - \frac{\Phi(u+\varepsilon v)-\Phi(u)}{\varepsilon}\right| \leq \|\dot{u}\|_\infty\|\dot{v}\|_\infty + \|\dot{v}\|_\infty^2 + \|v\|_\infty\|D\Phi\|_\infty < \infty, \tag{583}$$

which is integrable on $[0,T]$. This implies we may use the dominated convergence theorem to pull the limit as $\varepsilon \longrightarrow 0^+$ inside the integral, i.e., the Gâteaux derivative is

$$\begin{aligned} \lim_{\varepsilon\to 0^+}\frac{J(u+\varepsilon v)-J(u)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\int_0^T \dot{u}\cdot\dot{v} + \varepsilon|\dot{v}|^2 - \frac{\Phi(u+\varepsilon v)-\Phi(u)}{\varepsilon}\ \mathrm{d}\tau \\ &= \int_0^T \lim_{\varepsilon\to 0^+}\left[\dot{u}\cdot\dot{v} + \varepsilon|\dot{v}|^2 - \frac{\Phi(u+\varepsilon v)-\Phi(u)}{\varepsilon}\right]\ \mathrm{d}\tau \\ &= \int_0^T \dot{u}\cdot\dot{v} - D\Phi\cdot v\ \mathrm{d}\tau \\ &= \int_0^T [-\ddot{u} - D\Phi]\cdot v\ \mathrm{d}\tau. \end{aligned} \tag{584}$$

The final equality holds by integration by parts, recalling $v(0) = v(T) = 0$. Because this result holds for arbitrary v, it follows that for any extremizer u of J over $\mathcal{A}$

$$\ddot{u} = -D\Phi \quad \text{in } [0,T]. \tag{585}$$

Assuming $\Phi \equiv 0$, we see $u(t) = c_1 t + c_2$ for scalars $c_1, c_2 \in \mathbb{R}^d$. Applying the boundary conditions to solve for c_1 and c_2, we immediately deduce $u \equiv \sigma_0$ is the unique minimizer of J over $\mathcal{A}$.

b) We claim $J : \mathcal{A} \to \mathbb{R}$ is strictly convex. By way of contradiction, suppose there exists distinct minimizers u and v of J over $\mathcal{A}$. Then note $\frac{1}{2}(u+v) \in \mathcal{A}$ since $\mathcal{A}$ is convex and, by the strict convexity of J,

$$J\left(\frac{u+v}{2}\right) < \frac{1}{2}J(u) + \frac{1}{2}J(v) = J(u), \tag{586}$$

which contradicts the fact u is a minimizer of J over $\mathcal{A}$. This contradiction proves that if a minimizer of J over $\mathcal{A}$ exists, then it is unique.

All that remains is to verify J is strictly convex. First note for $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ and $i \in \{1,2,\ldots,n\}$

$$L_{v_i}(x,v) = v_i \quad \Longrightarrow \quad L_{v_i v_i}(x,v) = 1 \quad \Longrightarrow \quad \Delta_v L = n > 0, \tag{587}$$

and so L is strictly convex in v. Since Φ is concave in x, $-\Phi$ is convex in x. Because L is the sum of a convex function and a strictly convex function, we deduce L is strictly convex. Consequently,

$$\begin{aligned} J(\lambda u + (1-\lambda)v) &= \int_0^T L(\lambda u + (1-\lambda)v, \lambda\dot{u} + (1-\lambda)\dot{v}) \ \mathrm{d}\tau \\ &= \int_0^T |\lambda\dot{u} + (1-\lambda)\dot{v}|^2 - \Phi(\lambda u + (1-\lambda)v) \ \mathrm{d}\tau \\ &< \int_0^T \lambda|\dot{u}|^2 + (1-\lambda)|\dot{v}|^2 - \lambda\Phi(u) - (1-\lambda)\Phi(v) \ \mathrm{d}\tau \\ &= \lambda J(u) + (1-\lambda)J(v). \end{aligned} \tag{588}$$

The third line follows by the strict convexity of the integrands, making the integral in the third line larger than that in the second. Therefore J is strictly convex, and we are done.

□

2017 Fall

F17.1. Consider the differential equation

$$\ddot{x} + x^n \dot{x} + x = 0, \tag{589}$$

where n is a nonnegative integer.

a) If n is even, show that the equilibrium $(x, \dot{x}) = (0, 0)$ is asymptotically stable.

b) If $n = 1$, what can you say about the stability of $(x, \dot{x}) = (0, 0)$?

Solution:

a) First set $y = \dot{x}$ to obtain the autonomous ODE system

$$\dot{x} = y, \quad \dot{y} = -x^n y - x. \tag{590}$$

We then see $(0, 0)$ is an equilibrium point of the ODE system. Multiplying by y yields

$$y\dot{y} = -xy\left(x^{n-1}y + 1\right) = -x^n y^2 - x\dot{x} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x^2}{2} + \frac{y^2}{2}\right] = x\dot{x} + y\dot{y} = -x^n y^2 \leq 0, \tag{591}$$

where the final inequality holds since n is even. Thus, the Lyapunov function $V(x, y) := (x^2 + y^2)/2$ satisfies

$$\dot{V}(x, y) = -x^n y^2 \leq 0. \tag{592}$$

Moreover,

$$DV = \begin{pmatrix} x \\ y \end{pmatrix} \quad \Longrightarrow \quad D^2 V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{593}$$

which is positive definite. Because $DV(0, 0) = 0$ and $D^2V(0, 0)$ is positive definite, $V(x, y) > V(0, 0)$ in a neighborhood of $(0, 0)$. And, (592) reveals the only fixed point (x, y) of the ODE system for which $\dot{V}(x, y) = 0$ is $(x, y) = (0, 0)$. By Lasalle's theorem, we conclude $(0, 0)$ is asymptotically stable.

b) Observe

$$\frac{\dot{y}}{\dot{x}} = -\frac{x(y+1)}{y} \quad \Longrightarrow \quad -x \; \mathrm{d}x = \frac{y\mathrm{d}y}{1+y} = \frac{u-1}{u} \; \mathrm{d}u = \left(1 - \frac{1}{u}\right) \; \mathrm{d}u, \tag{594}$$

where we set $u := y + 1$ and note $du = dy$. This implies

$$0 = \frac{x^2}{2} + u - \ln(u) + C - 1, = \frac{x^2}{2} + y - \ln(y+1) + C \tag{595}$$

for some constant $C \in \mathbb{R}$. From this, define the Lyapunov function

$$F(x, y) := \frac{x^2}{2} + y - \ln(y+1), \tag{596}$$

and set $f(y) := \ln(1 + y)$. By Taylor's theorem, for each $y \in (-1, 1)$ there exists ξ_y between 0 and y such that

$$f(y) = f(0) + f'(0)(y-0) + \frac{f''(\xi_y)}{2}(y-0)^2 = 0 + 1(y-0) - \frac{y^2}{(1+\xi_y)^2}, \tag{597}$$

which implies

$$y - \ln(y+1) = y - f(y) = \frac{1}{2}\left(\frac{y}{1+\xi_y}\right)^2. \tag{598}$$

Therefore, for all $(x, y) \in \mathbb{R} \times (-1, 1)$ such that $(x, y) \neq (0, 0)$ we see

$$F(x, y) = \frac{x^2}{2} + \frac{1}{2}\left(\frac{y}{1+\xi_y}\right)^2 > 0, \tag{599}$$

and $F(0, 0) = 0$. Moreover,

$$\dot{F}(x, y) = x\dot{x} + \left(1 - \frac{1}{y+1}\right)\dot{y} = xy + \left(\frac{y}{y+1}\right) \cdot -x(y+1) = 0. \tag{600}$$

Hence Lyapunov's theorem asserts $(0, 0)$ is stable (n.b. $(0, 0)$ is *not* asymptotically stable).

□

F17.2. A chemical diffuses freely in 1D, satisfying the following PDE:

$$c_t = c_{xx} + \delta\Theta, \tag{601}$$

where $\Theta(t)$ is the Heaviside function and $\delta = \delta(x)$. Construct a similarity solution of the partial differential equation for $c(x,t)$. You may assume $c(x,0) = 0$ for $x \neq 0$ and $\lim_{|x|\to\infty} c(x,t) = 0$.

Solution:

We seek a solution[24] of the form $c(x,t) = t^\alpha v(xt^{-\beta})$, where α and β are to be determined. Throughout this work, we take $\eta = xt^{-\beta}$ and often omit writing the argument of v. Observe

$$c_t = \alpha t^{\alpha-1} v(\eta) - \beta t^{\alpha-\beta-1} x v'(\eta) = t^{\alpha-1}\left[\alpha v - \beta\eta v'\right] \quad \text{and} \quad c_{xx} = \partial_x\left[t^{\alpha-\beta} v'(\eta)\right] = t^{\alpha-2\beta} v''. \tag{602}$$

Thus,

$$t^{\alpha-1}\left[\alpha v - \beta\eta v\right] = t^{\alpha-2\beta} v'', \quad \text{for all } x \neq 0, \tag{603}$$

which implies, equating powers of t, that $\alpha - 1 = \alpha - 2\beta$, and so $\beta = -1/2$. The prompt states this PDE represents chemical diffusion. So, we may define $\phi : [0,\infty) \to \mathbb{R}$ by

$$\phi(t) := \int_{\mathbb{R}} c(x,t)\ \mathrm{d}x, \tag{604}$$

where, on physical grounds, we know $\phi(0)$ is well-defined and gives the initial amount of the chemical (note $c(x,0)$ forms some multiple of the Dirac δ). And, $\phi(t)$ is well-defined for all times since a finite amount of concentration is added per unit amount of time, as is illustrated by the fact

$$\dot{\phi} = \int_{\mathbb{R}} c_t\ \mathrm{d}x = \int_{\mathbb{R}} c_{xx} + \delta\Theta\ \mathrm{d}x = \int_{\mathbb{R}} \delta\Theta\ \mathrm{d}x + \underbrace{[c_x]_{x=-\infty}^{\infty}}_{=0} = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{605}$$

However,

$$\phi(t) = \int_{\mathbb{R}} t^\alpha v(\eta)\ \mathrm{d}x = t^{\alpha+1/2} \int_{\mathbb{R}} v(\eta)\ \mathrm{d}\eta = t^{\alpha+1/2}\phi(1) \quad \Longrightarrow \quad 1 = \dot{\phi} = \left(\alpha + \frac{1}{2}\right) t^{\alpha-1/2}\phi(1), \tag{606}$$

which implies $\alpha = 1/2$ and then also that $\phi(1) = 1$. Assuming ϕ is continuous, the facts $\phi(1) = 1$ and

[24] Credit is due here to insights from Dohyun Kim's notes.

$\dot{\phi} = 1$ for $t > 0$ imply $\phi(0) = 0$. Compiling our results, the PDE may be expressed via

$$\delta(x)\Theta(t) = c_t - c_{xx} = t^{-1/2}\left[\frac{v(\eta)}{2} - \frac{\eta v'(\eta)}{2}\right] - t^{-1/2}v''(\eta). \tag{607}$$

Multiplying by $t^{1/2}$ yields the ODE

$$\frac{v - \eta v'}{2} - v'' = t^{1/2}\delta, \quad \text{for all } (x,t) \in \mathbb{R} \times (0,\infty). \tag{608}$$

We shall construct a solution to the above ODE by utilizing a Green's function and the given conditions on c. Namely, $v(\pm\infty) = 0$. We see $v_1 = \eta$ is a solution to the associated homogeneous ODE. Using reduction of order, a second linearly independent solution is given by $v_2 = \eta w$, where w is to be determined. Differentiating reveals

$$v_2' = \eta w' + w \quad \implies \quad v_2'' = \eta w'' + 2w', \tag{609}$$

and so

$$0 = \frac{1}{2}\left[\eta w - \eta^2 w' - \eta w\right] - \eta w'' - 2w' = -\left(\frac{\eta^2}{2} + 2\right) w' - \eta w'' \quad \implies \quad 0 = w'' + \left(\frac{\eta^2 + 4}{2\eta}\right) w'. \tag{610}$$

Upon inclusion of an integrating factor, we see

$$0 = \left(w' \exp\left(\int^{\eta} \frac{z^2 + 4}{2z}\ \mathrm{d}z\right)\right)' = \left(w'\eta^2 \exp\left(\frac{\eta^2}{4}\right)\right)'. \tag{611}$$

Whence there exists $d \in \mathbb{R}$ such that

$$w'\eta^2 \exp\left(\frac{\eta^2}{4}\right) = d \quad \implies \quad w = d\int^{\eta} z^{-2} \exp\left(-\frac{z^2}{4}\right)\ \mathrm{d}z. \tag{612}$$

Therefore, the general solution to the homogeneous ODE is of the form

$$y = d_1\eta + d_2\eta\int^{\eta} z^{-2} \exp\left(-\frac{z^2}{4}\right)\ \mathrm{d}z. \tag{613}$$

Then the function v is of the form

$$v(\eta) = \begin{cases} d_1\eta + d_2\eta w & \text{if } \eta < 0, \\ d_3\eta + d_4\eta w & \text{if } \eta > 0, \end{cases} \tag{614}$$

where we note $w \longrightarrow \mp\sqrt{\pi}/2$ as $x \longrightarrow \pm\infty$ and the boundary conditions $v(\pm\infty) = 0$ imply

$$\sqrt{\pi}/2 \cdot d_1 + d_3 = 0 \quad \text{and} \quad -\sqrt{\pi}/2 \cdot d_2 + d_4 = 0. \tag{615}$$

The continuity of v implies

$$\lim_{\eta \longrightarrow 0^+} v(\eta) = \lim_{\eta \longrightarrow 0^-} v(\eta). \tag{616}$$

And, the final condition is given for determining the coefficients is given via

$$\begin{aligned} t^{1/2} &= t^{1/2} \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \delta(x)\ \mathrm{d}x \\ &= \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \frac{v - \eta v'}{2} - v''\ \mathrm{d}x \\ &= t^{1/2} \lim_{\varepsilon \to 0^+} \int_{-\varepsilon t^{-1/2}}^{\varepsilon t^{-1/2}} \frac{v - \eta v'}{2} - v''\ \mathrm{d}\eta \\ &= t^{1/2} \left[-v'(0^+) + v'(0^-) \right], \end{aligned} \tag{617}$$

i.e., $v'(0^-) = 1 + v'(0^+)$. For the v satisfying these conditions, we conclude

$$c(x,t) = t^{1/2} v\left(x t^{-1/2}\right). \tag{618}$$

□

F17.3. consider the initial value problem

$$y'' + \frac{yy'}{x^4} + y^2 = 0, \quad y(0) = y'(0) = 0. \tag{619}$$

Determine whether or not there exists a unique solution of this differential equation in a neighborhood of the origin.

Solution:

We claim this ODE does *not* admit a unique solution in neighborhood of the origin. First observe the zero function is a solution to the ODE. Through asymptotic analysis, we show below consistency of an ansatz solution asymptotic to $-2x^3$ as $x \to 0$. This (non-rigorously) establishes existence of two solutions within a neighborhood of the origin, i.e., the ODE does not admit a unique solution in a neighborhood of the origin.

Assume the ansatz $y \sim Ax^\alpha$ as $x \to 0$, for scalars $A \neq 0$ and α to be determined. Plugging this into the PDE yields

$$-A^2x^{2\alpha} \sim -y^2 \sim y'' + \frac{yy'}{x^4} \sim \alpha(\alpha-1)Ax^{\alpha-2} + \alpha A^2 x^{\alpha+(\alpha-1)-4} \quad \text{as } x \to 0. \tag{620}$$

Since $y(0) = 0$, we know $\alpha \geq 0$. Consequently,

$$-A^2x^{2\alpha} \ll \alpha(\alpha-1)Ax^{\alpha-2} \quad \text{and} \quad -A^2x^{2\alpha} \ll \alpha A^2 x^{2\alpha-5} \quad \text{as } x \to 0. \tag{621}$$

Thus

$$\alpha(\alpha-1)Ax^{\alpha-2} \sim -\alpha A^2 x^{2\alpha-5} \quad \text{as } x \to 0. \tag{622}$$

Equating powers of x, we see $\alpha - 2 = 2\alpha - 5$ implies $\alpha = 3$. Equating the coefficients, we obtain

$$3(2-1)A = -3A^2 \quad \Longrightarrow \quad A = -2, \tag{623}$$

since we assumed $A \neq 0$. Thus $y \sim -2x^3$ as $x \to 0$. □

F17.4. Solve for the entropy satisfying weak solution of Burgers' equation

$$u_t + uu_x = 0, \ \ x \in \mathbb{R}, t > 0, \tag{624}$$

with initial data

$$u(x,0) = \begin{cases} 1 - x & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{625}$$

Solution:

Let $g(x) := u(x,0)$ and $f(u) = \frac{1}{2}u^2$. Then the PDE becomes

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{626}$$

which is the form of a conservation law. We proceed via the method of characteristics. Define $F(p,q,z,x,t) := q + zp$. Taking $p = u_x$, $q = u_t$, and $z = u$, we see $F = 0$ and obtain the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z(s), & x(0) = x^0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = 0, & z(0) = g(x^0). \end{cases} \tag{627}$$

This implies $s = t$ and z is constant along characteristics. Hence

$$x(t) = tg(x^0) + x^0 = \begin{cases} (1-t)x^0 + t & \text{if } 0 \le x^0 \le 1, \\ x^0 & \text{otherwise.} \end{cases} \tag{628}$$

The characteristics originating in $[0,1] \times \{t = 0\}$ form line segments, ending at $(1,1)$. For such characteristics, it follows that $t \le x \le 1$ when $t \in [0,1]$ and

$$x = (1-t)x^0 + t \quad \implies \quad z(t) = g(x^0) = 1 - x^0 = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}. \tag{629}$$

Alternatively, for $0 < x < t$, we have $0 < x/t < 1$ and note that, if function u of the form $u(x,t) = v(x/t)$

solves the PDE, then[25]

$$0 = u_t + f(u)_x = -v'\left(\frac{x}{t}\right)\frac{x}{t^2} + f'(v)v'\left(\frac{x}{t}\right)\frac{1}{t} = v'\left(\frac{x}{t}\right)\frac{1}{t}\left[f'\left(v\left(\frac{x}{t}\right)\right) - \frac{x}{t}\right] \quad \Longrightarrow \quad v = (f')^{-1}, \tag{630}$$

assuming v' never vanishes. Whence

$$u(x,t) = (f')^{-1}\left(\frac{x}{t}\right) = \frac{x}{t}, \quad \text{whenever } 0 < x < t, \tag{631}$$

where we note $f'(u) = u$ is the identity function. Consequently, for all $(x,t) \in \mathbb{R} \times (0,1)$

$$u(x,t) = \begin{cases} \frac{x}{t} & \text{if } 0 \leq x \leq t, \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{632}$$

At $(x,t) = (1,1)$ the characteristics cross, and so a shock occurs. Using the Rankine-Hugenoit condition, we see the curve $(s(t), t)$ describing the shock satisfies $s(1) = 1$ and

$$\dot{s}(t) = \frac{\frac{1}{2}f(u_\ell) - \frac{1}{2}f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}\cdot (x/t)^2 - 0}{(x/t) - 0} = \frac{x}{2t}, \tag{633}$$

where u_ℓ and u_r denote the function values to the left and right of the shock, respectively. Moreover, note the entropy conditions are satisfied since

$$f'(u_\ell) = \frac{x}{t} > \frac{x}{2t} = \dot{s} > 0 = f'(u_r). \tag{634}$$

Using separation of variables and writing $x = s(t)$, we deduce

$$\frac{\mathrm{d}x}{x} = \frac{1}{2}\cdot\frac{\mathrm{d}t}{t} \quad \Longrightarrow \quad \ln(x) = \frac{1}{2}\ln(t) + C_0 = \ln(t^{1/2}) + C_0 \quad \Longrightarrow \quad x = C_1 t^{1/2}, \tag{635}$$

for some constants C_0 and $C_1 = \exp(C_0)$. The fact $s(1) = 1$ then implies $s(t) = t^{1/2}$. Thus for $(x,t) \in$

[25]The nifty idea for presenting this came from reading page 155 of Evan's text. This approach for the rarefaction wave is quite general, which will be useful for harder conservation law problems.

$\mathbb{R} \times (1, \infty)$ we obtain

$$u(x,t) = \begin{cases} \frac{x}{t} & \text{if } 0 \leq x < t^{1/2}, \\ 0 & \text{otherwise.} \end{cases} \tag{636}$$

Then (632) and (636) give the entropy satisfying weak solution u of the given PDE. □

F17.5

a) Solve for the Green's function $G(\cdot,\hat{x}) : [0,1] \to [0,1]$ with

$$-\frac{\partial^2 G}{\partial x^2}(x;\hat{x}) = \delta(x-\hat{x}), \quad \text{for } \hat{x} \in (0,1), \tag{637}$$

with $G(0;\hat{x}) = G(1;\hat{x}) = 0$.

b) Define

$$a(w,v) := \int_0^1 w_x(x) v_x(x) \ \mathrm{d}x \quad \text{and} \quad (v,f) := \int_0^1 v(x) f(x) \ \mathrm{d}x, \tag{638}$$

for some $f \in L^2(0,1)$. Let $u \in H^1(0,1)$ with $u(0) = u(1)$ and

$$a(u,v) = (v,f) \ \text{ for all } v \in H^1(0,1), \tag{639}$$

with $v(0) = v(1) = 0$. Similarly define $u^h \in W^k$ with $u^h(0) = u^h(1) = 0$ and

$$a(u^h, v^h) = (v^h, f) \ \text{ for all } v^h \in W^h, \tag{640}$$

where

$$W^h := \left\{ v^h \in H^1(0,1) : v_1, \dots, v_N \in \mathbb{R} \text{ such that } v^h(x) = \textstyle\sum_{i=1}^N v_i N_i(x) \right\}. \tag{641}$$

Here $h = 1/(N+1)$, $x_i = ih$, and

$$N_i(x) := \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in (x_{i-1}, x_i), \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}), \\ 0 & \text{otherwise.} \end{cases} \tag{642}$$

Show[26] that $G(\cdot, x_i) \in W^h$ and use this to show that $u^h(x_i) = u(x_i)$.

[26] This is a variation of Exercise 1.19 on page 43 of Johnson's *Numerical Solution of Partial Differential Equations by the Finite Element Method.*

Solution:

a) The general solution to $-y'' = 0$ is a linear function. Thus $G(x;\hat{x})$ may be expressed via

$$G(x;\hat{x}) = \begin{cases} c_1 x + c_2 & \text{if } x < \hat{x}, \\ d_1 x + d_2 & \text{if } x > \hat{x}. \end{cases} \tag{643}$$

The boundary condition $G(0;\hat{x}) = 0$ implies $c_2 = 0$. Similarly, the boundary condition $G(1;\hat{x}) = 0$ implies $d_2 = -d_1$, and so

$$G(x;\hat{x}) = \begin{cases} c_1 x & \text{if } x < \hat{x}, \\ d_1(x-1) & \text{if } x > \hat{x}. \end{cases} \tag{644}$$

By the continuity of G, we deduce

$$c_1\hat{x} = \lim_{x\to\hat{x}^-} G(x;\hat{x}) = \lim_{x\to\hat{x}^+} G(x;\hat{x}) = d_1(\hat{x}-1) \quad \Longrightarrow \quad c_1 = \frac{d_1(\hat{x}-1)}{\hat{x}}. \tag{645}$$

Integrating our differential equation, the jump discontinuity reveals

$$1 = G_x(\hat{x}^+;\hat{x}) - G_x(\hat{x}^-;\hat{x}) = d_1 - c_1 = d_1\left(1 - \frac{\hat{x}-1}{\hat{x}}\right) = \frac{d_1}{\hat{x}} \quad \Longrightarrow \quad d_1 = \hat{x}, \tag{646}$$

and so $c_1 = \hat{x} - 1$. Therefore,

$$G(x;\hat{x}) = \begin{cases} (\hat{x}-1)x & \text{if } x < \hat{x}, \\ (x-1)\hat{x} & \text{if } x > \hat{x}. \end{cases} \tag{647}$$

b) We first show $G(\cdot, x_i) \in W^h$. It suffices to show there exists scalars $\{v_1, \dots, v_N\}$ such that

$$G(x, x_i) = \sum_{j=1}^{N} v_j N_j(x), \tag{648}$$

for all $x \in [0,1]$. We shall verify (648), taking $v_j := G(x_j, x_i)$ for all $j \in \{1, 2, \dots, N\}$. Since each

$N_i(x)$ is the standard tent function, it follows directly that

$$\sum_{j=1}^N v_j N_j(x_\ell) = \sum_{j=1}^N G(x_j, x_i) N_j(x_\ell) = \sum_{j=1}^N G(x_j, x_i)\delta_{j\ell} = G(x_\ell, x_i), \quad \text{for all } \ell \in \{1, 2, \dots, N\}. \tag{649}$$

Thus, (648) holds at each x_ℓ. Now, if $x \in [0, x_1]$, then the equality in (648) holds since both the left and right hand sides are linear functions on $[0, x_1]$ that interpolate through $(0, 0)$ and $(x_1, G(x_1, x_i))$, and linear interpolations through two distinct points are unique. Similarly, (648) holds for $x \in [x_N, 1]$. Now suppose $x \in [x_1, x_N] - \{x_1, \dots, x_N\}$. Then there is an index ℓ such that $x > x_\ell$ and $x \in \operatorname{spt}(N_\ell)$ and $x \in \operatorname{spt}(N_{\ell+1})$ and $x \notin \operatorname{spt}(N_j)$ for all other $j \notin \{\ell, \ell+1\}$. If $\ell \geq i$, then this implies

$$\begin{aligned}
\sum_{j=1}^N v_j N_j(x) &= v_\ell N_\ell(x) + v_{\ell+1} N_{\ell+1}(x) \\
&= G(x_\ell, x_i) \cdot \frac{x_{\ell+1} - x}{h} + G(x_{\ell+1}, x_i) \cdot \frac{x - x_\ell}{h} \\
&= \frac{1}{h}\left[(x_\ell - 1)x_i \cdot (x_{\ell+1} - x) + (x_{\ell+1} - 1)x_i \cdot (x - x_\ell)\right] \\
&= \frac{1}{h}\left[(x_\ell - 1)x_i \cdot (x_\ell + h - x) + (x_\ell + h - 1)x_i \cdot (x - x_\ell)\right] \\
&= (x - 1)x_i \\
&= G(x; x_i),
\end{aligned} \tag{650}$$

where we have used the fact $x_{\ell+1} = x_\ell + h$. This shows (648) holds in this case. Likewise, for $\ell < i$, we see (648) again holds. This covers all cases, and so (648) holds for all $x \in [0, 1]$.

For notational compactness, set $G_i(x) := G(x; x_i)$. Then for each function $\phi \in H_0^1(0, 1)$ we see

$$\begin{aligned}
a(\phi, G_i) &= \int_0^1 \phi'(x) G'(x; x_i)\ \mathrm{d}x \\
&= -\int_0^1 \phi(x) G''(x; x_i)\ \mathrm{d}x + \left[\phi(x) G'(x; x_i)\right]_{x=0}^1 \\
&= \int_0^1 \phi(x)\delta(x - x_i)\ \mathrm{d}x \\
&= \phi(x_i),
\end{aligned} \tag{651}$$

where we use integration by parts, the fact G satisfies (637), and the fact $\phi = 0$ on $\partial(0,1)$. Since $G_i \in W^h \subset H^1(0,1)$, this further implies

$$u(x_i) - u^h(x_i) = a(u, G_i) - a(u^h, G_i) = (G_i, f) - (G_i, f) = 0, \quad \text{for } i = 1, 2, \ldots, N, \tag{652}$$

which completes the proof.

□

F17.6. Consider the wave equation

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = f & \text{on } \mathbb{R}^3 \times \{t=0\}, \\ u_t = g & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{653}$$

where the initial data f and g are only nonzero in the region $a < \|x\| < b$, with $\|\cdot\|$ the ℓ_1 norm of x. Given a point $x \in \mathbb{R}^3$, find the time $T > 0$ such that $u(x,t) = 0$ for $t \in (0,T)$ a) when $\|x\| > b$ and b) when $\|x\| < a$.

Solution:

a) We illustrate the the support of the initial data as follows.

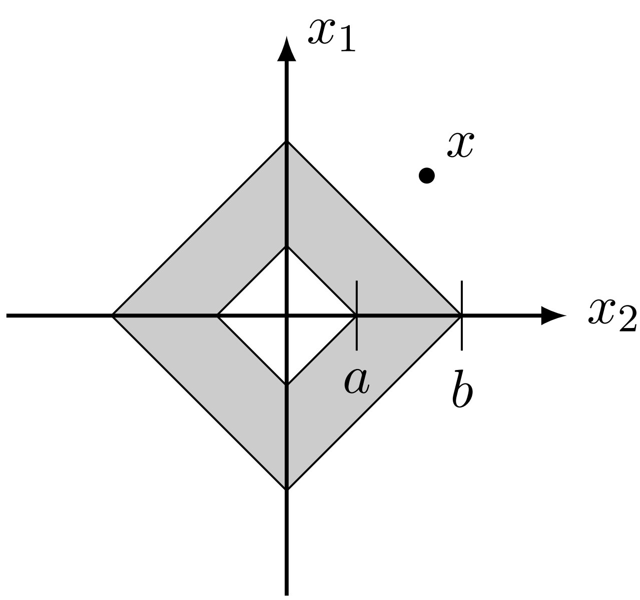

Figure 20: Illustration of the support of the initial data for F17.6b (the slice viewed at $x_3 = 0$).

Since the dimension of $\mathbb{R}^3$ is odd and at least three, Huygen's principle asserts, at a given point $x \in \mathbb{R}^3$, the data f and g only affect the solution v on the boundary $\partial C(x)$ of the cone[27]

$$C(x) := \{(y,t) \in \mathbb{R}^3 \times (0,\infty) : |x-y| < ct\}, \tag{654}$$

[27] See the PDE text by Evans on page 80.

i.e., a "disturbance" propagates along a sharp wavefront. Consider the problem[28]

$$\min_{y\in\mathbb{R}^3} \frac{1}{2}\|y-x\|_2^2 \text{ s.t. } \|y\|_1 = b. \tag{655}$$

The solution $y^\star$ to the problem (655) is such that $\|x-y^\star\|$ gives the minimum distance from x to a point in the maximal support of the initial data. One may also view $y^\star$ as the projection of x onto the ℓ_1 ball centered at the origin of radius b. Because we know the wave propagates at speed c, it follows that the minimum time T at which a disturbance may arrive is

$$T = \frac{\|x-y^\star\|}{c}. \tag{656}$$

b) In similar fashion to a), consider the problem

$$\min_{y\in\mathbb{R}^3} \frac{1}{2}\|y-x\|_2^2 \text{ s.t. } \|y\|_1 = a. \tag{657}$$

Then the solution $\tilde{y}$ to (657) reveals the time T for a disturbance to arrive is

$$T = \frac{\|x-\tilde{y}\|}{c}. \tag{658}$$

□

[28] Upon discussing this problem with Teran, it was noted that the exam graders did not expect students to fully solve the minimization problem (655). It apparently sufficed to illustrate the problem and prescribe an appropriate interpretation.

Remark: If someone has buckets of time to kill after having easily aced all the other problems on the exam (sarcasm intended), the following outlines the approach for solving the optimization problem (655). This primal problem has the associated Lagrangian $\mathcal{L}(y,\nu)$ given by

$$\mathcal{L}(y,\nu) = \frac{1}{2}\|y-x\|^2 + \nu\left(\|y\|_1 - b\right). \tag{659}$$

At a saddle point $(y^\star, \nu^\star)$,

$$0 \in \partial_y \mathcal{L} = y^\star - x + \nu \cdot \partial_y \|y^\star\|_1 \quad \text{and} \quad 0 \in \partial_\nu \mathcal{L} = \|y^\star\|_1 - b. \tag{660}$$

This implies $\|y^\star\|_1 = b$, as expected. Also, the subgradient of the absolute value $|\cdot|$ is

$$\partial|y_i| = \begin{cases} 1 & \text{if } y_i > 0, \\ [-1,1] & \text{if } y_i = 0, \\ -1 & \text{if } y_i < 0. \end{cases} \tag{661}$$

Then $\partial\|y\|_1$ is the component-wise application of the subgradient of the absolute value. Having x, we could carry this further forward in an explicit fashion (the limitation being that the subgradient is set-valued and not a single number, making the final result be expressed in several cases). $\triangle$

F17.7. For a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, consider a smooth solution of the parabolic PDE

$$\begin{cases} u_t - \Delta u = (M-u)_+ & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u = g & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{662}$$

where $f_+ := \max\{f, 0\}$ and g is a smooth function which vanishes on $\partial\Omega$. Show that if $g(x) \leq M$, then $u(x,t) \leq M$ for all $t > 0$.

Solution:

Fix a time $T > 0$. Then let $\varepsilon > 0$ and set $v := u - M - \varepsilon e^t$. Then

$$\begin{cases} v_t - \Delta v = (-v - \varepsilon e^t)_+ - \varepsilon e^t & \text{in } \Omega \times (0,T], \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T], \\ v \leq -\varepsilon & \text{on } \Omega \times \{t=0\}. \end{cases} \tag{663}$$

Set $\Omega_T := \Omega \times (0,T]$ and Γ_T to be the parabolic boundary. Since $\overline{\Omega}_T$ is compact and v is smooth, v attains its supremum over $\overline{\Omega}_T$. By way of contradiction, suppose

$$\sup_{\overline{\Omega}_T} v \geq 0. \tag{664}$$

By (663), it follows that the supremum occurs at a positive time since $v \leq -\varepsilon < 0$ at time $t = 0$. If the max were to occur at a point on $\partial\Omega \times (0,T]$, then by the ellipticity of the Laplacian operator[29] it would follow from Hopf's lemma that $\partial v/\partial\nu > 0$ at the maximizer, which would contradict (663). Therefore, the maximum is obtained at a point $(\overline{x}, \overline{t}) \in \Omega_T = \overline{\Omega}_T - \Gamma_T$, where we let $\overline{t}$ be the first time at which this maximum is obtained (which exists by the continuity of v). This implies $v_t(\overline{x}, \overline{t}) \geq 0$. Since $\overline{x}$ is a local maximizer of $v(\cdot, \overline{t})$, it follows that $0 \geq \Delta v(\overline{x}, \overline{t})$. Therefore, at $(\overline{x}, \overline{t})$,

$$0 \leq v_t - \Delta v = (-v - \varepsilon e^{\overline{t}})_+ - \varepsilon e^{\overline{t}} = 0 - \varepsilon e^{\overline{t}} \leq -\varepsilon < 0, \tag{665}$$

a contradiction. This contradiction proves the assumption in (664) was false. Therefore,

$$v = u - M - \varepsilon e^t < 0 \quad \Longrightarrow \quad u < M + \varepsilon e^t \leq M + \varepsilon e^T \ \text{ in } \overline{\Omega}_T. \tag{666}$$

[29] I use this phrase since I honestly don't know how we are supposed to apply Hopf's lemma.

Because this holds for arbitrary $\varepsilon > 0$, we may let $\varepsilon \to 0^+$ to deduce

$$u \leq M \quad \text{in } \overline{\Omega}_T. \tag{667}$$

And, since this result holds for arbitrary $T > 0$, we may let $T \to \infty$ to deduce $u(x,t) \leq M$ for all $t > 0$, as desired.

□

F17.8. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Suppose there exists a minimizer u of the functional

$$E(u) = \frac{1}{2}\int_\Omega |Du|^2 \ \mathrm{d}x \tag{668}$$

among smooth functions w in $\overline{\Omega}$ with the constraints

$$w = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega w^2 \ \mathrm{d}x = 1. \tag{669}$$

a) Show for any smooth function w in $\overline{\Omega}$ there exists a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that $w(\tau) := \tau w + \phi(\tau)u$ satisfies

$$\int_\Omega (u + w(\tau))^2 \ \mathrm{d}x = 1, \tag{670}$$

for sufficiently small $\tau > 0$. (Note the range of τ depends on the choice of w.)

b) Show that $\phi'(0) = -\int_\Omega uw \ \mathrm{d}x$.

c) One can use a) and b) to perturb the energy to derive a boundary value problem that u satisfies. Find the PDE problem, which involves the constant

$$\lambda := \int_\Omega |Du|^2 \ \mathrm{d}x. \tag{671}$$

Solution:

a) Observe (670) holds precisely when

$$\begin{aligned} 1 &= \int_\Omega u^2 + 2uw(\tau) + w(\tau)^2 \ \mathrm{d}x \\ &= 1 + 2\int_\Omega u[\tau w + \phi(\tau)u] + [\tau w + \phi(\tau)u]^2 \ \mathrm{d}x \\ &= 1 + \int_\Omega 2\tau uw + 2\phi(\tau)u^2 \ \mathrm{d}x + \int_\Omega \tau^2 w^2 + 2\tau\phi(\tau)uw + \phi(\tau)^2u^2 \ \mathrm{d}x \\ &= \phi(\tau)^2 + \phi(\tau)\left[2 + 2\tau\int_\Omega uw \ \mathrm{d}x\right] + \left[1 + 2\tau\int_\Omega uw \ \mathrm{d}x + \tau^2\int_\Omega w^2 \ \mathrm{d}x\right]. \end{aligned} \tag{672}$$

Using the quadratic formula, we see (672) holds precisely when

$$\phi(\tau) = \frac{-\left[2 + 2\tau \int_\Omega uw\ \mathrm{d}x\right] \pm \sqrt{\left[2 + 2\tau \int_\Omega uw\ \mathrm{d}x\right]^2 - 4\left[2\tau \int_\Omega uw\ \mathrm{d}x + \tau^2 \int_\Omega w^2\ \mathrm{d}x\right]}}{2}. \tag{673}$$

The choice of ϕ with the addition (rather than subtraction) is the only choice that yields the desired $\phi(\tau)$. Additionally, note

$$\begin{aligned} &\left[2 + 2\tau \int_\Omega uw\ \mathrm{d}x\right]^2 - 4\left[2\tau \int_\Omega uw\ \mathrm{d}x + \tau^2 \int_\Omega w^2\ \mathrm{d}x\right] \\ &= 4 + 8\tau \int_\Omega uw\ \mathrm{d}x + 4\tau^2 \left[\int_\Omega uw\ \mathrm{d}x\right]^2 - 8\tau \int_\Omega uw\ \mathrm{d}x - 4\tau^2 \int_\Omega w^2\ \mathrm{d}x \\ &= 4\ \left[1 + \tau^2 \left(\int_\Omega uw\ \mathrm{d}x\right)^2 - \tau^2 \int_\Omega w^2\ \mathrm{d}x\right]. \end{aligned} \tag{674}$$

Thus, by our earlier work, we see (670) holds for

$$\begin{aligned} \phi(\tau) &= \frac{-\left[2 + 2\tau \int_\Omega uw\ \mathrm{d}x\right] + 2\sqrt{1 + \tau^2 \left(\int_\Omega uw\ \mathrm{d}x\right)^2 - \tau^2 \int_\Omega w^2\ \mathrm{d}x}}{2} \\ &= -1 - \tau \int_\Omega uw\ \mathrm{d}x + \Bigg(1 + \tau^2 \underbrace{\left[\left(\int_\Omega uw\ \mathrm{d}x\right)^2 - \int_\Omega w^2\ \mathrm{d}x\right]}_{=:\mu}\Bigg)^{1/2} \\ &= -1 - \tau \int_\Omega uw\ \mathrm{d}x + \sqrt{1 + \tau^2 \mu}. \end{aligned} \tag{675}$$

provided $\tau > 0$ is sufficiently small to ensure the argument inside the square root is nonnegative.

b) Through direct computation we see

$$\phi'(0) = \left[-\int_\Omega uw\ \mathrm{d}x + \frac{1}{2}\left(1 + \tau^2 \mu\right)^{-1/2} \cdot 2\tau\mu\right]_{\tau = 0} = -\int_\Omega uw\ \mathrm{d}x. \tag{676}$$

c) We proceed by using Lagrange's theorem for multipliers. Observe, for each test function v,

$$\begin{aligned}\delta E(u,v) &= \lim_{\varepsilon\to 0^+}\frac{E(u+\varepsilon v)-E(u)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\left[\int_\Omega |Du+\varepsilon Dv|^2 - |Du|^2\ \mathrm{d}x\right]\\ &= \lim_{\varepsilon\to 0^+}\int_\Omega Du\cdot Dv + \frac{\varepsilon}{2}|Dv|^2\ \mathrm{d}x\\ &= \int_\Omega Du\cdot Dv\ \mathrm{d}x.\end{aligned} \tag{677}$$

Similarly, letting

$$J(u) := \int_\Omega u^2\ \mathrm{d}x, \tag{678}$$

we see

$$\delta J(u,v) = \lim_{\varepsilon\to 0^+}\frac{J(u+\varepsilon v)-J(u)}{\varepsilon} = \int_\Omega uv\ \mathrm{d}x. \tag{679}$$

Define the admissibility class $\mathcal{A} := \{u\in C^2(\Omega) : u = 0 \text{ on } \partial\Omega,\ J(u)=1\}$. Lagrange's theorem for multipliers asserts there exists $\lambda\in\mathbb{R}$ such that the minimizer u of E over $\mathcal{A}$ satisfies

$$\delta E(u,v) = \lambda\delta J(u,v), \quad \text{for all test functions } v. \tag{680}$$

Thus, for each test function v,

$$0 = \int_\Omega Du\cdot Dv - \lambda uv\ \mathrm{d}x = \int_\Omega(-\Delta u-\lambda u)v\ \mathrm{d}x + \underbrace{\int_{\partial\Omega} u\frac{\partial v}{\partial n}\ \mathrm{d}\sigma}_{=0} = \int_\Omega(-\Delta u - \lambda u)v\ \mathrm{d}x, \tag{681}$$

from which it follows that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{682}$$

Furthermore,

$$\int_\Omega |Du|^2\ \mathrm{d}x = -\int_\Omega u\Delta u\ \mathrm{d}x + \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\ \mathrm{d}\sigma}_{=0} = \int_\Omega \lambda u^2\ \mathrm{d}x = \lambda, \tag{683}$$

as desired.

□

2017 Spring

S17.1. Let us consider the continuity equation $\rho_t + \nabla \cdot (\rho v) = 0$ in $\mathbb{R}^3 \times (0, \infty) \to \mathbb{R}$ with $v(x) : \mathbb{R}^3 \to \mathbb{R}^3$ and initial data $\rho_0(x)$.

a) Represent ρ in terms of ρ_0 using the method of characteristics, assuming that v is Lipschitz continuous. Explain where the Lipschitz continuity assumption is used in the argument.

b) Suppose $-1 < \nabla \cdot v$ in $\mathbb{R}^3$ and $\rho_0 = \chi_{|x|<1}$, where χ_A denotes the characteristic function of a set A. Show that then $\Omega := \{x : \rho(x, 1) > 0\}$ has its volume greater than 4/3.

Solution:

a) Define $F(p, q, z, x, t) := q + p \cdot v + z(\nabla \cdot v)$. Then, using the method of characteristics, taking $p = D\rho$, $z = \rho$, and $q = \rho_t$, we obtain $F = 0$ and the ODE system

$$\begin{cases} \dot{x}(s) & = F_p = v, & x(0) = x^0, \\ \dot{t}(s) & = F_q = 1, & t(0) = 0, \\ \dot{z}(s) & = F_p \cdot p + F_q q = v \cdot p + q = -z(\nabla \cdot v), & z(0) = \rho_0(x^0). \end{cases} \tag{684}$$

This implies $t = s$ and

$$x(t) = x^0 + \int_0^t v(x(\tau))\ \mathrm{d}\tau. \tag{685}$$

Since v is Lipschitz, elementary theory of differential equations tells us there exists a unique solution to (685). Then

$$\rho(x, t) = z(t) = \rho_0(x^0) \exp\left(-\int_0^t \nabla \cdot v(x(\tau))\ \mathrm{d}\tau\right), \tag{686}$$

which is well-defined by the existence of the unique path $x(t)$ that passes through x and originates at x^0 (n.b. this is where the Lipschitz continuity was used).

b) Let u be a solution to the PDE $u_t + \nabla u \cdot v = 0$ with initial data ρ_0. Then the method of characteristics reveals that if $F(p, q, z, x, t) = q + p \cdot v$, then taking $z = u$, $p = Du$, and $q = u_t$ implies $F = 0$ and gives the ODE system

$$\begin{cases} \dot{x}(s) &= F_p = v, & x(0) = x^0, \\ \dot{t}(s) &= F_q = 1, & t(0) = 0, \\ \dot{z}(s) &= F_p \cdot p + F_q q = v \cdot p + q = 0, & z(0) = \rho_0(x^0). \end{cases} \tag{687}$$

Notice u has the same characteristics as ρ. Moreover, ρ is positive along each characteristic that starts in the support of ρ_0 since ρ_0 is nonnegative. Consequently, for each $t \in (0, \infty)$ the equality

$$\{x : u(x,t) > 0\} = \{x : \rho(x,t) > 0\} \tag{688}$$

holds. Since u is constant along its characteristics and ρ_0 is the characteristic function, it follows that

$$|\{x : u(x,t) > 0\}| = w(t) := \int_{\mathbb{R}^3} u(x,t)\ \mathrm{d}x. \tag{689}$$

Then

$$w(0) = \int_{\mathbb{R}^3} \rho_0(x)\ \mathrm{d}x = |\{x : |x| < 1\}| = \frac{4\pi}{3}. \tag{690}$$

Differentiating in time reveals

$$\dot{w}(t) = \int_{\mathbb{R}^3} u_t\ \mathrm{d}x = \int_{\mathbb{R}^3} -\nabla u \cdot v\ \mathrm{d}x = \int_{\mathbb{R}^3} u\left(\nabla \cdot v\right)\ \mathrm{d}x > -\int_{\mathbb{R}^3} u\ \mathrm{d}x = -w(t), \tag{691}$$

where we have used integration by parts. We now prove a variation of Gronwall's inequality. Observe (691) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[we^t\right] = \left[\dot{w} + w\right]e^t > 0, \tag{692}$$

which shows we^t is increasing. Thus,

$$w(t)e^t \geq w(0)e^0 = w(0) \quad\implies\quad w(t) \geq w(0)e^{-t} = \frac{4\pi}{3e^t}. \tag{693}$$

Combining the above results with this fact, we see

$$\{x : \rho(x,1) > 0\} = \{x : u(x,1) > 0\} = w(1) \geq \frac{4\pi}{3e} > \frac{4}{3}, \tag{694}$$

noting $\pi > 3 > e$. This completes the proof.

□

S17.2. Consider the following parabolic equation

$$\theta_t = \Delta\left(\left(|x|^2+1\right)\theta\right) + |D\theta| - 4n\theta \ \text{ in } \mathbb{R}^n \times (0,\infty). \tag{695}$$

a) Let $\theta_1(x,t)$ and $\theta_2(x,t)$ be two smooth, nonnegative solutions of the above equation which vanishes at infinity, with ordered initial data $\theta_1(x,0) \le \theta_2(x,0)$. Show that $\theta_1(x,t) \le \theta_2(x,t)$ for all $t > 0$.

b) Let θ be a smooth, nonnegative, integrable solution of the above equation, where all its derivatives and its products with $|x|^2$ vanish as $|x| \to \infty$. Show that $\int \theta(\cdot,t)\ \mathrm{d}x$ exponentially decays to zero as $t \to \infty$.

Solution:

a) Set $w(x,t) := \theta_2(x,t) - \theta_1(x,t)$. It suffices to show $w \ge 0$ in $\mathbb{R}^n \times (0,\infty)$, noting we are given that $w \ge 0$ on $\mathbb{R}^n \times \{t = 0\}$. Now let $T > 0$ and $\varepsilon > 0$. Then choose $R > 0$ sufficiently large that

$$\theta_1, \theta_2 \le \varepsilon \quad \text{in } (\mathbb{R}^n - B(0,R)) \times [0,T]. \tag{696}$$

We presume such a choice is possible by the hypotheses in the prompt. Since θ_1 and θ_2 are nonnegative, this implies

$$w \ge -\varepsilon \quad \text{in } (\mathbb{R}^n - B(0,R)) \times [0,T]. \tag{697}$$

Letting $\Omega_T := B(0,R) \times (0,T]$, we claim

$$\inf_{\Omega_T} w > -\varepsilon, \tag{698}$$

which implies

$$\inf_{\overline{\Omega}_T} w = \min\left\{\inf_{\Omega_T} w,\ \inf_{\Gamma_T} w\right\} \ge \min\{-\varepsilon,-\varepsilon\} \ge -\varepsilon, \tag{699}$$

where Γ_T is the parabolic boundary of Ω_T and we have used (697) for the boundary portion assertion. Together (697) and (699) imply

$$\inf_{\mathbb{R}^n\times[0,T]} w = \inf\left\{\inf_{\overline{\Omega}_T} w,\ \inf_{(\mathbb{R}^n - B(0,R))\times[0,T]} w\right\} \ge \inf\{-\varepsilon,-\varepsilon\} = -\varepsilon \quad \Longrightarrow \quad \inf_{\mathbb{R}^n\times[0,T]} w \ge -\varepsilon. \tag{700}$$

Because the right hand inequality holds for arbitrary $\varepsilon > 0$, we may let $\varepsilon \to 0^+$ to deduce

$$\inf_{\mathbb{R}^n \times [0,T]} w \geq 0. \tag{701}$$

Finally, because $T > 0$ was also arbitrarily chosen, we may let $T \to \infty$ to deduce

$$w \geq 0 \quad \text{in } \mathbb{R}^n \times [0, \infty), \tag{702}$$

as desired.

All that remains is to verify (698). By way of contradiction, suppose there exists a point $(\overline{x}, \overline{t}) \in \Omega_T$ for which $w(\overline{x}, \overline{t}) = -\varepsilon$, with $\overline{t} > 0$ the first time at which this occurs (since $w(\overline{x}, 0) \geq 0$). This implies $w_t(\overline{x}, \overline{t}) \leq 0$. And, since $\overline{x}$ is a local minimizer of $w(\cdot, \overline{t})$ we know $\Delta w(\overline{x}, \overline{t}) \geq 0$ and

$$0 = Dw(\overline{x}, \overline{t}) = D\theta_2(\overline{x}, \overline{t}) - D\theta_1(\overline{x}, \overline{t}) \quad \Longrightarrow \quad D\theta_2(\overline{x}, \overline{t}) = D\theta_1(\overline{x}, \overline{t}). \tag{703}$$

Consequently, at $(\overline{x}, \overline{t})$,

$$\begin{aligned} 0 \geq w_t &= \Delta\left([|\overline{x}|^2 + 1]w\right) + |D\theta_2| - D|\theta_1| - 4nw \\ &= \Delta\left([|\overline{x}|^2 + 1]w\right) - 4nw \\ &= w\Delta(|\overline{x}|^2 + 1) + 2D\left[|\overline{x}|^2 + 1\right] \cdot Dw + (|\overline{x}|^2 + 1)\Delta w - 4nw \\ &= -2nw + 4\overline{x} \cdot Dw + (|\overline{x}|^2 + 1)\Delta w \\ &\geq -2nw \\ &= 2n\varepsilon \\ &> 0, \end{aligned} \tag{704}$$

which implies $0 > 0$, a contradiction. This completes the proof.

b) Define $e : [0, \infty) \to \mathbb{R}$ by

$$e(t) := \int_{\mathbb{R}^n} \theta(x, t) \, \mathrm{d}x. \tag{705}$$

Since θ is integrable, this function e is well-defined. Differentiating in time, we see

$$\begin{aligned}
\dot{e} &= \int_{\mathbb{R}^n} \theta_t \ \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \Delta\left((|x|^2+1)\theta\right) + |D\theta| - 4n\theta \ \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \underbrace{D1 \cdot D\left((|x|^2+1)\theta\right)}_{=0} + |D\theta| - 4n\theta \ \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \mathrm{sgn}(D\theta)\cdot D\theta - 4n\theta \ \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \underbrace{[\nabla \cdot \mathrm{sgn}(D\theta)]}_{=0}\theta - 4n\theta \ \mathrm{d}x \\
&= -4ne(t),
\end{aligned} \tag{706}$$

where sgn is the signum function and is applied component-wise on vector inputs and the boundary terms vanish during the integration by parts steps since θ by our hypothesis about θ vanishing. Note the "=0" statement on the fifth line is accurate up to a set of measure zero. This result implies

$$e(t) = e(0)\exp(-4nt), \tag{707}$$

from which we immediately deduce $e(t) \to 0$ exponentially as $t \to \infty$.

□

Remark: We offer the following as an alternative approach to asserting

$$\int_{\mathbb{R}^n} |D\theta| \ \mathrm{d}x = 0. \tag{708}$$

The third equality holds using integration by parts, where the boundary terms vanish by our hypotheses. We next show the integral term in the final line of (706) equals zero. Employing the use of polar coordinates (r, ϕ) with $r \in \mathbb{R}$ the radial distance and $\phi \in \mathbb{R}^{n-1}$ giving the direction of each point from the origin, we see

$$\theta(r_2, \phi) - \theta(r_1, \phi) = \int_\Gamma D\theta(r, \phi) \cdot \mathrm{d}s = \int_{r_1}^{r_2} -|D\theta(\ell, \phi)| \mathrm{d}\ell, \tag{709}$$

where Γ is the straight path from (r_1, ϕ) to (r_2, ϕ). The final equality holds since the fact θ is radially symmetric[30] implies $D\theta$ points radially so that $D\theta \cdot \mathrm{d}s = -|D\theta| \mathrm{d}\ell$ along Γ. Since $\lim_{|x|\to\infty} \theta = 0$, for each $\varepsilon > 0$ we see

$$-\theta(\varepsilon, \phi) = \lim_{r\to\infty} \theta(r, \phi) - \theta(\varepsilon, \phi) = \int_\varepsilon^\infty -|D\theta| \ \mathrm{d}\ell \quad \Longrightarrow \quad \theta(\varepsilon, \phi) = \int_\varepsilon^\infty |D\theta| \ \mathrm{d}\ell. \tag{710}$$

Substituting in this result, the integral of θ over the boundary of $B(0, \varepsilon)$ becomes

$$\int_{\partial B(0,\varepsilon)} \theta(r, \phi) \ \mathrm{d}x = \int_{\partial B(0,\varepsilon)} \int_\varepsilon^\infty |D\theta(r, \phi)| \ \mathrm{d}r = \int_{\mathbb{R}^n - B(0,\varepsilon)} |D\theta| \ \mathrm{d}x \tag{711}$$

However,

$$0 \leq \lim_{\varepsilon\to 0^+} \int_{\partial B(0,\varepsilon)} \theta(r, \phi) \ \mathrm{d}x = \lim_{\varepsilon\to 0^+} C \left|\partial B(0, \varepsilon)\right| = \lim_{\varepsilon\to 0^+} Cn\alpha(n)\varepsilon^{n-1} = 0, \tag{712}$$

where $C := \|\theta\|_{L^\infty(\partial B(0,1))}$ and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Thus (711) and (712) together imply

$$\int_{\mathbb{R}^n} |D\theta| \ \mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_{\partial B(0,\varepsilon)} \theta(r, \phi) \ \mathrm{d}x = 0. \tag{713}$$

Consequently, (706) and (713) imply

$$\dot{e}(t) = -4ne(t) \quad \Longrightarrow \quad e(t) = e(0) \exp\left(-4nt\right), \tag{714}$$

and we conclude e decays exponentially to zero as $t \to \infty$, as desired. △

[30] This should be true, but we do not verify the details here.

S17.3. Let u solve the following boundary value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \{(x,t) \in \mathbb{R}^3 \times (0,\infty) : x_1 > t/2\}, \\ u_t = 4u_{x_1} & \text{on } \mathbb{R}^3 \times \{x_1 = t/2\}. \end{cases} \tag{715}$$

Show that $u = 0$ in $\{|x| < R - t\} \cap \{x_1 > t/2\}$ when $u(x,0) = u_t(x,0) = 0$ in $\{|x| < R\} \cap \{x_1 > 0\}$. Explain where the boundary condition $\{x_1 = t/2\}$ has been used.

Solution:

We proceed via an energy argument. Define the local wave energy $e : [0,R) \to \mathbb{R}$ by

$$e(t) := \frac{1}{2} \int_{S(t)} u_t^2 + |Du|^2 \ \mathrm{d}x, \tag{716}$$

where $S(t) := B(0, R-t) \cap \{x \in \mathbb{R}^3 : x_1 > t/2\}$. By our hypothesis, $Du = 0$ and $u_t = 0$ on $S(0) \times \{t = 0\}$, and so $e(0) = 0$. Differentiating in time yields

$$\begin{aligned} \dot{e}(t) &= \int_{S(t)} u_t u_{tt} + Du \cdot Du_t \ \mathrm{d}x + \int_{\partial S(t)} \frac{1}{2}\left(u_t^2 + |Du|^2\right) v \cdot n \ \mathrm{d}\sigma \\ &= \underbrace{\int_{S(t)} u_t(u_{tt} - \Delta u) \ \mathrm{d}x}_{=0} + \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} + \frac{1}{2}\left(u_t^2 + |Du|^2\right) v \cdot n \ \mathrm{d}\sigma \\ &= \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} + \frac{1}{2}\left(u_t^2 + |Du|^2\right) v \cdot n \ \mathrm{d}\sigma, \end{aligned} \tag{717}$$

where $v(t) \in \mathbb{R}^3$ is the Eulerian velocity of the boundary $S(t)$ and n is the outward normal along the boundary $S(t)$. Now observe

$$\partial S(t) = \underbrace{\Big(\partial B(0, R-t) \cap \{x_1 > t/2\}\Big)}_{=:A(t)} \cup \underbrace{\Big(B(0, R-t) \cap \{x_1 = t/2\}\Big)}_{=:B(t)} = A(t) \cup B(t). \tag{718}$$

Along $A(t)$ we have $v = -n$ and along $B(t)$ we have $v = -n/2$. This implies

$$\begin{aligned}\int_{B(t)} u_t \frac{\partial u}{\partial n} + \frac{1}{2}\left(u_t^2 + |Du|^2\right) v \cdot n \, \mathrm{d}\sigma &= \int_{B(t)} u_t u_{x_1} + \frac{1}{2}\left(|u_t|^2 + |Du|^2\right) \cdot -\frac{1}{2} \, \mathrm{d}\sigma \\ &= \int_{B(t)} 4u_{x_1}^2 - \frac{1}{4}\left(16u_{x_1}^2 + |Du|^2\right) \, \mathrm{d}\sigma \\ &= -\int_{B(t)} |Du|^2 \, \mathrm{d}\sigma.\end{aligned} \tag{719}$$

Also, by the Cauchy-Schwarz inequality

$$u_t \frac{\partial u}{\partial n} \leq |u_t| \left|\frac{\partial u}{\partial n}\right| = |u_t||Du \cdot n| \leq |u_t||Du| \leq \frac{1}{2}\left(u_t^2 + |Du|^2\right), \tag{720}$$

where we note $|n| = 1$ and

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2 \quad \Longrightarrow \quad ab \leq \frac{1}{2}\left(a^2 + b^2\right), \quad \text{for all } a, b \in \mathbb{R}. \tag{721}$$

Thus,

$$\int_{A(t)} u_t \frac{\partial u}{\partial n} + \frac{1}{2}\left(u_t^2 + |Du|^2\right) v \cdot n \, \mathrm{d}\sigma = \int_{A(t)} u_t \frac{\partial u}{\partial n} - \frac{1}{2}\left(u_t^2 + |Du|^2\right) \, \mathrm{d}\sigma \leq \int_{A(t)} 0 \, \mathrm{d}\sigma = 0. \tag{722}$$

Together (717), (718), (719), and (722) imply

$$\dot{e}(t) = \frac{1}{2} \int_{\partial S(t)} u_t \frac{\partial u}{\partial \nu} + \frac{1}{2}\left(u_t^2 + |Du|^2\right) v \cdot n \, \mathrm{d}\sigma \leq -\int_{B(t)} |Du|^2 \, \mathrm{d}\sigma \leq 0, \tag{723}$$

i.e., e is nonincreasing, and so $e(t) \leq e(0) = 0$. Since the integrand in the definition of $e(t)$ is nonnegative, we then deduce $e(t) = 0$. Thus, $u_t = 0$ and $Du = 0$ on $S(t) \times \{t\}$ for each for each $t \in [0, R)$. That is, $u_t = 0$ and $Du = 0$ in $K(R) := \{(x,t) : 0 \leq t < R, \ x \in S(t)\}$. This implies u is constant in $K(R)$. Since $K(R)$ is connected and we are given that $u = 0$ on $S(0) \subset K(R)$, it follows that $u = 0$ in $K(R)$. This completes the proof. □

S17.4. Let $\mathcal{V}^k = \text{span}\{q_1, q_2, \ldots, q_k\}$ with $q_i \neq 0 \in H^1(0,1)$ and

$$\int_0^1 q_i q_j \ \mathrm{d}x = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{724}$$

Define $A = (a_{ij}) \in \mathbb{R}^{k \times k}$ by

$$a_{ij} = \int_0^1 q_i' q_j' \ \mathrm{d}x \tag{725}$$

with eigenvalue decomposition[31] $A = V^T \Lambda V$, where $\Lambda = \text{diag}(\lambda_i)$ and $V = (v_{ij})$ is orthogonal. Show that

$$r_i \in (\mathcal{V}^k)^\perp = \left\{ f \in L^2(0,1) : \int_0^1 fq \ \mathrm{d}x = 0 \ \ \forall \ q \in \mathcal{V}^k \right\}, \tag{726}$$

where $r_i := -w_i'' - \lambda_i w_i$ and $w_i := \sum_j v_{ij} q_j$.

Solution:

Define the inner products $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ by

$$\langle f, g \rangle := \int_0^1 fg \ \mathrm{d}x \quad \text{and} \quad (f,g) := \int_0^1 f'g' \ \mathrm{d}x. \tag{727}$$

Define the matrix $M = (m_{ij})$ by $m_{ij} := \langle r_i, q_j \rangle$. Now observe

$$m_{ij} = \langle r_i, q_j \rangle = \sum_\ell v_{i\ell} \left[(q_\ell, q_j) - \lambda_i \langle q_\ell, q_j \rangle \right] = \sum_\ell v_{i\ell} \left[a_{\ell j} - \lambda_i \delta_{\ell j} \right] = \sum_\ell v_{i\ell} a_{\ell j} - \lambda_i \delta_{j\ell} v_{ij}, \tag{728}$$

where the second equality holds since

$$\int_0^1 (-q_\ell'' - \lambda_i q_\ell) q_j \ \mathrm{d}x = \int_0^1 q_\ell' q_j' - \lambda_i q_\ell) q_j \ \mathrm{d}x + \cancelto{0}{\left[-q_\ell' q_j \right]_0^1} = (q_\ell, q_j) - \lambda_i \langle q_\ell, q_j \rangle. \tag{729}$$

Using the definition of the matrix product and V, Λ, and A, we see

$$m_{ij} = \left(\sum_\ell v_{i\ell} a_{\ell j} \right) - \lambda_i v_{ij} = (VA)_{ij} - (\Lambda V)_{ij} = (VA - \Lambda V)_{ij}. \tag{730}$$

[31] I believe a typo was made here since originally the prompt was given as $A = V \Lambda V^T$.

However, since V is orthogonal,

$$A = V^T \Lambda V \quad \Longrightarrow \quad VA = VV^T \Lambda V = \Lambda V \quad \Longrightarrow \quad 0 = VA - \Lambda V. \tag{731}$$

Then (730) and (731) imply $M = 0$. Now let $q \in \mathcal{V}^k$. Then there exist scalars $\alpha_1, \dots, \alpha_k$ such that $q = \alpha_1 q_1 + \cdots + \alpha_k q_k$. For each r_i, the linearity of the scalar product yields

$$\langle r_i, q \rangle = \langle r_i, \alpha_1 q_1 + \cdots + \alpha_k q_k \rangle = \alpha_1 \langle r_i, q_1 \rangle + \cdots \alpha_k \langle r_i, q_k \rangle = \alpha_1 0 + \cdots + \alpha_k 0 = 0, \tag{732}$$

where the third equality holds since $M = 0$. This shows $r_i \in \left(\mathcal{V}^k\right)^\perp$ and completes the proof. $\square$

S17.5. Consider the PDE

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), & \text{in } (0,1)\times(0,\infty), \\ u(x,0) = (s-1)x & \text{on } (0,1)\times\{t=0\}, \\ u_t(x,0) = 0 & \text{on } (0,1)\times\{t=0\}, \\ \dfrac{\partial u}{\partial x} = 0 & \text{on } \partial(0,1)\times(0,\infty), \end{cases} \tag{733}$$

for a constant $s \in \mathbb{R}$.

a) Solve the PDE. Hint: solve in terms of the even extension $u^e : \mathbb{R} \to \mathbb{R}$ of the initial data where

$$u^e(x) = \begin{cases} (s_1)\hat{x}, & \text{if } \hat{x} \in [0,1), \\ (s-1(2-\hat{x})), & \text{if } \hat{x} \in [1,2), \end{cases} \tag{734}$$

with $\hat{x} = 2(x/2 - \text{floor}(x/2))$ for $x \in \mathbb{R}$. The function floor(y) is the closest integer to y with floor$(y) \leq y$.

b) Define

$$e(t) := \int_0^1 u_t^2(x,t) + u_x^2(x,t)\ \mathrm{d}x. \tag{735}$$

Show that $e(t) = (s-1)^2$.

Solution:

a) (Return and complete.)

b) From the given PDE (733), we see

$$e(0) = \int_0^1 u_t^2(x,0) + u_x^2(x,0)\ \mathrm{d}x = \int_0^1 0 + (s-1)^2\ \mathrm{d}x = (s-1)^2. \tag{736}$$

Additionally,

$$\dot{e}(t) = \int_0^1 2u_t u_{tt} + 2u_x u_{xt}\ \mathrm{d}x = \int_0^1 2u_t \underbrace{(u_{tt} - u_{xx})}_{=0}\ \mathrm{d}x + [2u_x u_t]_0^1 = 0 + 0 = 0, \tag{737}$$

where we have again utilized the PDE (733) and note the boundary terms vanish since $u_x = 0$ on $\partial(0,1)\times(0,\infty)$.

This shows that $e(t)$ is constant in time. Together with the fact that $e(0) = (s-1)^2$, we conclude

$$e(t) = (s-1)^2, \quad \text{for all } t \in (0, \infty). \tag{738}$$

□

S17.6. Explain whether the ordinary differential equation

$$5y'' + \left(\frac{y'}{x}\right)^2 + 4y^2 = 0 \tag{739}$$

has a unique smooth solution in a neighborhood of the origin when initial conditions $y(0) = 1$ and $y'(0) = 0$ are applied.

Solution:

We claim the given ODE does *not* admit a unique smooth solution in a neighborhood of the origin with the given conditions. Suppose y is a smooth solution to the given ODE. Then the initial conditions imply $y(x) = 1 + cx^2 + \delta(x)$ for some scalar c and $\delta(x) = o(x^2)$ as $x \to 0$. We verify our claim by showing c can consistently take on either of two possible values when deriving the local behavior of y near the origin. We do not hope to find y exactly as that would be equivalent to solving the ODE. Instead we seek an asymptotic estimate of y, and so, as $x \to 0$, observe

$$\begin{aligned} 5y'' &= -4y^2 - \left(\frac{y'}{x}\right)^2 \\ &= -4(1 + cx^2 + v)^2 - \left(\frac{2cx + v'}{x}\right)^2 \\ &\sim -4(1 + cx^2)^2 - 4c^2 \\ &= -4(1 + 2cx^2 + c^2x^4) - 4c^2 \\ &\sim -4(1 + c^2). \end{aligned} \tag{740}$$

Since $v = o(x^2)$ as $x \to 0$, we know

$$y'' = 2c + v'' \sim 2c \quad \text{as } x \to 0. \tag{741}$$

Combining the previous two results reveals

$$10c \sim -4(1 + c^2), \quad \text{as } x \to 0, \tag{742}$$

which occurs when

$$4c^2 + 10c + 4 = 0 \quad \Longrightarrow \quad c = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 4 \cdot 4}}{2 \cdot 4} = -\frac{1}{2},\ -2. \tag{743}$$

This reveals either $y \sim 1 - 2x^2$ as $x \to 0$ or $y \sim 1 - \frac{1}{2}x^2$ as $x \to 0$. These two possibilities reveal that the ODE does not admit a unique smooth solution in a neighborhood of the origin. □

S17.7. Evolutionary rock-paper-scissors are used to model interactions among bacteria. Consider three species of bacteria, with relative abundances R (rock), P (paper), and S (scissors), respectively. You may assume that $P + R + S = 1$. A R-type bacteria tends to out compete S-type bacteria, but is itself out competed by P-type bacteria. The growth rate of the R-population is therefore proprotional to the number of interactions in each R-type bacteria has with S-types, minus the number of interactions of P-types, i.e.,

$$\dot{R} = R(S - P). \tag{744}$$

Similarly,

$$\dot{S} = S(P - R) \quad \text{and} \quad \dot{P} = P(R - S). \tag{745}$$

a) Describe all of the possible behaviors of the system if $R = 0$ at $t = 0$.

b) Show that , if all three populations are present in the system at $t = 0$, then none of the types of bacteria will go extinct.

Solution:

a) Since the expression for $\dot{R}$ is a multiple of R, it follows that $R = 0$ for all time $t \in [0, \infty)$. Additionally,

$$\begin{aligned} \dot{S} &= S(P - 0) = SP, \\ \dot{P} &= P(0 - S) = -SP. \end{aligned} \tag{746}$$

from which we see the only fixed points occur when either $S = 0$ or $P = 0$. Together with the fact $1 = R + S + P = S + P$, we deduce the only equilibrium points are $(R, S, P) = (0, 1, 0)$ and $(R, S, P) = (0, 0, 1)$.

We now deduce the possible behaviors. If $S = 0$ at time $t = 0$, then $(R, S, P) = (0, 0, 1)$ for all time. If $P = 0$ at time $t = 0$, then $(R, S, P) = (0, 1, 0)$ for all time. If $S, P > 0$ at time $t = 0$, then $\dot{S} > 0$ for all time, such that

$$\lim_{t \to \infty} (R, S, P) = (0, 1, 0). \tag{747}$$

Indeed, in this case $\dot{S} > 0$ and S is bounded above, and the monotone convergence theorem implies S converges. The only possible limit is the fixed point at $(0, 1, 0)$, and so the result (747) follows.

b) Our hypothesis asserts $R > 0$, $S > 0$, and $P > 0$ at time $t = 0$. We seek to show this same set of inequalities holds for all time. It then suffices to show $RSP > 0$ for all time. Observe

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[RSP] &= \dot{R}SP + R\dot{S}P + RS\dot{P} \\ &= RSP(S-P) + RSP(P-R) + RSP(R-S) \\ &= RSP[S-P+P-R+R-S] \\ &= 0.\end{aligned} \tag{748}$$

Therefore, the quantity RSP is constant in time and, thus, positive for all time. Hence none of the types of bacteria will go extinct.

□

S17.8. The space $y > 0$ is filled with non-Newtonian fluid, initially at rest. A plate at $y = 0$ is set into motion at time $t = 0$. The fluid velocity $u(t, y)$ obeys the equation

$$u_t = -\tau_y \quad \text{in } (0, \infty) \times (0, \infty), \tag{749}$$

with boundary conditions

$$u(t, 0) = 1 \quad \text{and} \quad u(t, +\infty) = 0, \tag{750}$$

and the initial condition $u(0, y) = 0$ for $y > 0$. The variable τ obeys the constitutive equation

$$\tau = (u_y)^2 \,. \tag{751}$$

a) Try to derive a similarity solution, i.e., look for a solution of the form $u(t, y) = f(\eta)$, where $\eta = y/\delta(t)$ for some function δ that you should determine, by applying only the boundary condition $u(t, 0) = 1$. Show that this similarity solution can neither be compatible with the other boundary condition nor with the initial condition.

b) To find a solution that is compatible with all boundary and initial conditions, we modify the constitutive equation to

$$\tau = \begin{cases} (u_y)^2 & \text{if } u_y < 0, \\ 0 & \text{if } u_y \geq 0. \end{cases} \tag{752}$$

Derive a similarity solution that satisfies all of the initial and boundary conditions.

Hint: Start by assuming that the solution breaks down into two parts: $0 < y < Y(t)$, in which $\tau \neq 0$ and $y > Y(t)$ in which $\tau = 0$. Derive continuity conditions that must be applied at $y = Y(t)$. You need to solve for the function $Y(t)$ as well as for $f(\eta)$.

Solution:

a) First suppose $u(t, y) = f(\eta)$ and observe

$$u_t = f'(\eta)\eta_t = f'(\eta) \cdot -\frac{y\delta'}{\delta^2} = -\frac{\delta'}{\delta} \cdot \eta f'(\eta). \tag{753}$$

Similarly,

$$\tau_y = \frac{\partial}{\partial y}\left[(u_y)^2\right] = \frac{\partial}{\partial y}\left[\left(\frac{f'(\eta)}{\delta}\right)^2\right] = \frac{2f'(\eta)}{\delta}\cdot\frac{f''(\eta)}{\delta}\cdot\frac{1}{\delta} = \frac{2}{\delta^3}f'(\eta)f''(\eta). \tag{754}$$

Consequently, the similarity solution satisfies

$$0 = u_t + \tau_y = -\frac{\delta'}{\delta}\cdot\eta f' + \frac{2}{\delta^3}f'f'' = \frac{f'}{\delta}\left[\frac{2f''}{\delta^2} - \delta'\eta\right]. \tag{755}$$

Assuming $f' \neq 0$, we deduce

$$\frac{f''(\eta)}{\eta} = \frac{\delta^2\delta'}{2}. \tag{756}$$

Since the left and right hand sides are functions of different variables, there is $\alpha \in \mathbb{R}$ such that

$$\frac{\delta^2\delta'}{2} = \alpha \quad\Longrightarrow\quad \frac{\delta^3}{3} = 2\alpha t \quad\Longrightarrow\quad \delta = (6\alpha t)^{1/3} = \beta t^{1/3}, \tag{757}$$

where $\beta = (6\alpha)^{1/3}$. We may assume $\beta = 1$ so that $\alpha = 1/6$. Then integrating the left hand side of (756) yields

$$f = \frac{\eta^3}{36} + c_1\eta + c_2, \tag{758}$$

for scalars $c_1, c_2 \in \mathbb{R}$. Applying the boundary condition $u(t,0) = 1$ yields

$$1 = u(t,0) = f(0) = \frac{0}{36} + c_1 0 + c_2 \quad\Longrightarrow\quad c_2 = 1. \tag{759}$$

The other boundary condition does not hold since

$$|u(t,+\infty)| = \lim_{\eta\to\infty}|f(\eta)| = \lim_{\eta\to\infty}\left|\frac{\eta^3}{36} + c_1\eta + 1\right| = +\infty. \tag{760}$$

b) To satisfy the boundary conditions, the required continuity conditions are that $u(Y(t),t) = 0$ and $u_y(Y(t),t) = 0$. This implies there is $\eta_0 > 0$ such that $f(\eta_0) = 0$ and $f'(\eta_0) = 0$. Consequently,

$$0 = f'(\eta_0) = \frac{\eta_0^2}{12} + c_2 \quad\Longrightarrow\quad c_2 = -\frac{\eta_0^2}{12}, \tag{761}$$

and so

$$0 = f(\eta_0) = \frac{\eta_0^3}{36} + c_2\eta_0 + 1 = -\frac{\eta_0^3}{18} + 1 \quad \Longrightarrow \quad \eta_0 = 18^{1/3} \quad \Longrightarrow \quad c_2 = -\frac{18^{2/3}}{12}. \tag{762}$$

Therefore,

$$f(\eta) = \frac{\eta^3}{36} - \frac{18^{2/3}}{12} \cdot \eta + 1 \tag{763}$$

and

$$\eta_0 = \frac{Y}{\delta} \quad \Longrightarrow \quad Y = \eta_0\delta = 18^{1/3}t^{1/2}. \tag{764}$$

Compiling our results, we conclude

$$u(x,t) = \begin{cases} f(\eta) & \text{if } y < Y(t), \\ 0 & \text{otherwise,} \end{cases} \tag{765}$$

where $\eta = yt^{-1/3}$, Y is given in (764), and f is given in (763).

□

2016 Fall

F16.1 Show that $u(x) = -\frac{1}{4\pi|x|}$ with $x \in \mathbb{R}^3$ satisfies $\Delta u(x) = \delta(x)$ in the sense of distribution, i.e., for each smooth ϕ with compact support

$$\int_{\mathbb{R}^3} u(x)\Delta\phi(x)\ \mathrm{d}x = \phi(0). \tag{766}$$

Solution:

First we compute the partial derivatives of u at $x \neq 0$. For $x \neq 0$ and $i \in \{1,2,3\}$ observe

$$\partial_{x_i} u = -\frac{1}{4\pi} \cdot -\frac{1}{|x|^2} \cdot \frac{x_i}{|x|} = \frac{x_i}{4\pi|x|^3}, \tag{767}$$

which implies

$$\partial_{x_i x_i} u = \frac{1}{4\pi}\left[x_i \cdot -\frac{3}{|x|^4} \cdot \frac{x_i}{|x|} + \frac{1}{|x|^3}\right] = \frac{1}{4\pi}\left[\frac{-3x_i^2}{|x|^5} + \frac{1}{|x|^3}\right]. \tag{768}$$

Thus for $x \neq 0$

$$\Delta u = \sum_{i=1}^{3} \partial_{x_i x_i} u = \frac{1}{4\pi}\left[\frac{-3|x|^2}{|x|^5} + \frac{3}{|x|^3}\right] = \frac{1}{4\pi}\left[-\frac{3}{|x|^3} + \frac{3}{|x|^3}\right] = 0. \tag{769}$$

Now fix $\varepsilon > 0$, choose any smooth ϕ with compact support, and observe

$$\int_{\mathbb{R}^3} u(x)\Delta\phi(x)\ \mathrm{d}x = \underbrace{\int_{\mathbb{R}^3 - B(0,\varepsilon)} u(x)\Delta\phi(x)\ \mathrm{d}x}_{I_\varepsilon} + \underbrace{\int_{B(0,\varepsilon)} u(x)\Delta\phi(x)\ \mathrm{d}x}_{J_\varepsilon} = I_\varepsilon + J_\varepsilon, \tag{770}$$

where I_ε and J_ε are defined to be the underbraced quantities. Integrating by parts, we see

$$\begin{aligned} I_\varepsilon &= \int_{\mathbb{R}^3 - B(0,\varepsilon)} -Du(x) \cdot D\phi(x)\ \mathrm{d}x - \int_{\partial B(0,\varepsilon)} u(x)\frac{\partial \phi}{\partial \nu}\ \mathrm{d}\sigma \\ &= \int_{\mathbb{R}^3 - B(0,\varepsilon)} \Delta u(x)\phi(x)\ \mathrm{d}x + \int_{\partial B(0,\varepsilon)} \frac{\partial u}{\partial \nu}\phi - u(x)\frac{\partial \phi}{\partial \nu}\ \mathrm{d}\sigma \\ &= \underbrace{\int_{\partial B(0,\varepsilon)} \frac{\partial u}{\partial \nu}\phi\ \mathrm{d}\sigma}_{K_\varepsilon} + \underbrace{\int_{\partial B(0,\varepsilon)} -u(x)\frac{\partial \phi}{\partial \nu}}_{N_\varepsilon}\ \mathrm{d}\sigma \\ &= K_\varepsilon + N_\varepsilon, \end{aligned} \tag{771}$$

where K_ε and N_ε are the underbraced quantities, and where ν is the outward normal along $B(0,\varepsilon)$, thereby making $-\nu$ the outward normal along the boundary of $\mathbb{R}^3 - B(0,\varepsilon)$. Note the third line above holds since $\Delta u = 0$ in $\mathbb{R}^3 - B(0,\varepsilon)$. Next observe

$$|N_\varepsilon| \leq \int_{\partial B(0,\varepsilon)} u(x) \left|\frac{\partial u}{\partial \nu}\right| \ \mathrm{d}\sigma \leq \|D\phi\|_\infty \int_{\partial B(0,\varepsilon)} u(x)\ \mathrm{d}\sigma = \|D\phi\|_\infty \int_{\partial B(0,\varepsilon)} \frac{1}{4\pi\varepsilon}\ \mathrm{d}\sigma, \tag{772}$$

and, since $|\partial B(0,\varepsilon)| = 4\pi\varepsilon^2$,

$$|N_\varepsilon| \leq \|D\phi\|_\infty \cdot \frac{4\pi\varepsilon^2}{4\pi\varepsilon} = \|D\phi\|_\infty \varepsilon. \tag{773}$$

We note $\|D\phi\|_\infty < \infty$ since ϕ is smooth and has compact support, thereby making $\phi_{x_i}(\partial B(0,\varepsilon))$ compact for each i. Hence $0 \leq \lim_{\varepsilon\to 0^+} |N_\varepsilon| \leq \lim_{\varepsilon\to 0^+} \|D\phi\|_\infty \varepsilon = 0$, which implies by the squeeze lemma that $\lim_{\varepsilon\to 0^+} N_\varepsilon = 0$.

For each $x \in \partial B(0,\varepsilon)$ observe

$$\frac{\partial u}{\partial \nu}(x) = Du \cdot \nu = \sum_{i=1}^{3} \frac{x_i}{4\pi\varepsilon^3} \cdot \frac{x_i}{\varepsilon} = \frac{|x|^2}{4\pi\varepsilon^4} = \frac{1}{4\pi\varepsilon^2} = \frac{1}{|\partial B(0,\varepsilon)|}. \tag{774}$$

Consequently,

$$\lim_{\varepsilon\to 0^+} K_\varepsilon = \lim_{\varepsilon\to 0^+} \int_{\partial B(0,\varepsilon)} \frac{\partial u}{\partial \nu}\phi\ \mathrm{d}\sigma = \lim_{\varepsilon\to 0^+} \fint_{\partial B(0,\varepsilon)} \phi\ \mathrm{d}\sigma = \phi(0). \tag{775}$$

Next employing the use of polar coordinates reveals

$$|J_\varepsilon| \leq \int_{B(0,\varepsilon)} u|\Delta\phi|\ \mathrm{d}x \leq \|D^2\phi\|_\infty \int_{B(0,\varepsilon)} \frac{\mathrm{d}x}{4\pi|x|} = \|D^2\phi\|_\infty \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{4\pi r}\ \mathrm{d}\sigma \mathrm{d}r, \tag{776}$$

and the integrand simplifies to yield

$$|J_\varepsilon| \leq \|D^2\phi\|_\infty \int_0^\varepsilon r\ \mathrm{d}r = \|D^2\phi\|_\infty \frac{\varepsilon^2}{2} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0. \tag{777}$$

Whence $\lim_{\varepsilon\to 0^+} J_\varepsilon = 0$. Compiling our results, we conclude

$$\int_{\mathbb{R}^3} u(x)\Delta\phi(x)\ \mathrm{d}x = \lim_{\varepsilon\to 0^+} I_\varepsilon + J_\varepsilon = \lim_{\varepsilon\to 0^+} K_\varepsilon + N_\varepsilon + J_\varepsilon = \phi(0) + 0 + 0 = \phi(0), \tag{778}$$

as desired. □

F16.2. Consider the following nonlinear drift-diffusion equation

$$\theta_t = \Delta(\theta^2) + \nabla \cdot (x\theta) \ \text{ in } \mathbb{R}^n \times (0,\infty). \tag{779}$$

a) Let θ_1 and θ_2 be smooth nonnegative solutions of the given PDE with ordered initial data $\theta_1(x,0) \leq \theta_2(x,0)$. Show then that $\theta_1(x,t) \leq \theta_2(x,t)$ for all $t > 0$.

b) Show that for any constant $C > 0$ the function $U(x) := \max\left[(C - \frac{|x|^2}{4}), 0\right]$ is a weak stationary solution of the above equation, i.e.,

$$\int_{\mathbb{R}^n} \big(\nabla(U^2) + xU\big) \cdot \nabla\phi(x) \ \mathrm{d}x = 0 \tag{780}$$

for any compactly support and smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$.

Solution:

a) Set $w := \theta_2 - \theta_1$. It suffices to show $w \geq 0$ in $\mathbb{R} \times (0,\infty)$. By our hypothesis, $w \geq 0$ on $\mathbb{R} \times \{t = 0\}$, and

$$w_t = \Delta\big(w \underbrace{(\theta_2 + \theta_1)}_{=:\phi}\big) + \nabla \cdot (xw) = \Delta(w\phi) + \nabla \cdot (wx). \tag{781}$$

Assume $\Delta\phi \in L^\infty(\mathbb{R} \times (0,\infty))$ and choose $M > \|\Delta\phi\|_{L^\infty(\mathbb{R}^n)\times(0,\infty)} + n + 1$. Now let $\varepsilon > 0$ and set $v := w + \varepsilon \exp(Mt)$. This implies $v \geq \varepsilon > 0$ on $\mathbb{R}^n \times \{t = 0\}$. By way of contradiction, suppose there exist a point in $\mathbb{R}^n \times (0,\infty)$ at which $v = 0$. Let $(\overline{x}, \overline{t}) \in \mathbb{R}^n \times (0,\infty)$ be such a point with $\overline{t} > 0$ the smallest time at which this occurs. Then $v_t(\overline{x}, \overline{t}) \leq 0$ and, since $\overline{x}$ is a local minimizer of $v(\cdot, \overline{t})$, we see $\Delta v(\overline{x},\overline{t}) \geq 0$. Consequently, at $(\overline{x},\overline{t})$,

$$\begin{aligned}
0 \geq v_t &= w_t + M\varepsilon \exp(M\overline{t}) \\
&= \Delta(w\phi) + \nabla\cdot(w\overline{x}) + M\varepsilon\exp(M\overline{t}) \\
&= w\Delta\phi + 2Dw\cdot D\phi + \phi\Delta w + \nabla\cdot(w\overline{x}) + M\varepsilon\exp(M\overline{t}) \\
&= w\Delta\phi + \phi\Delta v + nw + Dw\cdot\overline{x} + M\varepsilon\exp(M\overline{t}) \\
&\geq w\Delta\phi + nw + M\varepsilon\exp(M\overline{t}) \\
&= \varepsilon\exp(M\overline{t})\,(M - \Delta\phi - n) \\
&\geq \varepsilon \\
&> 0,
\end{aligned} \tag{782}$$

which implies $0 > 0$, a contradiction. Note the fourth and fifth lines hold since $Dw = Dv = 0$ at $(\overline{x},\overline{t})$ and since $\phi \geq 0$, by hypothesis, and $\Delta w = \Delta v \geq 0$. The sixth line holds since $v = 0$ and the seventh line holds due to our choice of M and the fact the exponential term is at least unity. This contradiction proves $v > 0$ in $\mathbb{R}^n \times (0,\infty)$. Letting $\varepsilon \longrightarrow 0^+$, we deduce

$$w(x,t) = \lim_{\varepsilon\to 0^+} w(x,t) + \varepsilon\exp(Mt) = \lim_{\varepsilon\to 0^+} v(x,t) \geq \lim_{\varepsilon\to 0^+} 0 = 0, \quad \text{for all } (x,t)\in\mathbb{R}^n\times(0,\infty), \tag{783}$$

and the proof is complete.

b) Let $r := 2\sqrt{C}$ and note the support of U is precisely $B(0,r)$. Thus, for each $\phi \in C_c^\infty(\mathbb{R}^n\times(0,\infty))$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\nabla(U^2) + xU\right)\cdot\nabla\phi\ \mathrm{d}x &= \int_{\mathbb{R}^n} U\,(2\nabla U + x)\cdot\nabla\phi\ \mathrm{d}x \\
&= \int_{B(0,r)} U\,(2\nabla U + x)\cdot\nabla\phi\ \mathrm{d}x \\
&= \int_{B(0,r)} U\cdot 0\cdot\nabla\phi\ \mathrm{d}x \\
&= 0,
\end{aligned} \tag{784}$$

where, in $B(0,r)$,

$$x + 2\nabla U = x + 2\nabla\left[C - \frac{|x|^2}{4}\right] = x + 2\cdot -\frac{x}{2} = 0. \tag{785}$$

□

F16.3. Consider the system of ODE for the pair $(x(t), v(t))$ of real-valued functions

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -\frac{1}{2}x(x^2-1) - v, \end{cases} \tag{786}$$

for $t > 0$, with initial conditions $x(0) = x_0$ and $v(0) = v_0$.

a) Find all stationary points and sketch the local trajectories.

b) Define a nonzero function $E(a, b)$ such that $E \geq 0$ and $\dot{E}(x(t), v(t)) \leq 0$ for all $t > 0$ when $(x(t), v(t))$ solves the ODE system.

Solution:

a) Since $\dot{x} = 0$ if and only if $v = 0$, we see the equilibrium points are $(0, 0)$ and $(\pm 1, 0)$. The Jacobian matrix is

$$J(x, v) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial v \\ \partial\dot{v}/\partial x & \partial\dot{v}/\partial v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{3x^2+1}{2} & -1 \end{pmatrix}. \tag{787}$$

This implies

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -1 \end{pmatrix}, \tag{788}$$

which has eigenvalues λ that satisfy

$$0 = \lambda(\lambda+1) - \frac{1}{2} \quad \Longrightarrow \quad 0 = 2\lambda^2 + 2\lambda - 1 \quad \Longrightarrow \quad \lambda = \frac{-2 \pm \sqrt{4 - 4(-2)}}{4} = \frac{-1 \pm \sqrt{3}}{2}, \tag{789}$$

and so $(0, 0)$ is a saddle. Also,

$$J(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \tag{790}$$

which has eigenvalues λ that satisfy

$$0 = \lambda(\lambda+1) + 1 = \lambda^2 + \lambda + 1 \quad \Longrightarrow \quad \lambda = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1}}{2} = \frac{-1 \pm i\sqrt{3}}{2}, \tag{791}$$

and so around $(\pm 1, 0)$ are inward pointing spirals. A sketch is given in the figure below.

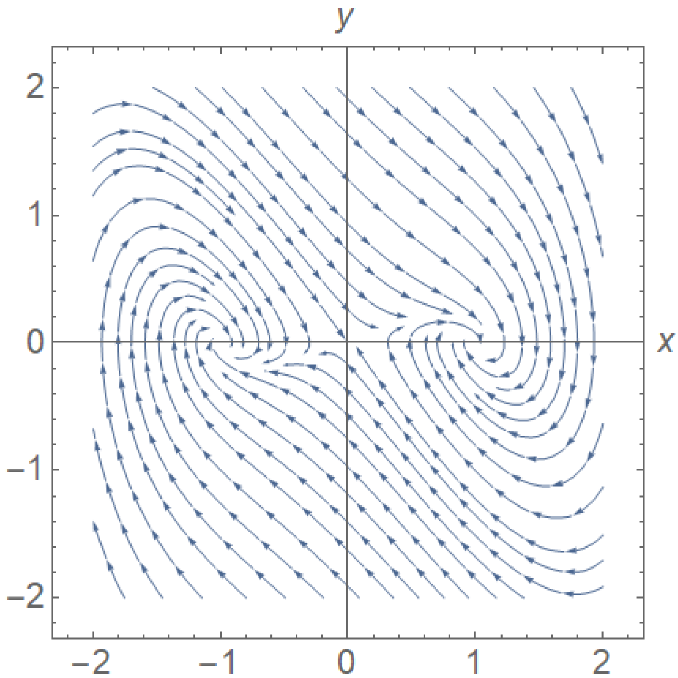

Figure 21: Phase plane for F16.3.

b) Consider an analogous system $(z(t), w(t))$ without the damping term, i.e.,

$$\begin{cases} \dot{z} &= w, \\ \dot{w} &= -\frac{1}{2} z(z^2 - 1). \end{cases} \tag{792}$$

Then

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\dot{w}}{\dot{z}} = \frac{-\frac{1}{2}z(z^2-1)}{w} \implies 0 = w\mathrm{d}w + \left(\frac{z^3}{2} - \frac{z}{2}\right)\mathrm{d}z \implies C = \frac{w^2}{2} + \frac{z^4}{8} - \frac{z^2}{4}. \tag{793}$$

From this, we define the energy

$$E(a,b) := \frac{b^2}{2} + \frac{a^2}{4}\left(\frac{a^2}{2} - 1\right) + 1. \tag{794}$$

If $|a| \geq 2$, then

$$E(a,b) \geq \frac{b^2}{2} + \frac{a^2}{4} \cdot 0 + 1 \geq 0. \tag{795}$$

If $|a| < 2$ then

$$E(a,b) \geq \frac{b^2}{2} + \frac{a^2}{4}\cdot(-1) + 1 \geq 0 - \frac{2^2}{4}\cdot -1 + 1 = 0. \tag{796}$$

This shows $E(a,b) \geq 0$. Lastly, observe

$$\dot{E}(x,v) = v\dot{v} + \frac{x}{2}(x^2-1)\dot{x} = v\left(-\frac{x}{2}(x^2-1) - v\right) + \frac{x}{2}(x^2-1)v = -v^2 \leq 0. \tag{797}$$

□

Remark: Here we give an alternative solution to F16.3b. Note the ODE system models a damped equation, with the undamped form being

$$\dot{\tilde{x}} = \tilde{v}, \quad \dot{\tilde{v}} = -\frac{1}{2}\tilde{x}\left(\tilde{x}^2 - 1\right), \tag{798}$$

and note the undamped $(\tilde{x}, \tilde{v})$ system is Hamiltonian since

$$\frac{\partial \dot{\tilde{x}}}{\partial \tilde{x}} + \frac{\partial \dot{\tilde{v}}}{\partial \tilde{v}} = 0 + 0 = 0. \tag{799}$$

Consequently, there exists a function $H(\tilde{x}, \tilde{v})$ such that $H_{\tilde{x}} = -\dot{\tilde{v}}$ and $H_{\tilde{v}} = \dot{\tilde{x}}$. Integrating reveals

$$H = \int -\dot{\tilde{v}}\ \mathrm{d}\tilde{x} = \int \frac{1}{2}\tilde{x}\left(\tilde{x}^2 - 1\right)\ \mathrm{d}x = \frac{1}{8}\left(\tilde{x}^2 - 1\right)^2 + f(\tilde{v}), \tag{800}$$

for some function $f(\tilde{v})$. Similarly, there exists $g(\tilde{x})$ such that

$$H = \int \dot{\tilde{x}}\ \mathrm{d}\tilde{v} = \frac{\tilde{v}^2}{2} + g(\tilde{x}). \tag{801}$$

Combining these results we write

$$H(\tilde{x}, \tilde{v}) = \frac{1}{8}\left(\tilde{x}^2 - 1\right)^2 + \frac{\tilde{v}^2}{2}. \tag{802}$$

Returning to the damped ODE system, set $E(x,v) := H(x,v)$ and note $E(x,v) \geq 0$ since both terms in (802) are squared. Moreover,

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{8}(x^2-1)^2 + \frac{v^2}{2}\right] = \frac{x}{2}(x^2-1)\dot{x} + v\dot{v} = \frac{x}{2}(x^2-1)v + v\left[-\frac{x}{2}(x^2-1) - v\right] = -v^2 \leq 0. \tag{803}$$

△

F16.4. Consider $\phi : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ solving the Hamilton-Jacobi equation

$$\phi_t + |\nabla \phi| = 0, \tag{804}$$

with initial data $\phi(x, 0) = \max(|x|^2 - 1, 0)$. Show that $\phi(x, t) = 0$ when $t = |x| - 1$.

Solution:

Define $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) := \max(|x|^2 - 1, 0)$ and the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ by $H(p) = |p|$. Then the PDE may be rewritten as

$$\begin{cases} \phi_t + H(D\phi) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \phi = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{805}$$

Then the associated Lagrangian L is given by the Fenchel transform

$$L(v) = \max_{p \in \mathbb{R}^n} p \cdot v - H(p) = \max_{p \in \mathbb{R}^n} p \cdot v - |p|. \tag{806}$$

If $|v| > 1$, then observe letting $p = \mu v$ for a scalar $\mu \in \mathbb{R}$ yields

$$\lim_{\mu \longrightarrow \infty} p \cdot v - |p| = \lim_{\mu \longrightarrow \infty} \mu |v|^2 - \mu |v| = |v|(|v| - 1) \lim_{\mu \longrightarrow \infty} \mu = +\infty. \tag{807}$$

Alternatively, if $|v| < 1$, then

$$p \cdot v - |p| \leq |p||v| - |p| = |p|(|v| - 1) \leq 0, \tag{808}$$

with equality holding on the right hand side when $p = 0$. This implies

$$L(v) = \begin{cases} +\infty & \text{if } |v| > 1, \\ 0 & \text{if } |v| \leq 1. \end{cases} \tag{809}$$

By the Hopf-Lax formula,

$$\phi(x, t) = \min_{y \in \mathbb{R}^n} t \cdot L\left(\frac{x - y}{t}\right) + g(y) = \min_{y \in \mathcal{A}} g(y), \tag{810}$$

where $\mathcal{A} := \{y \in \mathbb{R}^n : |x-y| \leq t\}$. Now, if $t = |x| - 1$ and $y \in \mathcal{A}$, then

$$|x| - |y| \leq |x-y| \leq t = |x| - 1 \quad \Longrightarrow \quad |y| \geq 1 \quad \Longrightarrow \quad |y|^2 - 1 \geq 0 \quad \Longrightarrow \quad g(y) = |y|^2 - 1. \tag{811}$$

Of course, $\min_{|y|\geq 1} g(y) = \min_{|y|\geq 1} |y|^2 - 1 = 0$. Whence for $t = |x| - 1$ we conclude

$$u(x,t) = \min_{|x-y|\leq t} g(y) = 0, \tag{812}$$

as desired. □

F16.5. Consider the smooth solution u of the Dirichlet problem

$$\begin{cases} -\nabla \cdot (\beta(x)\nabla u(x)) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{813}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^n$ and the functions $\beta : \Omega \to (0,\infty)$ and $g : \partial\Omega \to \mathbb{R}$ are smooth. Suppose there is a bijective map $\phi : \Omega \to \hat{\Omega}$ satisfying

$$D\phi(x) = \frac{1}{\beta(x)} Q(x), \tag{814}$$

where Q satisfies $\det(Q) = 1$ and $Q^T Q = I$. Show that then $u(x) := \hat{u}(\phi(x))$ is the smooth solution of

$$\begin{cases} -\Delta \hat{u}(\hat{x}) = 0 & \text{in } \hat{\Omega}, \\ \hat{u}(\hat{x}) = g(\phi^{-1}(\hat{x})) & \text{on } \partial\hat{\Omega}. \end{cases} \tag{815}$$

Solution:

For each function $f : \Omega \to \mathbb{R}$, let $\hat{f} : \hat{\Omega} \to \mathbb{R}$ be defined by $\hat{f}(\hat{x}) := f(\phi(x))$. For differentiable f,

$$f_{x_i}(x) = \sum_{j=1}^{n} \tilde{f}_{x_j}(\phi(x))(\phi_j)_{x_i}(x) \quad \Longrightarrow \quad f_i = \hat{f}_j \phi_{j,i} \ \text{ in } \Omega, \tag{816}$$

where on the right hand side we have adopted the compact tensor notation and repeated index summation convention. Now let $q \in C_c^\infty(\Omega)$. Integration by parts yields

$$0 = \int_\Omega -\nabla \cdot (\beta \nabla u) q \ \mathrm{d}x = \int_\Omega \beta u_i q_i \ \mathrm{d}x = \int_\Omega \beta \hat{u}_j \phi_{j,i} \hat{q}_k \phi_{k,i} \ \mathrm{d}x = \int_\Omega \frac{1}{\beta} \hat{u}_j Q_{j,i} Q^T_{i,k} \hat{q}_k \ \mathrm{d}x, \tag{817}$$

where the final equality holds by (814). Since Q is orthogonal, it follows that

$$0 = \int_\Omega \frac{1}{\beta} \hat{u}_j \delta_{j,k} \hat{q}_k \ \mathrm{d}x = \int_\Omega \frac{1}{\beta} \hat{u}_j \hat{q}_j \ \mathrm{d}x = \int_{\hat{\Omega}} \hat{u}_j \hat{q}_j \ \mathrm{d}\hat{x}, \tag{818}$$

where the final equality holds since the determinant of the Jacobian matrix is $|D\phi| = |Q/\beta| = 1/\beta$, again noting Q is orthogonal. Integrating by parts once more, we see

$$0 = \int_{\hat{\Omega}} -\hat{u}_{jj}\hat{q}\ \mathrm{d}\hat{x}. \tag{819}$$

Because (819) holds for arbitrarily chosen q, this holds for all $\hat{q} \in C_c^\infty(\hat{\Omega})$, thereby revealing $-\Delta\hat{u} = 0$ in $\hat{\Omega}$. Lastly, let $\hat{x} \in \partial\hat{\Omega}$. Since ϕ is a smooth bijection, there exists $x \in \partial\Omega$ such that $\hat{x} = \phi(x)$, and so

$$\hat{u}(\hat{x}) = \hat{u}(\phi(x)) = u(x) = g(x) = g(\phi^{-1}(\hat{x})). \tag{820}$$

This completes the proof. $\square$

F16.6. Let D be a subset of $\mathbb{R}^n$. Show that the smallest C for the Poincaré Inequality

$$\int_D u^2 \, \mathrm{d}x \leq C \int_D |\nabla u|^2 \, \mathrm{d}x, \quad \text{for all } u \in H_0^1(D), \tag{821}$$

can be obtained from an eigenvalue problem. State the eigenvalue problem and explain why.

Solution:

Consider[32] the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } D, \\ \quad\; u = 0 & \text{on } \partial D, \end{cases} \tag{822}$$

which has the weak formulation for $u \in H := H_0^1(D)$ given by

$$\int_D Du \cdot Dv - \lambda uv \, \mathrm{d}x = 0, \quad \text{for all } v \in H. \tag{823}$$

The Laplacian operator $L = -\Delta$ is symmetric and elliptic since

$$Lu = -\sum_{i,j=1}^n \delta_{ij} u_{x_i x_j} \quad \Longrightarrow \quad \sum_{i,j=1}^n \delta_{ij}\xi_i\xi_j = |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{824}$$

By the theory of elliptic operators,[33] this implies L has a countable set of orthogonal eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}} \subseteq H$ that forms a basis for $L^2(D)$ and each $\phi_n \in H$ has an associated eigenvalue λ_n. Additionally, $\lambda_n > 0$ for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \lambda_n = +\infty$. This Hilbert space H is equipped with the scalar product $\langle \cdot, \cdot \rangle$ given by

$$\langle u, v \rangle := \int_D \nabla u \cdot \nabla v \, \mathrm{d}x, \tag{825}$$

with the norm denoted $\|u\|_H := \sqrt{\langle u, u \rangle}$.

Now let $u \in H \subset L^2(D)$. Since $\{\phi_n\}_{n\in\mathbb{N}}$ forms a basis for $L^2(D)$, there exists scalars $\{c_n\}_{n\in\mathbb{N}}$ such that

$$u = \sum_{n\in\mathbb{N}} c_n \phi_n. \tag{826}$$

[32] This is similar to F11.8.

[33] See §6.5 of the PDE text by Evans.

We may assume the collection $\{\phi_n\}_{n\in\mathbb{N}}$ is orthonormal. For each $m, n \in \mathbb{N}$, this implies

$$\int_D \lambda_n \phi_n \phi_m \ \mathrm{d}x = \int_D \nabla\phi_n \cdot \nabla\phi_m \ \mathrm{d}x = \langle \phi_n, \phi_m\rangle = \delta_{nm}, \tag{827}$$

and so

$$\|u\|^2_{L^2(D)} = \int_D u^2 \ \mathrm{d}x = \sum_{n,m\in\mathbb{N}} c_n c_m \int_D \phi_n\phi_m \ \mathrm{d}x = \sum_{n,m\in\mathbb{N}} c_n c_m \frac{\delta_{nm}}{\lambda_n} = \sum_{n\in\mathbb{N}} \frac{c_n^2}{\lambda_n}. \tag{828}$$

Again using the orthonormality of $\{\phi_n\}_{n\in\mathbb{N}}$, we see

$$\|u\|^2_H = \langle \sum_{n\in\mathbb{N}} c_n\phi_n, \sum_{m\in\mathbb{N}} c_m\phi_m\rangle = \sum_{n,m\in\mathbb{N}} c_n c_m \langle\phi_n, \phi_m\rangle = \sum_{n,m\in\mathbb{N}} c_n c_m \delta_{nm} = \sum_{n\in\mathbb{N}} c_n^2. \tag{829}$$

If $u \in H$ is nonzero and such that the inequality in (821) is an equality, then

$$\|u\|^2_H = C\|u\|^2_{L^2(D)} = C\sum_{n\in\mathbb{N}} \frac{c_n^2}{\lambda_n} \leq \frac{C}{\lambda_1}\sum_{n\in\mathbb{N}} c_n^2 = \frac{C}{\lambda_1}\|u\|^2_H, \tag{830}$$

noting $\{\lambda_n\}_{n\in\mathbb{N}}$ is an increasing sequence. Dividing by $\|u\|^2_H$ and multiplying by λ_1, the above inequality becomes $\lambda_1 \leq C$. Since the inequality in (830) becomes an equality precisely when $C = \lambda_1$, we see the choice of $C = \lambda_1$ is possible. We therefore deduce the smallest possible value for C is $C = \lambda_1$. This shows the smallest value of C has been obtained from the eigenvalue problem, as desired. □

F16.7. Consider the one dimensional reaction diffusion equation[34]

$$u_t - u_{xx} = f(u) \quad \text{in } \mathbb{R} \times (0, \infty), \tag{831}$$

where $f(0) = f(1) = 0$, $f'(0) > 0$, and $0 < f(u) < f'(0)u$ for $0 < u < 1$. We would like to show there is a positive traveling wave solution with speed c, i.e., of the form

$$u(x,t) = U(x - ct) > 0 \tag{832}$$

satisfying

$$u(-\infty) = 1, \quad u(+\infty) = 0, \tag{833}$$

for the range of speeds $c \geq 2\sqrt{f'(0)}$. To show our claim, first observe U and $V := U'$ satisfy

$$U' = V, \quad V' = -cV - f(U). \tag{834}$$

We will analyze the (U,V) phase plane to find the traveling wave solution, which connects the steady states $(0,0)$ and $(1,0)$ of the system.

a) By studying the ODE near $(0,0)$, conclude no positive wave solution exists if $c < 2\sqrt{f'(0)}$. By studying the ODE system near $(1,0)$, show that there is at most one traveling wave solution.

b) For $c \geq 2\sqrt{f'(0)}$, let λ be one of the eigenvalues at $(0,0)$. Show that $V'/U' < \lambda$ on the line $V = \lambda U$.

c) Using b) and other observations from the phase plane, show there is exactly one trajectory connecting $(1,0)$ to $(0,0)$ in the (U,V) system when $c \geq 2\sqrt{f'(0)}$.

Solution:

a) The Jacobian matrix $J(U,V)$ for this system is given by

$$J(U,V) = \begin{pmatrix} \partial U'/\partial U & \partial U'/\partial V \\ \partial V'/\partial U & \partial V'/\partial V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f'(U) & -c \end{pmatrix}, \tag{835}$$

[34] This prompt is excessively wordy and is quite similar to the style of prompts given by Roper in 266A in Fall 2016.

which has eigenvalues that satisfy

$$0 = \lambda(\lambda + c) + f'(u) = \lambda^2 + \lambda c + f'(u) \quad \Longrightarrow \quad \lambda = \frac{-c \pm \sqrt{c^2 - 4f'(U)}}{2}. \tag{836}$$

Note $(U,V) = (0,0)$ and $(U,V) = (1,0)$ are fixed points of the ODE system since $f(0) = f(1) = 0$. If $c < 2\sqrt{f'(0)}$, then $c^2 - 4'f(0) < 0$, which implies $J(0,0)$ has eigenvalues λ with $\text{Re}(\lambda) = -c/2$ and $\text{Im}(\lambda) = \pm\sqrt{4f'(0) - c^2}/2$, which implies the origin forms a spiral. Regardless of its stability, the origin forming a spiral would imply there exists points along each trajectory where $U \leq 0$, a contradiction to the fact U is assume to be positive. Thus, no positive wave solution exists if $c < 2\sqrt{f'(0)}$.

We now analyze the behavior near $(U,V) = (1,0)$. Taylor's theorem asserts for each $u \in (0,1)$ there exists $\xi_u \in (0,u)$ such that

$$f(u) = f(0) + f'(0)u + \frac{f''(\xi_u)}{2} \cdot u^2. \tag{837}$$

Combined with the fact $f(0) = 0$ and $0 < f(u) < f'(0)u$, we see $f'' < 0$ in $(0,1)$. Thus, as f is concave down on $(0,1)$ and decreasing as it approaches $u = 1$ where $f(1) = 0$, we deduce $f'(1) < 0$. Consequently, (836) implies the eigenvalues of $J(1,0)$ are real-valued, with one positive and one negative. This shows $(1,0)$ forms a saddle, for which there is a single stable manifold originating along a trajectory from $(0,0)$. (The single other stable manifold originates at infinity.)

b) For $c \geq 2\sqrt{f'(0)}$ and $v = \lambda u$, observe

$$\begin{aligned} \frac{v'}{u'} &= \frac{-cv - f(u)}{v} \\ &= \frac{-c\lambda u - f(u)}{\lambda u} \\ &= -c - \frac{1}{\lambda}\frac{f(u)}{u} \\ &< -c - \frac{1}{\lambda} f'(0) \\ &= \frac{1}{\lambda}\big(-c\lambda - f'(0)\big) \\ &= \lambda. \end{aligned} \tag{838}$$

The first equality follows from the prompt. The second equality holds since we are analyzing along the line $v = \lambda u$. The fourth equality follows from the fact $0 < f(u) < f'(0)u$. The final equality then follows from the eigenvalue equation (836).

c) Note the eigenvalues of $J(0,0)$ are nonpositive (for simplicity, we assume they are both negative). Let us consider the region R enclosed by the line $V=0$, $U=1$, and $V=\lambda U$, with λ an eigenvalue of $J(0,0)$. Along the curve $V=\lambda U$ in $\{U>0\}\times\{V<0\}$ we have $U'=V<0$, and so our earlier result reveals

$$\frac{\mathrm{d}V}{\mathrm{d}U}=\frac{V'}{U'}<\lambda<0 \quad\Longrightarrow\quad \left|\frac{\mathrm{d}V}{\mathrm{d}U}\right|>\lambda \quad\text{and}\quad V'>0. \tag{839}$$

In plain words, this reveals trajectories along $V=\lambda U$ point up and to the left *into* R. Additionally, along the line $U=1$ in $\{U>0\}\times\{V<0\}$ we have $U'<0$ and $V'<0$. Again, trajectories remain in R. Along $V=0$ we see $U'=0$ and $V'=-f(U)<0$ for $U\in(0,1)$. These results imply the unstable trajectory leaving $(1,0)$ enters into R and is contained inside R, with the only possible termination at the fixed point $(0,0)$. An illustration is provided below.

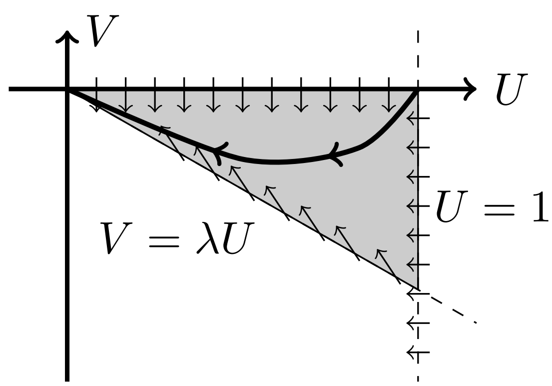

Figure 22: Illustration for F16.7c with the region R shaded.

□

F16.8. Let u be a solution of the wave equation

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = \phi & \text{on } \mathbb{R}^3 \times \{t=0\}, \\ u_t = \psi & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{840}$$

where the initial data is supported in the ball of radius R about the origin. Let x_0 be a point in $\mathbb{R}^3$ with $|x_0| > R$.

a) Find the largest time T_1 for which we can guarantee that $u(x_0,t)$ must be zero for all $t \in [0, T_1)$.

b) Find the smallest time T_2 for which we can guarantee that $u(x_0,t)$ must be zero for all $t > T_2$.

Solution:

a) First set $v(x,t) := u(|c|x,t)$. Then

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ v = \tilde{\phi} & \text{on } \mathbb{R}^3 \times \{t=0\}, \\ v_t = \tilde{\psi} & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{841}$$

where $\tilde{\phi}(x) := \phi(|c|x)$ and $\tilde{\psi}(x) := \psi(|c|x)$. Now set

$$r := \frac{|x_0| - R}{|c|}, \tag{842}$$

and define the wave energy $e : [0,r) \to \mathbb{R}$ by

$$e(t) := \frac{1}{2} \int_{S(t)} v_t^2 + |Dv|^2 \text{ d}x, \tag{843}$$

where $S(t) := B(x_0/|c|, r-t)$ and the arguments of v in the integrand are implicit. Then, by the definition of R and choice of the constant r and function v, $e(0) = 0$. Differentiating in time yields

$$\dot{e} = \int_{S(t)} v_t v_{tt} + Dv \cdot Dv_t \text{ d}x + \frac{1}{2} \int_{\partial S(t)} \left(v_t^2 + |Dv|^2\right) \cdot (-\nu) \cdot \nu \text{ d}\sigma, \tag{844}$$

where $-\nu$ is the Eulerian velocity of the boundary $\partial S(t)$ and ν is the outward normal along the boundary of $S(t)$. Integrating by parts yields

$$\dot{e} = \int_{S(t)} v_t \underbrace{(v_{tt} - \Delta v)}_{=0} \, \mathrm{d}x + \int_{\partial S(t)} -\frac{1}{2}\left(v_t^2 + |Dv|^2\right) + v_t \frac{\partial v}{\partial \nu} \mathrm{d}\sigma, \tag{845}$$

where the first integral evaluates to zero by (841). Noting that

$$\left| v_t \frac{\partial v}{\partial \nu} \right| \leq |v_t||Dv| \leq \frac{1}{2}\left(v_t^2 + |Dv|^2\right), \tag{846}$$

we see

$$\dot{e} = \int_{\partial S(t)} -\frac{1}{2}\left(v_t^2 + |Dv|^2\right) + v_t \frac{\partial v}{\partial \nu} \mathrm{d}\sigma \leq 0. \tag{847}$$

This shows e is monotonically decreasing in time. Thus $e(t) \leq e(0) = 0$. However, since the integrand of e is nonnegative, $e(t) \geq 0$ for all $t \in [0, r]$. Therefore $e(t) = 0$ for all $t \in [0, r)$. This implies v is constant in the cone $\{(x,t) : t \in [0,r), \ x \in S(t)\}$. The fact $v(x_0/|c|, 0) = 0$ then implies $v = 0$ inside the cone. Therefore, at the tip of the cone, we see $\lim_{t \to r^-} v(x_0/|c|, t) = 0$, and so the longest time T_1 at which we can guarantee $0 = u(x_0, t) = v(x_0/|c|, t)$ for $t \in [0, T_1)$ is $T_1 = r$.

b) The result in a) showed that it would take at least time

$$T_1 = \frac{|x_0| - R}{|c|}. \tag{848}$$

This is simply derived from the distance of x_0 to the closest possible point in the support of the initial data divided by the wave's propagation speed. Similarly, the smallest time T_2 for which we can guarantee u must be zero for all $t > T_2$ is the distance of x_0 to the farthest away point that can be in the support of the initial data divided by the wave's propagation speed, i.e.,

$$T_2 = \frac{|x_0| + R}{|c|}. \tag{849}$$

Note this follows since, by Huygen's principle, the fact our PDE is in $\mathbb{R}^3$ implies the solution u only depends on the boundary information of the wavefront. So, once both wavefronts pass a point x, $u(x,t)$ will thenceforth be zero.

□

2016 Spring

S16.1.

a) Show that the point $(x, y) = (-1, 0)$ is a stable fixed point for the system of ODE

$$\dot{x} = 4y^3, \quad \dot{y} = -2(x+1). \tag{850}$$

b) Now consider the following modification of the system of ODE:

$$\begin{aligned} \dot{x} &= 4y^3 + (x+1) - (x+1)\left[(x+1)^2 + y^4\right], \\ \dot{y} &= -2(x+1) + 2y^3 - 2y^3\left[(x+1)^2 + y^4\right]. \end{aligned} \tag{851}$$

Show the modified system has a limit cycle.

Solution:

a) First note

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial x}\left[4y^3\right] + \frac{\partial}{\partial x}\left[-2(x+1)\right] = 0, \tag{852}$$

and so the system in Hamiltonian. Therefore, there exists a function $H(x, y)$ such that $H_x = -\dot{y}$ and $H_y = \dot{x}$. Integrating reveals

$$H(x, y) = \int \dot{x}\ \mathrm{d}y = y^4 + f(x), \tag{853}$$

for some function $f(x)$. Similarly, there exists a function $g(y)$ such that

$$H(x, y) = \int -\dot{y}\ \mathrm{d}x = (x+1)^2 + g(y). \tag{854}$$

Combining these results, we may assume

$$H(x, y) = y^4 + (x+1)^2. \tag{855}$$

Note $H(x, y) > 0$ for all $(x, y) \neq (-1, 0)$, $H(-1, 0) = 0$, and

$$\dot{H}(x, y) = H_x\dot{x} + H_y\dot{y} = -\dot{y}\dot{x} + \dot{x}\dot{y} = 0. \tag{856}$$

In consideration of the Lyapunov function $H(x,y)$, we conclude, by Lyapunov's theorem, that $(-1,0)$ is a stable fixed point.

b) Consider the same function $H(x,y)$ as in a). Differentiating in time reveals

$$\begin{aligned}
\dot{H}(x,y) &= 2(x+1)\dot{x} + 4y^3\dot{y} \\
&= 2(x+1)\left(4y^3 + (x+1) - (x+1)\left[(x+1)^2 + y^4\right]\right) \\
&\quad + 4y^3\left(-2(x+1) + 2y^3 - 2y^3\left[(x+1)^2 + y^4\right]\right) \\
&= 2(x+1)^2 - 2(x+1)^2\left[(x+1)^2 + y^4\right] + 8y^6 - 8y^6\left[(x+1)^2 + y^4\right] \\
&= \left[2(x+1)^2 + 8y^6\right]\left(1 - \left[(x+1)^2 + y^4\right]\right) \\
&= 2\left[(x+1)^2 + 4y^6\right]\left[1 - H(x,y)\right].
\end{aligned} \tag{857}$$

Now define $S := \{(x,y) : H(x,y) = 1\}$. Consider any trajectory originating from a point in $(a,b) \in S$. Since $H(a,b) = 1$, we see

$$\dot{H}(a,b) = 2\left[(a+1)^2 + 4b^6\right]\left[1 - H(a,b)\right] = 2\left[(a+1)^2 + 4b^6\right] \cdot 0 = 0. \tag{858}$$

This implies the H will remain constant in time, i.e., the trajectory will be contained in S. Now let $R := \{(x,y) : \frac{1}{2} \leq H(x,y) \leq \frac{3}{2}\}$. Note R is closed since H is convex and the level sets of convex sets are closed. Also, R is bounded since $|H| \to \infty$ as $\|(x,y)\| \to \infty$. Additionally, R contains a trajectory since $S \subset R$. Lastly, R does not contain any fixed point since $(-1,0) \notin R$ as $H(-1,0) = 0 < \frac{1}{2}$. Whence the Poincaré-Bendixson theorem asserts R contains a closed orbit, i.e., a limit cycle.

□

S16.2. Consider the conservation law

$$g(u)_t + f(u)_x = 0 \text{ in } \mathbb{R} \times (0, \infty), \tag{859}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

a) Derive the Rankine-Hugoniot condition for shock speeds.

b) Use the result from a) to solve the following Riemann problem with $g(u) = u^2/2$, $f(u) = u^3/3$, and with initial data

$$u(x, 0) = \begin{cases} 1 & \text{if } x \in (0, 1/3), \\ 0 & \text{otherwise.} \end{cases} \tag{860}$$

Solution:

a) (Return and complete.)

b) We proceed by using the method of characteristics. Let $F(p, q, z, x, t) = z(q + zp)$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the ODE system for the characteristics:

$$\begin{cases} \dot{x}(s) = F_p = z^2 & x(0) = x_0, \\ \dot{t}(s) = F_q = 1 & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = z^2 p + zq = 0, & z(0) = g(x_0), \end{cases} \tag{861}$$

where we take $g(\alpha) := u(\alpha, 0)$. This implies $t = s$ and z is constant along characteristics. Thus,

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t z^2(\tau)\ \mathrm{d}\tau = x_0 + tg^2(x_0) = \begin{cases} x_0 + t & \text{if } x_0 \in (0, 1/3), \\ x_0 & \text{otherwise.} \end{cases} \tag{862}$$

So, if $0 \le x_0 = x - t < 1/3$, then $u(x,t) = g(x_0) = 1$. And, if $x \le 0$ or $x > 1/3$, then $u(x,t) = g(x_0) = 0$. Lastly, we must consider the case when $0 \le x \le t$. If u is of the form $u(x,t) = v(x/t)$ in this region, then

$$0 = g(u)_t + f(u)_x = vv' \cdot -\frac{x}{t^2} + v^2 v' \cdot \frac{x}{t} = \frac{vv'}{t}\left[v - \frac{x}{t}\right]. \tag{863}$$

Thus taking $v = x/t$ yields a solution of the desired form, and this is the unique option of this form that would yield continuity of u at the boundaries $x = 0$ and $x = t$.

We now observe the characteristics crash immediately at $(1, 0)$, and so a shock occurs. Let $(s(t), t)$ be a parameterization of this curve. Then $s(0) = 0$ and our RH condition from above yields

$$\dot{s}(t) = \frac{f(u_\ell) - f(u_r)}{g(u_\ell) - g(u_r)} = \frac{\frac{1}{3} - 0}{\frac{1}{2} - 0} = \frac{2}{3}. \tag{864}$$

Consequently,

$$s(t) = \frac{2t + 1}{3} \quad \text{in } [0, 1], \tag{865}$$

and so, in $\mathbb{R} \times (0, 1)$,

$$u(x, t) = \begin{cases} x/t & \text{if } 0 \leq x \leq t, \\ 1 & \text{if } t \leq x \leq (2t + 1)/3, \\ 0 & \text{otherwise.} \end{cases} \tag{866}$$

This limited time interval is because at time $t = 1$ we also find another collision (see Figure 23 below) at $(s(1), 1) = (1, 1)$. Again using the RH condition reveals

$$\dot{s}(t) = \frac{f(u_\ell) - f(u_r)}{g(u_\ell) - g(u_r)} = \frac{\frac{1}{3}(x/t)^3 - 0}{\frac{1}{2}(x/t)^2 - 0} = \frac{2x}{3t}. \tag{867}$$

Taking $x = s(t)$, using separation of variables, and then using the condition $s(1) = 1$ reveals $s(t) = t^{2/3}$ in $[1, \infty)$. Whence in $\mathbb{R} \times (1, \infty)$

$$u(x, t) = \begin{cases} x/t & \text{if } 0 \leq x \leq s(t) = t^{2/3}, \\ 0 & \text{otherwise.} \end{cases} \tag{868}$$

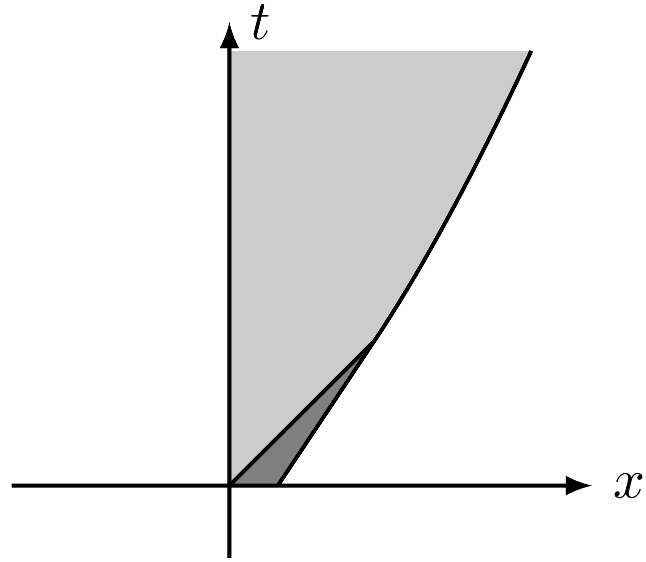

Figure 23: Plot of $u(x,t)$. In the light gray region, $u = x/t$. In the dark gray region, $u = 1$, and $u = 0$ elsewhere.

$\square$

S16.3. Solve

$$\Delta u = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in } [-1, 1] \times \mathbb{R} \times \mathbb{R}. \tag{869}$$

You may assume $u \longrightarrow 0$ as $|y| \longrightarrow +\infty$ or $|z| \longrightarrow +\infty$, and the boundary conditions on the walls $x = 0$ and $x = 1$ are

$$u(0, y, z) = \left.\frac{\partial u}{\partial x}\right|_{1,y,z} = 0. \tag{870}$$

Seek a solution of the form

$$u(x, y, z) = \frac{1}{(2\pi)^2} \iint e^{i(\ell y + mz)} \hat{u}(x, \ell, m) \ \mathrm{d}\ell \mathrm{d}m, \tag{871}$$

and find the function $\hat{u}$. You do not need to evaluate this integral for u.

Solution:

It suffices to identify $\hat{u}$ since then u is given by (871). Using the properties of derivatives with the Fourier transform, we see

$$\Delta u = \frac{1}{(2\pi)^2} \iint e^{i(\ell y + mz)} \left[\hat{u}_{xx} - (m^2 + \ell^2)\hat{u}\right] \mathrm{d}\ell \mathrm{d}m. \tag{872}$$

This implies

$$\hat{u}_{xx} - (m^2 + \ell^2)\hat{u} = \iint e^{-i(\ell y + mz)} \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \ \mathrm{d}y\mathrm{d}z = e^{-i(\ell y_0 + mz_0)} \delta(x - x_0). \tag{873}$$

Note also

$$u(0, y, z) = 0 \quad \Longrightarrow \quad \hat{u}(0, \ell, m) = 0 \quad \text{and} \quad u_x(1, y, z) = 0 \quad \Longrightarrow \quad \hat{u}_{xx}(1, \ell, m) = 0. \tag{874}$$

Therefore, we seek a Green's function $\hat{u}(x, \ell, m)$ such that

$$\begin{cases} \hat{u}_{xx} - \alpha^2 \hat{u} = e^{-i(\ell y_0 + mz_0)} \delta(x - x_0) & \text{in } [0, 1] \times \mathbb{R} \times \mathbb{R}, \\ \hat{u} = 0 & \text{on } \{x = 0\} \times \mathbb{R} \times \mathbb{R}, \\ \hat{u}_x = 0 & \text{on } \{x = 1\} \times \mathbb{R} \times \mathbb{R}, \end{cases} \tag{875}$$

where $\alpha := \sqrt{\ell^2 + m^2}$. The associated homogeneous ODE has a solution of the form $ce^{\pm\alpha x}$, and so

$$\hat{u}(x,\ell,m) = \begin{cases} c_1 e^{\alpha x} + c_2 e^{-\alpha x} & \text{if } x < x_0, \\ c_3 e^{\alpha x} + c_4 e^{-\alpha x} & \text{if } x > x_0. \end{cases} \tag{876}$$

The first boundary condition reveals

$$0 = c_1 e^0 + c_2 e^0 = c_1 + c_2 \quad \implies \quad c_2 = -c_1. \tag{877}$$

Additionally,

$$0 = \alpha\left(c_3 e^{\alpha} - c_4 e^{-\alpha}\right) \quad \implies \quad c_4 = c_3 e^{2\alpha}. \tag{878}$$

By the continuity of $\hat{u}$ in x, we see

$$c_1\left(e^{\alpha x_0} - e^{-\alpha x_0}\right) = \lim_{x\to x_0^-} \hat{u} = \lim_{x\to x_0^+} \hat{u} = c_3\left(e^{\alpha x_0} + e^{\alpha(2-x_0)}\right) \quad \implies \quad c_1 = \frac{e^{\alpha x_0} + e^{\alpha(2-x_0)}}{e^{\alpha x_0} - e^{-\alpha x_0}} \cdot c_3. \tag{879}$$

Lastly, considering an arbitrarily small neighborhood about x_0 reveals

$$\begin{aligned} e^{-i(\ell y_0 + m z_0)} &= \lim_{\varepsilon\to 0^+} \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-i(\ell y_0 + m z_0)} \delta(x - x_0)\ \mathrm{d}x \\ &= \lim_{\varepsilon\to 0^+} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \hat{u}_{xx} - \alpha^2 \hat{u}\ \mathrm{d}x \\ &= \hat{u}_x(x_0^+, \ell, m) - \hat{u}_x(x_0^-, \ell, m). \end{aligned} \tag{880}$$

Thus,

$$e^{-i(\ell y_0 + m z_0)} = c_3 \alpha\left(e^{\alpha x_0} - e^{\alpha(2-x_0)}\right) - c_1 \alpha (e^{\alpha x_0} + e^{\alpha x_0}) \tag{881}$$

Combining (879) and (881), one may readily solve for c_1 and c_3 (upon application of some tedious algebra). For such values c_1 and c_2, we deduce

$$\hat{u}(x,\ell,m) = \begin{cases} c_1\left(e^{\alpha x} - e^{-\alpha x}\right) & \text{if } x < x_0, \\ c_3\left(e^{\alpha x} + e^{\alpha(2-x_0)}\right) & \text{if } x > x_0, \end{cases} \tag{882}$$

and we are done. □

S16.4. Solve the Hamilton-Jacobi equation

$$\phi_t + |\phi_x| = 0, \quad x \in \mathbb{R}, \; t > 0 \tag{883}$$

with initial data

$$\phi(x, 0) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{884}$$

Solution:

Define the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ by $H(p) := |p|$ and set $g(x)$ to be zero if $x \leq 0$ and 0 otherwise. Then the PDE may be written as

$$\begin{cases} \phi_t + H(D\phi) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ \phi = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{885}$$

Taking the Fenchel transform gives the Lagrangian L to be

$$L(v) := \sup_{p \in \mathbb{R}} pv - H(p) = \sup_{p \in \mathbb{R}} pv - |p|. \tag{886}$$

If $v > 1$, then observe

$$\lim_{p \longrightarrow +\infty} pv - H(p) = (v - 1) \lim_{p \longrightarrow +\infty} p = +\infty, \tag{887}$$

and if $v < -1$, then

$$\lim_{p \longrightarrow -\infty} pv - H(p) = (v + 1) \lim_{p \longrightarrow -\infty} p = +\infty. \tag{888}$$

Now, if $|v| \leq 1$, then

$$pv - H(p) = pv - \text{sgn}(p) \cdot p = p\,(v - \text{sgn}(p)) \leq 0, \tag{889}$$

with strict equality precisely when $p = 0$. Combining (886), (887), (888), and (889), we deduce

$$L(v) = \begin{cases} +\infty & \text{if } |v| > 1, \\ 0 & \text{if } |v| \leq 1. \end{cases} \tag{890}$$

From the Hopf-Lax formula, we then know

$$u(x,t) = \min_{y\in\mathbb{R}} \left(t \cdot L\left(\frac{x-y}{t}\right) + g(y) \right). \tag{891}$$

By (890), the Lagrangian simplifies this to

$$u(x,t) = \min_{|x-y|\leq t} g(y). \tag{892}$$

Note $|x-y| \leq t$ implies y is bounded below by $x - t$. Thus there is $y \leq 0$ in the set of all feasible y (i.e., points such that $|x-y| \leq t$) precisely when $x - t \leq 0$. Consequently, using this and the definition of g reveals

$$u = \begin{cases} 0 & \text{if } x \leq t, \\ 1 & \text{if } x \geq t. \end{cases} \tag{893}$$

□

S16.5. A toy model of the propagation of an action potential along a neuron is given by the PDE

$$u_t = u_{xx} + f(u), \tag{894}$$

where $f(u)$ may be assumed to be continuously differentiable. Propagating action potential solutions of this PDE are given by traveling waves, i.e., solutions of the form $u(x,t) = u(x-ct)$, that tend to (different) constant values: $u \longrightarrow u_\star^-$ as $x \longrightarrow -\infty$ and $u \longrightarrow u_\star^+$ as $x \longrightarrow +\infty$.

a) Explain why the limits as $x \longrightarrow \pm\infty$ must correspond to values $u_\star^\pm$ at which $f(u_\star^\pm) = 0$ and $f'(u_\star^\pm) \leq 0$.

b) Suppose $u_\star^- < u_\star^+$ and $u(x-ct)$ is monotone increasing in $\eta = x - ct$. prove the wave moves leftward (i.e., $c < 0$) or rightward according to whether

$$\int_{u_\star^-}^{u_\star^+} f(u)\ \mathrm{d}u \gtreqless 0. \tag{895}$$

c) Now consider the specific function $f(u) = -(u-u_0)(u-u_1)(u-u_2)$, where $u_0 < u_1 < u_2$ are all constants. Guessing that $\mathrm{d}u/\mathrm{d}\eta = B(u-u_0)(u-u_2)$ for some constant B, find the traveling wave solution $u(\eta)$ and its velocity c.

Solution:

a) Assume $u(x,t) = v(x-ct) = v(\eta)$, where $\eta := x - ct$. Plugging this into the PDE reveals

$$0 = u_t - u_{xx} - f(u) = -cv' - v'' - f(v) \quad \Longrightarrow \quad v'' = -cv' - f(v). \tag{896}$$

This may be rewritten as the ODE system

$$v' = w, \quad w' = -cw - f(v), \tag{897}$$

Note the fixed points of the ODE system occur at all points of the form $(\overline{v}, 0)$, where $f(\overline{v}) = 0$. For fixed t, since $\eta \longrightarrow \pm\infty$ as $x \longrightarrow \pm\infty$, it follows that in order for v to converge to a value $u_\star^\pm$ we must have $f(u_\star^\pm) = 0$. The associated Jacobian matrix is given by

$$J(v,w) := \begin{pmatrix} \partial v'/\partial v & \partial v'/\partial w \\ \partial w'/\partial v & \partial w'/\partial w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f'(v) & -c \end{pmatrix}. \tag{898}$$

The eigenvalues to $J(v,w)$ satisfy

$$0 = \lambda(\lambda + c) + f'(v) \quad \Longrightarrow \quad \lambda = \frac{-c \pm \sqrt{c^2 - 4f'(v)}}{2}. \tag{899}$$

For a fixed point $(\overline{v}, 0)$, this shows that if $f'(\overline{v}) > 0$ then $(\overline{v}, 0)$ forms a stable node. However, this cannot be the case if there is a trajectory leaving the fixed point $(u_\star^-, 0)$ and terminating at $(u_\star^+, 0)$. Consequently, we must have $f'(u_\star^\pm) \leq 0$.

b) Since $u = u(\eta)$, we may use the change of variables with $du = u'(\eta)\ \mathrm{d}\eta$ to write

$$\int_{u_\star^-}^{u_\star^+} f(u)\ \mathrm{d}u = \int_{-\infty}^{\infty} f(u)u'\ \mathrm{d}\eta = \int_{-\infty}^{\infty} [-u'' - cu']u'\ \mathrm{d}\eta. \tag{900}$$

However, because $u' \longrightarrow 0$ as $\eta \longrightarrow \pm\infty$, we see

$$\int_{-\infty}^{\infty} u'u''\ \mathrm{d}\eta = -\int_{-\infty}^{\infty} u''u'\ \mathrm{d}\eta + \underbrace{\left[(u')^2\right]_{-\infty}^{\infty}}_{=0} \quad \Longrightarrow \quad 2\int_{-\infty}^{\infty} u'u''\ \mathrm{d}\eta = 0. \tag{901}$$

Therefore,

$$\int_{u_\star^-}^{u_\star^+} f(u)\ \mathrm{d}u = -c\int_{-\infty}^{\infty} (u')^2\ \mathrm{d}\eta, \tag{902}$$

and because u' is not identically zero (as $u_\star^- \neq u_\star^+$), the integral on the right hand side is nonzero and, thus, positive as the integrand is nonnegative. Whence

$$c = -\frac{\int_{u_\star^-}^{u_\star^+} f(u)\ \mathrm{d}u}{\int_{-\infty}^{\infty} (u')^2\ \mathrm{d}\eta} \quad \Longrightarrow \quad \mathrm{sgn}(c) = \mathrm{sgn}\left(-\int_{u_\star^-}^{u_\star^+} f(u)\ \mathrm{d}u\right) = -\mathrm{sgn}\left(\int_{u_\star^-}^{u_\star^+} f(u)\ \mathrm{d}u\right). \tag{903}$$

This shows the sign of c is opposite that of the integral in (895), as desired.

c) First note our hypothesis implies

$$u'' = B\left[u'(u - u_2) + (u - u_0)u'\right] = Bu'\left[2u - u_0 - u_2\right] = B^2(u - u_0)(u - u_2)(2u - u_0 - u_2). \tag{904}$$

Then (896) implies

$$\begin{aligned} 0 &= u'' + cu' + f(u) \\ &= B^2(u-u_0)(u-u_2)(2u-u_0-u_2) + cB(u-u_0)(u-u_2) - (u-u_0)(u-u_1)(u-u_2) \\ &= (u-u_0)(u-u_2)\left[B^2(2u-u_0-u_2) + cB - (u-u_1)\right]. \end{aligned} \tag{905}$$

Because this holds for each u, this implies

$$0 = B^2(2u-u_0-u_2)+cB-(u-u_1) \quad\Longrightarrow\quad (1-2B^2)u = cB - B^2(u_0+u_2)+u_1 \quad\Longrightarrow\quad B = \pm\frac{1}{\sqrt{2}}, \tag{906}$$

where the final implication holds through equating powers of u. Because u' goes to zero in the limits as $\eta \longrightarrow \pm\infty$, we know $u \in [u_0, u_2]$; otherwise, $|u'|$ would be increasing as $\eta \longrightarrow \pm\infty$. Since $u \in (u_0, u_2)$ and u is monotone increasing, the formula for u' reveals $B < 0$, and so $B = -1/\sqrt{2}$. Then

$$0 = -\frac{c}{\sqrt{2}} + \frac{u_0+u_2}{2} + u_1 \quad\Longrightarrow\quad c = \sqrt{2}\left[\frac{u_0+u_2}{2} + u_1\right]. \tag{907}$$

All that remains is to solve for u in terms of η, c, and B. Observe

$$B\mathrm{d}\eta = \frac{\mathrm{d}u}{(u-u_0)(u-u_2)}. \tag{908}$$

Using partial fraction expansions, we know there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\frac{1}{(u-u_0)(u-u_2)} = \frac{\alpha_1}{u-u_0} + \frac{\alpha_2}{u-u_2} \quad\Longrightarrow\quad 1 = \alpha_1(u-u_2) + \alpha_2(u-u_0). \tag{909}$$

Plugging in $u = u_2$ and $u = u_0$ reveals

$$\alpha_1 = \frac{1}{u_0-u_2} \quad\text{and}\quad \alpha_2 = \frac{1}{u_2-u_0}. \tag{910}$$

Thus,

$$\begin{aligned} \int \frac{\mathrm{d}u}{(u-u_0)(u-u_2)} &= \frac{1}{u_2-u_0}\int \left[\frac{1}{u-u_2} - \frac{1}{u-u_0}\right]\,\mathrm{d}u \\ &= \frac{1}{u_2-u_0}\left[\ln(|u-u_2|) - \ln(|u-u_0|)\right] \\ &= \frac{1}{u_2-u_0}\ln\left(\frac{u_2-u}{u-u_0}\right), \end{aligned} \tag{911}$$

and, integrating the left hand side of (908), there exists $\alpha_3 \in \mathbb{R}$ such that

$$\ln\left(\frac{u_2 - u}{u - u_0}\right) = (u_2 - u_0)\,[B\eta + \alpha_3] \quad \Longrightarrow \quad \frac{u_2 - u}{u - u_0} = \underbrace{\exp\left((u_2 - u_0)\,[B\eta + \alpha_3]\right)}_{=:\varphi(\eta)} = \varphi(\eta), \tag{912}$$

which implies

$$u_2 - u = (u - u_0)\varphi \quad \Longrightarrow \quad u = \frac{u_2 + u_0\varphi}{\varphi + 1}. \tag{913}$$

This completes the proof.

□

S16.7. Consider the system of PDEs:[35]

$$\begin{cases} \rho_t + \rho_x u + \rho u_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \rho(u_t + u_x u) = k\partial_x[\rho^0/\rho] & \text{in } \mathbb{R}\times(0,\infty), \end{cases} \tag{914}$$

for $u(x,t)$ and $\rho(x,t)$, where $\rho^0(x) := \rho(x,0)$ and $k > 0$. Also consider $\phi : \mathbb{R}\times(0,\infty) \to \mathbb{R}$ defined by

$$\begin{cases} \phi_t(X,t) := u(\phi(X,t),t) & \text{in } \mathbb{R}\times(0,\infty), \\ \phi(X,t) = X & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{915}$$

where u and ρ are solutions of the above.

a) Define $R(X,t) := \rho(\phi(X,t),t)$ and $J(X,t) = \phi_X(X,t)$. Show that $R(X,t)J(X,t) = R(X,0)$.

b) Show that $R(X,0)\phi_{tt}(X,t) = k\phi_{XX}(X,t)$ in $\mathbb{R}\times(0,\infty)$.

c) Use the results of a) and b) to solve the system with initial conditions $u(x,0) = c\sin(x)$ and $\rho(x,0) = 1$, where $|c| < \sqrt{k}$. You can express you answer in terms of the inverse (f^{-1}) of the function $f(X) := X + c\sin(X)$, which exists provided $|c| < 1$.

Solution:

a) Observe

$$J(X,0) = \phi_X(X,0) = \partial_X[X] = 1 \quad\Longrightarrow\quad R(X,0)J(X,0) = R(X,0). \tag{916}$$

This verifies the equality at time $t = 0$. Differentiating in time, we see

$$\partial_t[RJ] = \dot{R}J + R\dot{J} = [\rho_X\phi_t + \rho_t]J + R[\phi_{Xt}] = [\rho_X u + \rho_t]\phi_X + \rho[u_x\phi_X] = \phi_X[\rho_X u + \rho_t + \rho u_x] = 0. \tag{917}$$

This shows RJ is constant in time, from which the result follows.

b) Observe

$$\rho^0(X) = R(X,0) = R(X,t) = J(X,t) = \rho(X,t)\phi_X(X,t) \tag{918}$$

[35] Although I applaud Teran's amazing abilities as a mathematician, his notation can be quite hard to follow. Here we diverge slightly from the prompt, but don't make it too 'easy' to read so that it might mirror the notation of a future exam...

and

$$\phi_{tt} = u_x\phi_t + u_t = u_x u + u_t = \frac{k}{\rho}\partial_x\left[\frac{\rho^0}{\rho}\right]. \tag{919}$$

Thus,

$$\rho^0\phi_{tt} = \frac{k\rho^0}{\rho}\partial_x\left[\frac{\rho^0}{\rho}\right] = k\phi_x\partial_x[\phi_x] = k\phi_x\phi_{xx}. \tag{920}$$

(There is an error here.)

c)

□

S16.8. Consider the parabolic PDE

$$\begin{cases} u_t - \Delta u + |u_{x_1}| = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{921}$$

where g is a continuous function with compact support. Show that there is at most one solution of the above problem that tends to zero as $|x| \to \infty$.

Solution:

Let u and v be two solutions to the given PDE that tend to zero as $|x| \to \infty$. Set $w := u - v$. It suffices to show $w = 0$ in $\mathbb{R}^n \times (0,\infty)$. Fix $\varepsilon > 0$ and $T > 0$. Since u and v are continuous in time and $\lim_{|x|\to\infty} |u(x,t)| = \lim_{|x|\to\infty} |v(x,t)| = 0$ for each $t \in (0,\infty)$, there exists a continuous function $\xi : [0,\infty) \to \mathbb{R}$ such that

$$|x| \geq \xi(t) \quad \Longrightarrow \quad \Longrightarrow \quad |u(x,t)|, |v(x,t)| \leq \varepsilon/2. \tag{922}$$

Set $R := \max_{[0,T]} \xi$, which is well-defined due to the compactness of $[0,T]$ and continuity of ξ. Then

$$|x| \geq R \quad \Longrightarrow \quad |u(x,t)|, |v(x,t)| \leq \varepsilon/2, \quad \text{for all } t \in [0,T], \tag{923}$$

and so application of the triangle inequality yields

$$|w(x,t)| \leq |u(x,t)| + |v(x,t)| \leq \varepsilon, \quad \text{for all } |x| \geq R \text{ and } t \in [0,T]. \tag{924}$$

This implies

$$\sup_{(\mathbb{R}^n - B(0,R))\times[0,T]} |w| \leq \varepsilon \quad \Longrightarrow \quad \sup_{\mathbb{R}^n\times[0,T]} w = \max\left\{ \sup_{\overline{U_T}} w,\ \varepsilon \right\}. \tag{925}$$

We claim

$$\sup_{\overline{U_T}} w = \sup_{\Gamma_T} w \leq \varepsilon, \tag{926}$$

where the inequality holds by (924) and the fact $w = g - g = 0$ on $B(0,R) \times \{t=0\}$, and so

$$\varepsilon \geq \sup_{\mathbb{R}^n\times[0,T]} w = \sup_{\mathbb{R}^n\times[0,T]} u - v - \varepsilon t \geq \sup_{\mathbb{R}^n\times[0,T]} u - v. \tag{927}$$

Now letting $\varepsilon \to 0^+$, we see

$$\sup_{\mathbb{R}^n\times[0,T]} u - v = \lim_{\varepsilon\to 0^+}\left(\sup_{\mathbb{R}^n\times[0,T]} u-v\right) \le \lim_{\varepsilon\to 0^+} \varepsilon = 0. \tag{928}$$

This shows

$$\sup_{\mathbb{R}^n\times[0,T]} u - v \le 0. \tag{929}$$

Since this holds for arbitrary $T > 0$, we may let $T \to \infty$ to deduce

$$\sup_{\mathbb{R}^n\times(0,\infty)} u - v \le 0. \tag{930}$$

Through likewise argument with u and v in the definition of w, we find

$$\sup_{\mathbb{R}^n\times(0,\infty)} v - u \le 0 \quad\Longrightarrow\quad \sup_{\mathbb{R}^n\times(0,\infty)} u - v \ge 0. \tag{931}$$

Therefore $u = v$ in $\mathbb{R}^n \times (0,\infty)$.

All that remains is to verify the equality in (926). Observe

$$\begin{cases} w_t - \Delta w = |v_{x_1}| - |u_{x_1}| - \varepsilon & \text{in } \mathbb{R}^n\times(0,\infty), \\ w = 0 & \text{on } \mathbb{R}^n\times\{t=0\}. \end{cases} \tag{932}$$

By way of contradiction, suppose w attains its max over $\overline{U_T}$ in its parabolic interior at a point $(x,t)\in U_T$. Then $w_t(x,t)\ge 0$ and, since x is a local maximizer of $w(\cdot,t)$, we know $\Delta w(x,t)\le 0$. Furthermore,

$$0 = Dw = Du - Dv \quad\Longrightarrow\quad Du = Dv \quad\Longrightarrow\quad u_{x_1} = v_{x_1}. \tag{933}$$

Thus, at (x,t), we obtain the inequality

$$0 \le w_t - \Delta w = |v_{x_i}| - |u_{x_i}| - \varepsilon = -\varepsilon < 0, \tag{934}$$

a contradiction. Whence the equality in (926) must hold. □

2015 Fall

F15.1. Consider the autonomous differential equation system

$$\dot{x} = -x + y^2, \quad \dot{y} = y - x^2. \tag{935}$$

a) Identify the fixed points of this equation and show (either by linearizing the equation or some other method) whether they are stable or unstable.

b) Sketch the trajectories for this differential equation in the (x, y) phase plane. You sketch should include the eigenvectors of the fixed points identified in a) if they correspond to features that can be seen in the phase plane. It should also include the asymptotic behavior of trajectories for large x and/or y.

Solution:

a) First note $\dot{x} = 0$ if either $x = y = 0$ or $x > 0$ and $y \neq 0$. Similarly, $\dot{y} = 0$ if $y = x = 0$ or $y > 0$ and $x \neq 0$. Thus the two fixed points of this system are $(0, 0)$ and $(1, 1)$. The Jacobian matrix $J(x, y)$ for this system is

$$J(x,y) := \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} -1 & 2y \\ -2x & 1 \end{pmatrix}. \tag{936}$$

This implies

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{937}$$

which has eigenvalues $\lambda = \pm 1$. Therefore $(0, 0)$ is a saddle point, which is unstable. Now observe

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = -1 + 1 = 0, \tag{938}$$

which implies the system is Hamiltonian, i.e., there exists $H(x, y)$ such that $H_y = \dot{x}$ and $H_x = -\dot{y}$. Integrating reveals

$$H(x,y) = \int \dot{x}\ \mathrm{d}y = \frac{y^3}{3} - xy + f(y) \quad \text{and} \quad H(x,y) = -\int \dot{y}\ \mathrm{d}x = \frac{x^3}{3} - xy + h(x), \tag{939}$$

for some functions $f(y)$ and $h(x)$. Combining these results, we deduce

$$H(x,y) = \frac{x^3+y^3}{3} - xy + \frac{1}{3}. \tag{940}$$

Observe $H(1,1) = (1+1)/3 - 1 + 1/3 = 0$ and

$$\nabla H(x,y) = \begin{pmatrix} x^2 - y \\ y^2 - x \end{pmatrix} \implies \nabla^2 H(x,y) = \begin{pmatrix} 2x & -1 \\ -1 & 2y \end{pmatrix} \implies \nabla^2 H(1,1) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{941}$$

where $\nabla^2 H$ is the Hessian matrix. The eigenvalues of $\nabla^2 H(1,1)$ satisfy

$$0 = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 \implies \lambda = \frac{4 \pm \sqrt{4^2 - 4\cdot 1\cdot 3}}{2} = 1, 3. \tag{942}$$

This shows $\nabla^2 H(1,1)$ is positive definite, and so $H(x,y) > H(1,1) = 0$ in a neighborhood of $(1,1)$ (excluding $(1,1)$). Moreover,

$$\dot{H}(x,y) = x^2\dot{x} + y^2\dot{y} - \dot{x}y - x\dot{y} = \dot{x}(x^2-y) + \dot{y}(y^2-x) = (y^2-x)(x^2-y) - (x^2-y)(y^2-x) = 0. \tag{943}$$

We have thus shown $H(1,1) = 0$, $H(x,y) > 0$ in a neighborhood of $(1,1)$, and $\dot{H}(x,y) \le 0$. Therefore Lyapunov's theorem tells us $(1,1)$ is stable.

b) The desired figure is below.

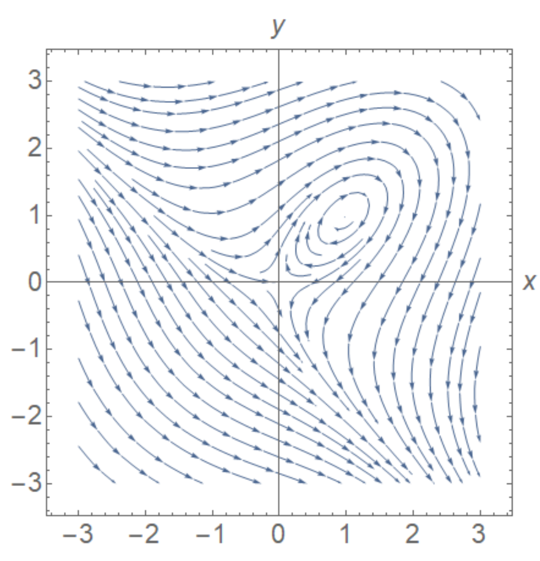

Figure 24: Phase Plane for F15.1.

Lastly, we analyze the behavior for large x and y, which reveals

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{y - x^2}{-x + y^2} \sim -\frac{x^2}{y^2} \quad \Longrightarrow \quad y^2 \ \mathrm{d}y \sim -x^2 \ \mathrm{d}x, \tag{944}$$

which implies there exists $C \in \mathbb{R}$ such that large trajectories approximately are of the form

$$x^3 + y^3 = C. \tag{945}$$

Alternatively, this behavior could be deduced directly from the Hamiltonian in (940) by considering level curves of H for large x and y.

□

F15.2. The LWR models the density of cars $\rho(x,t)$ on an infinite 1D road with flow in one direction via the PDE

$$\partial_t \rho + \partial_x \left(\rho(1-\rho)\right) = 0 \quad \text{in } \mathbb{R} \times [0,\infty), \tag{946}$$

where $F = \rho(1-\rho)$ denotes the flux of cars past a point in the road. The formula for F comes from the assumption that $u(x,t) := 1-\rho$ is the mean speed of cars on the road.

a) Under the assumption that (946) admits a unique solution, show that any state with uniform traffic $\rho(x,0) = \rho_0$ is stable, taking $\rho_0 \in (0,1)$. Specifically, show that if $\rho(x,t)$ is a solution of (946), then $\|\rho(\cdot,t) - \rho_0\|_{L^\infty(\mathbb{R})} \leq \|\rho(\cdot,0) - \rho_0\|_{L^\infty(\mathbb{R})}$.

b) Show that any step discontinuous function of the form

$$\rho(x,t) := \begin{cases} \rho_\ell & \text{if } x < vt, \\ \rho_r & \text{if } x > vt, \end{cases} \tag{947}$$

where $\rho_\ell, \rho_r \in (0,1)$ are both constants, satisfies the weak form of (946) (which you should derive) so long as the velocity of the discontinuity v satisfies a condition, which you should also derive.

c) The weak solutions of this PDE are not unique. To impose uniqueness, it is often assumed that car accelerations must be bounded. Namely, if $x = x(t)$ is the trajectory of a car on this road, then

$$\dot{x}(t) = u(x(t),t) \tag{948}$$

and

$$\ddot{x}(t) < \infty. \tag{949}$$

Note cars can decelerate infinitely quickly so that $\ddot{x}$ can be arbitrarily large and negative. Show that under this assumption, solutions of the form in b) are allowed only if $\rho_\ell < \rho_r$.

Solution:

a) We proceed using the method of characteristics, assuming ρ is a solution with uniform initial data ρ_0. Set $F(p,q,z,x,t) = q + p(1-2z)$. Taking $p = \rho_x$, $q = \rho_t$, and $z = \rho$, we deduce $F = 0$ in $\mathbb{R} \times [0,\infty)$

and obtain the ODE system

$$\begin{cases} \dot{x}(s) = F_p = 1 - 2z, & x(0) = x^0, \\ \dot{t}(s) = F_q = 1 & t(0) = 0, \\ \dot{z}(s) = F_q q + F_p p = 1q + (1-2z) = 0, & z(0) = rho_0. \end{cases} \tag{950}$$

This implies $t = s$ and ρ is constant along characteristics. Moreover, this implies

$$x = (1 - 2\rho_0)t + x^0, \tag{951}$$

i.e., the characteristics are parallel lines. Hence for $(x,t) \in \mathbb{R} \times (0,\infty)$ we deduce $\rho(x,t) = \rho_0$, and thus

$$\|\rho(\cdot,t) - \rho_0\|_{L^\infty(\mathbb{R})} = \|\rho_0 - \rho_0\|_{L^\infty(\mathbb{R})} = \|\rho(\cdot,0) - \rho_0\|_{L^\infty(\mathbb{R})}. \tag{952}$$

b) Since this PDE is a conservation law, the velocity v of the discontinuity must satisfy the Rankine-Hugenoit condition

$$v = \frac{f(\rho_\ell) - f(\rho_r)}{\rho_\ell - \rho_r}, \tag{953}$$

where we take $f(\rho) := \rho(1-\rho)$. We derive this as follows. First assume u is a smooth solution of the given PDE and let v be a test function, i.e., $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is smooth with compact support. Integration by parts reveals

$$0 = \int_0^\infty \int_{-\infty}^\infty v\left(u_t + f(u)_x\right)\ \mathrm{d}x\mathrm{d}t = -\int_0^\infty \int_{-\infty}^\infty uv_t + f(u)v_x\ \mathrm{d}x\mathrm{d}x - \int_{-\infty}^\infty uv|_{t=0}\ \mathrm{d}x, \tag{954}$$

where the nonlisted boundary terms are zero since v has compact support. Observe the right hand side makes sense even if u is only bounded. Consequently, we say $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is a weak solution of the PDE (946) provided for all test functions v

$$0 = -\int_0^\infty \int_{-\infty}^\infty uv_t + f(u)v_x\ \mathrm{d}x\mathrm{d}x - \int_{-\infty}^\infty uv|_{t=0}\ \mathrm{d}x. \tag{955}$$

Now suppose ρ is as given in (947). Set C to be the curve of the discontinuity so that $(x,t) = (vt,t)$ along C, and set $V_\ell := \{(x,t) \in \mathbb{R} \times (0,\infty) : x < vt\}$ and set $V_r := \{(x,t) \in \mathbb{R} \times (0,\infty) : x < vt\}$. For each test function v with compact support in V_ℓ we see

$$\iint_{V_\ell} \rho v_t + f(\rho) v_x \, \mathrm{d}x\mathrm{d}t = -\iint_{V_\ell} \left[\rho_t + f(\rho)_x\right] v \, \mathrm{d}x\mathrm{d}t = -\iint_{V_\ell} [0+0]\, v \, \mathrm{d}x\mathrm{d}t = 0. \tag{956}$$

Similarly, for all test functions with compact support in V_r

$$\iint_{V_r} \rho v_t + f(\rho) v_x \, \mathrm{d}x\mathrm{d}t = 0. \tag{957}$$

For any test function v with compact support (not necessarily vanishing along the discontinuity), we see

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \rho v_t + f(\rho) v_x \, \mathrm{d}x\mathrm{d}t &= \iint_{V_\ell} \rho v_t + f(\rho) v_x \, \mathrm{d}x\mathrm{d}t + \iint_{V_r} \rho v_t + f(\rho) v_x \, \mathrm{d}x\mathrm{d}t \\ &= \int_{\partial V_\ell} \left(\rho_\ell \nu^2 + f(\rho_\ell)\nu^1\right) v \, \mathrm{d}s - \int_{\partial V_\ell} \left(\rho_r \nu^2 + f(\rho_r)\nu^1\right) v \, \mathrm{d}s \\ &= \int_{\partial V_\ell} \left((\rho_\ell - \rho_r)\nu^2 + \left[f(\rho_\ell) - f(\rho_r)\right]\nu^1\right) v \, \mathrm{d}s, \end{aligned} \tag{958}$$

where $\nu = (\nu^1, \nu^2)$ is the outward normal along ∂V_ℓ, making $-\nu$ the outward normal along ∂V_r, and the integrals over the interior of V_ℓ and V_r vanish by (956) and (957). Note (958) shows ρ is a weak solution of the PDE precisely the expression on the right hand side is zero. Since v was an arbitrary test function, this occurs precisely when

$$0 = (\rho_\ell - \rho_r)\nu^2 + \left[f(\rho_\ell) - f(\rho_r)\right]\nu^1 \quad \text{along } C. \tag{959}$$

Along the discontinuity C we have $(x,t) = (vt, t)$, and so $\nu = \frac{1}{\sqrt{(vt)^2+t^2}}(-t, vt)$. Substituting in for ν and dividing by t yields

$$\boxed{v = \frac{f(\rho_\ell) - f(\rho_r)}{\rho_\ell - \rho_r},} \tag{960}$$

as claimed above. Consequently, we conclude ρ is a weak solution of the PDE provided (960) holds.

c) Suppose ρ is a solution of the form in b). Then

$$\dot{x}(t) = u(x(t), t) = \begin{cases} 1 - \rho_\ell & \text{if } x < vt, \\ 1 - \rho_r & \text{if } x > vt. \end{cases} \tag{961}$$

Consider any car trajectory for which there is a time $t^* \in (0,\infty)$ at which the trajectory crosses the discontinuity, i.e., $(x(t^*), t^*) \in C$. Then

$$\ddot{x}(t^*) = \lim_{\varepsilon \to 0^+} \frac{\dot{x}(t^* + \varepsilon) - \dot{x}(t^* - \varepsilon)}{2\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\rho_\ell - \rho_r}{2\varepsilon} = \begin{cases} +\infty & \text{if } \rho_\ell - \rho_r > 0, \\ -\infty & \text{if } \rho_\ell - \rho_r < 0, \end{cases} \tag{962}$$

where the final equality holds since $\rho_\ell \neq \rho_r$. By our assumption in (949), we conclude $\rho_\ell - \rho_r < 0$.

□

F15.3. Consider the porous medium equation with drift

$$\rho_t - \Delta(\rho^3) - \nabla\cdot(2x\rho) = 0 \ \text{ in } \mathbb{R}^2 \times [0,t], \tag{963}$$

where the initial data $\rho_0(x) \geq 0$ is compactly supported and $\int \rho_0 \mathrm{d}x = 1$. Let us assume that $\rho(\cdot,t)$ stays nonnegative and compactly supported for all times $t > 0$. Using formal calculations, show the following.

a) $\int \rho(\cdot,t)\ \mathrm{d}x = 1$ for all $t > 0$.

b) Show that the energy

$$E(t) := \iint_{\mathbb{R}^2} \frac{1}{2}\rho^3 + \rho|x|^2 + C\rho\ \mathrm{d}x \tag{964}$$

decreases for all times $t > 0$, for any constant C.

c) Using a) and b), show that ρ converges as $t \to \infty$ to the stationary profile

$$\left(\max(0, A - B|x|^2/2)\right)^{1/2}, \tag{965}$$

for appropriate A and B.

Solution:

a) We point out the notation in the prompt is likely incorrect, and we instead assume

$$\rho_t - \Delta(\rho^3) - \nabla\cdot(2x\rho) = 0 \ \text{ in } \mathbb{R}^2 \times [0,\infty). \tag{966}$$

Now define the energy $e : [0,\infty) \to \mathbb{R}$ via

$$e(t) := \iint_{\mathbb{R}^2} \rho(x,t)\ \mathrm{d}x. \tag{967}$$

Then, by hypothesis, $e(0) = 1$. For $t \in (0,\infty)$ we see

$$\dot{e}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^2} \rho\ \mathrm{d}x = \iint_{\mathbb{R}^2} \rho_t\ \mathrm{d}x = \iint_{\mathbb{R}^2} \Delta(\rho^3) + \nabla\cdot(2x\rho)\ \mathrm{d}x = \iint_{\mathbb{R}^2} \nabla\cdot\left[3\rho^2 D\rho + 2x\rho\right]\ \mathrm{d}x. \tag{968}$$

Since ρ is compactly support at time t, there is $R_t > 0$ such that $\text{spt}(\rho(\cdot,t)) \subseteq R_t$. Consequently,

$$\begin{aligned}
\dot{e}(t) &= \iint_{\mathbb{R}^2} \nabla \cdot \left[3\rho^2 D\rho - 2x\rho\right] \ \mathrm{d}x \\
&= \iint_{B(0,2R_t)} \nabla \cdot \left[3\rho^2 D\rho - 2x\rho\right] \ \mathrm{d}x \\
&= \int_{\partial B(0,2R_t)} \left[3\rho^2 D\rho - 2x\rho\right] \cdot \nu \ \mathrm{d}\sigma \\
&= \int_{\partial B(0,2R_t)} 0 \ \mathrm{d}\sigma \\
&= 0,
\end{aligned} \tag{969}$$

where the third equality holds by Gauss's law (a.k.a. the "divergence theorem") and ν denotes the outward normal along $\partial B(0, 2R_t)$. This shows $\dot{e}(t) = 0$ for all $t \in (0,\infty)$, from which we conclude $e(t) = 1$ for all $t \in (0,\infty)$.

b) Differentiating yields

$$\begin{aligned}
\dot{E}(t) &= \iint_{\mathbb{R}^2} \partial_t \left(\frac{1}{2}\rho^3 + \rho|x|^2\right) \ \mathrm{d}x + C\dot{e}(t) \\
&= \iint_{\mathbb{R}^2} \rho_t \left[\frac{3}{2}\rho^2 + |x|^2\right] \ \mathrm{d}x \\
&= \iint_{\mathbb{R}^2} \left[\nabla \cdot \left(3\rho^2 D\rho + 2x\rho\right)\right] \left[\frac{3}{2}\rho^2 + |x|^2\right] \ \mathrm{d}x \\
&= -\iint_{\mathbb{R}^2} \left(3\rho^2 D\rho + 2x\rho\right) \cdot \left[3\rho D\rho + 2x\right] \ \mathrm{d}x \\
&= -\iint_{\mathbb{R}^2} \rho \left|3\rho D\rho + 2x\right|^2 \ \mathrm{d}x \\
&\leq 0.
\end{aligned} \tag{970}$$

The first equality holds by linearity of the integral and definition of $e(t)$. The second holds by differentiating and noting $\dot{e}(t) = 0$. The third holds by using the PDE to substitute for ρ_t. The fourth holds by integration by parts, where the boundary terms cancel since ρ has compact support. The final inequality holds since $\rho \geq 0$ and $|3\rho D\rho + 2x|^2 \geq 0$, making the integrand nonnegative. This shows $\dot{E}(t) \leq 0$ for all $t \in (0,\infty)$, from which we conclude $E(t)$ is monotonically decreasing.

c) Note the terms $\rho^3/2$ and $\rho|x|^2$ in the integrand of $E(t)$ are nonnegative. And, using a), we see $E(t)$

is bounded below by C. Since we showed in b) that $E(t)$ is monotonically decreasing, it follows from the monotone convergence theorem that $\lim_{t\to\infty} E(t)$ exists. Together with the fact $\dot{E} \leq 0$, we see

$$0 = \lim_{t\to\infty} \dot{E}(t) = -\iint_{\mathbb{R}^2} \rho \left|3\rho D\rho + 2x\right|^2 \ \mathrm{d}x. \tag{971}$$

Let ρ_∞ be the function to which ρ converges. We know ρ_∞ is nonnegative since $\rho(\cdot, t)$ is nonnegative for all $t \in (0,\infty)$.[36]. Wherever $\rho_\infty \neq 0$ in $\mathbb{R}^2$, we obtain

$$0 = |3\rho_\infty D\rho_\infty + 2x|^2 \quad \Longrightarrow \quad 0 = 3\rho_\infty D\rho_\infty + 2x \quad \Longrightarrow \quad 0 = \nabla\left(3\rho_\infty^2 + |x|^2 + c\right), \tag{972}$$

for some constant A. Thus, where $\rho_\infty \neq 0$,

$$\rho_\infty^2 = -\frac{|x|^2}{3} - \frac{c}{3} \quad \Longrightarrow \quad \rho_\infty = \left(-\frac{c}{3} - \frac{|x|^2}{3}\right)^{1/2}. \tag{973}$$

Combining our results, we conclude

$$\rho_\infty = \left(\max\left(0, -\frac{c}{3} - \frac{|x|^2}{3}\right)\right)^{1/2}. \tag{974}$$

From the problem statement, we conclude $A = c/3$ and $B = 2/3$, and c is chosen so that $\iint_{\mathbb{R}^2} \rho_\infty \ \mathrm{d}x = 1$ since $\dot{e}(t) = 0$ for all $t \in (0,\infty)$.

□

[36] This can proven by way of contradiction using elementary analysis.

F15.5. Let $U := \{|x| \leq 1\} \subset \mathbb{R}^n$. For a given $T > 0$, consider a smooth solution to the PDE

$$u_t - \Delta u = u(u-1) \quad \text{in } U_T = U \times (0,T], \tag{975}$$

with boundary data $0 \leq u < 1$ on the parabolic boundary of U_T, i.e., on $U \times \{t = 0\}$ and $\partial U \times (0,T]$. With these assumptions, prove that $0 \leq u < 1$ in the entire domain U_T. You should show the proof of any maximum principle you use.

Solution:

Let Γ_T be the parabolic boundary of U_T. Since Γ_T is closed and u is smooth, u attains its supremum over Γ_T. Let $\mu := \max_{\Gamma_T} u$. Then our hypothesis implies $0 \leq \mu < 1$. Now let $\varepsilon \in (0, 1-\mu)$ and define $w(x,t) := u(x,t) + \varepsilon|x|^2$. Then $w < 1$ on Γ_T since

$$\max_{\Gamma_T} w = \max_{\Gamma_T} u + \varepsilon|x|^2 \leq \max_{\Gamma_T} u + \varepsilon = \mu + \varepsilon < \mu + (1-\mu) = 1. \tag{976}$$

We now use a "first time" argument. By way of contradiction, suppose $w = 1$ somewhere in U_T. Let $(x^*, t^*) \in U_T$ be such a point that $w(x^*, t^*) = 1$ and t^* is the first time at which this occurs. Then $w_t(x^*, t^*) \geq 0$ and $\Delta w(x^*, t^*) \leq 0$ because x^* is a local maximizer of the function $w(\cdot, t^*)$. Using our PDE for u, this implies

$$0 \leq w_t - \Delta w = u(u-1) - 2n\varepsilon = (w - \varepsilon|x|^2)\left(w - \varepsilon|x|^2 - 1\right) - 2n\varepsilon = -\varepsilon|x|^2(1 - \varepsilon|x|^2) - 2n\varepsilon < 0, \tag{977}$$

a contradiction. Note $1 - \varepsilon|x|^2 > 0$ since $\varepsilon < 1$ and $|x|^2 \leq 1$ in U, and note $\Delta|x|^2 = 2n\varepsilon$. This contradiction reveals $w < 1$ in U_T. Consequently, $u < u + \varepsilon = w < 1$ in U_T.

Now we show $u \geq 0$ in U_T using another "first time" argument. Let $\delta > 0$ and suppose there is $(x', t') \in U_T$ such that $u(x', t') = -\delta$ with t' the first time at which this occurs. Then $u_t(x', t') \leq 0$ and $\Delta u(x', t') \geq 0$ since x' is a local minimizer of $u(\cdot, t')$. This implies

$$0 \geq u_t - \Delta u = u(u-1) = -\delta(-\delta - 1) = \delta(1+\delta) > 0, \tag{978}$$

a contradiction. This proves $u > -\delta$ in U_T. Since $\delta > 0$ was arbitrarily chosen, we may let $\delta \to 0^+$ to deduce $u \geq 0$ in U_T, and the proof is complete. $\square$

F15.6. Let $g : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, and let u be the unique weak solution of the Hamilton-Jacobi equation

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{979}$$

a) Show there exists $C > 0$ such that for all $t > 0$, $|u(x,t) - g(x)| \leq Ct$.

b) Suppose $|g(x)| \leq M|x|^{-1}$ with M a constant. Show that $u(x,t)$ converges to zero as $t \longrightarrow \infty$.

Solution:

a) The Hamiltonian for the given PDE is $H(p) = |p|^2$. Substituting this into our PDE yields that u is a weak solution to

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{980}$$

The associated Lagrangian is given by the Fenchel dual, i.e.,

$$L(v) = \sup_{p \in \mathbb{R}^n} p \cdot v - H(p) = \sup_{p \in \mathbb{R}^n} p \cdot v - |p|^2. \tag{981}$$

Since the expression to be maximized is quadratic and concave down, the maximizer is the unique critical point of the expression. Thus differentiating yields

$$0 = D_p \left[p \cdot v - |p|^2 \right] = v - 2p \quad \Longrightarrow \quad p = \frac{v}{2}, \tag{982}$$

and so

$$L(v) = \left(\frac{v}{2}\right) \cdot v - \left|\frac{v}{2}\right|^2 = \frac{|v|^2}{4}. \tag{983}$$

The Hopf-Lax formula tells us

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left(t \cdot L\left(\frac{x-y}{t}\right) + g(y) \right) \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{984}$$

To obtain our desired inequality, we use this formula and the fact g is Lipschitz. For all $x, y \in \mathbb{R}^n$

$$|g(y) - g(x) \leq \text{Lip}(g)|y - x| \quad \Longrightarrow \quad g(y) \geq g(x) - \text{Lip}(g)|y - x|. \tag{985}$$

This implies

$$u(x,t) \geq \min_{y\in\mathbb{R}^n}\left(t\cdot L\left(\frac{x-y}{t}\right) + g(x) - \text{Lip}(g)|y-x|\right) = g(x) - t\cdot \max_{y\in\mathbb{R}^n}\left(\text{Lip}(g)\cdot\frac{|x-y|}{t} - L\left(\frac{x-y}{t}\right)\right). \tag{986}$$

Letting $\xi = (x-y)/t$, we see

$$\max_{y\in\mathbb{R}^n}\left(\text{Lip}(g)\cdot\frac{|x-y|}{t} - L\left(\frac{x-y}{t}\right)\right) = \max_{\xi\in\mathbb{R}^n}\left(\text{Lip}(g)|\xi| - \frac{|\xi|^2}{4}\right) = \max_{\alpha\in\mathbb{R}} \text{Lip}(g)\alpha - \frac{\alpha^2}{4}. \tag{987}$$

The critical point α satisfies

$$0 = \text{Lip}(g) - \frac{\alpha}{2} \quad\implies\quad \alpha = 2\text{Lip}(g). \tag{988}$$

Plugging this value for α back in yields

$$u(x,t) \geq g(x) - t\cdot\text{Lip}(g) \quad\implies\quad u(x,t) - g(x) \geq -t\cdot\text{Lip}(g). \tag{989}$$

Now observe

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left(t\cdot L\left(\frac{x-y}{t}\right) + g(y)\right) \leq t\cdot L(0) + g(x) = g(x), \tag{990}$$

where we take $y = x$ to obtain the inequality. This shows, for all $x, y \in \mathbb{R}^n$,

$$u(x,t) - g(x) \leq 0 \leq \text{Lip}(g)\cdot t. \tag{991}$$

Combining our results, we conclude

$$\|u(\cdot,t) - g\|_{L^\infty(\mathbb{R}^n)} \leq \text{Lip}(g)\cdot t. \tag{992}$$

b) Using our inequality with the Hopf-Lax formula reveals, for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$,

$$\begin{aligned}
|u(x,t)| &= \left| \min_{y\in\mathbb{R}^n} \left(t\cdot L\left(\frac{x-y}{t}\right) + g(y)\right)\right| \\
&\leq \min_{y\in\mathbb{R}^n} \left(t\cdot \left| L\left(\frac{x-y}{t}\right)\right| + |g(y)|\right) \\
&\leq \min_{y\in\mathbb{R}^n} \frac{|x-y|^2}{4t} + \frac{M}{|y|} \\
&\leq \min_{y\in\mathbb{R}^n} \frac{|x|^2+2|x||y|+|y|^2}{4t} + \frac{M}{|y|} \\
&= \min_{\alpha\in\mathbb{R}} \frac{|x|^2+2|x|\alpha+\alpha^2}{4t} + \frac{M}{\alpha} \\
&\leq \frac{|x|^2}{4t} + \frac{2|x|(2M)^{1/3}}{t^{2/3}} + \frac{(2M)^{2/3}}{t^{1/3}} + \frac{M}{(2Mt)^{1/3}},
\end{aligned} \tag{993}$$

where we substitute $\alpha = |y|$ for notational convenience and take $\alpha = (2Mt)^{1/3}$ in the final line. Noting the powers of t in the denominator of each term on the right hand side, we obtain

$$\lim_{t\to\infty} |u(x,t)| \leq \lim_{t\to\infty} \frac{|x|^2}{4t} + \frac{2|x|(2M)^{1/3}}{t^{2/3}} + \frac{(2M)^{2/3}}{t^{1/3}} + \frac{M}{(2Mt)^{1/3}} = 0. \tag{994}$$

Whence $u \longrightarrow 0$ point-wise as $t \longrightarrow \infty$.

□

F15.7. Consider the set K of functions $u : [0,2] \to \mathbb{R}$ of the form

$$u(x) = \begin{cases} v(x), & \text{if } x \in [0,1), \\ w(x), & \text{if } x \in (1,2], \end{cases} \tag{995}$$

with $v \in C^2[0,1)$ and $w \in C^2(1,2]$, and with the property that $v(0) = w(2) = 0$ and $w(1) - v(1) = a$, for a given constant a. Define the energy[37]

$$E(u) := \frac{1}{2}\int_0^1 (u_x)^2 \ \mathrm{d}x + \frac{1}{2}\int_1^2 (u_x)^2 \ \mathrm{d}x + \overline{u}b, \tag{996}$$

where $\overline{u} := \frac{1}{2}(w(1) + v(1))$, and b is a given constant. Show that there exists $h(x)$ that minimizes E over K, and solve for h.

Solution:

Our admissibility class K is defined in the prompt. Of course, $f(x) := x^2$ is convex since $f''(x) = 2 \geq 0$. We claim E is convex. Indeed, if $\hat{u}, \tilde{u} \in K$ and if $\lambda \in (0,1)$, then

$$\begin{aligned} E\left(\lambda\hat{u} + (1-\lambda\tilde{u})\right) &= \frac{1}{2}\int_0^1 f\left(\lambda\hat{v} + (1-\lambda\tilde{v})\right) \ \mathrm{d}x + \frac{1}{2}\int_1^2 f\left(\lambda\hat{u} + (1-\lambda\tilde{w})\right) \ \mathrm{d}x + \overline{\lambda\hat{u} + (1-\lambda\tilde{u})}b \\ &\leq \frac{1}{2}\int_0^1 \lambda f(\hat{v}) + (1-\lambda)f(\tilde{v}) \ \mathrm{d}x + \frac{1}{2}\int_1^2 \lambda f(\hat{w}) + (1-\lambda)f(\tilde{w}) \ \mathrm{d}x + \lambda\overline{\hat{u}}b + (1-\lambda)\overline{\tilde{u}}b \\ &= \lambda E(\hat{u}) + (1-\lambda)E(\tilde{u}), \end{aligned} \tag{997}$$

where we have used the convexity of f, the linearity of integration, and the fact $\overline{\lambda\hat{u} + (1-\lambda\tilde{u})} = \lambda\overline{\hat{u}} + (1-\lambda)\overline{\tilde{u}}$. This shows any extremal of E over K is a minimizer. Therefore, it suffices to identify an extremal of E over K since such an extremal is necessarily a minimizer. Let $u \in K$, $\varepsilon \in (0,\infty)$, and $z \in S := \{\phi \in C^2[0,2] : \phi(0) = \phi(2) = 0\}$. Then $u + \varepsilon z \in K$ since

$$(v + \varepsilon z) \in C^2[0,1), \quad (w + \varepsilon z) \in C^2(1,2], \quad (v+\varepsilon z)(0) = 0, \quad (w + \varepsilon z)(2) = 0. \tag{998}$$

[37] I'm not sure if there was a typo or not, but I added a 1/2 factor in front of the second integral. If this wasn't a typo, then the same procedure can be used, but a slightly different minimizer will be obtained.

Now observe

$$E(u+\varepsilon z) = \frac{1}{2}\int_0^1 (v'+\varepsilon z')^2 \, \mathrm{d}x + \frac{1}{2}\int_1^2 (w'+\varepsilon z')^2 \, \mathrm{d}x + b\left(\overline{u} + \varepsilon z(1)\right). \tag{999}$$

Consequently,

$$\begin{aligned}
\frac{E(u+\varepsilon z)-E(u)}{\varepsilon} &= \frac{1}{\varepsilon}\left[\frac{1}{2}\int_0^1 2\varepsilon v'z' + \varepsilon^2 (z')^2 \, \mathrm{d}x + \frac{1}{2}\int_1^2 2\varepsilon w'z' + \varepsilon^2 z'^2 \, \mathrm{d}x + \varepsilon b z(1)\right] \\
&= \int_0^1 v'z' + \varepsilon (z')^2 \, \mathrm{d}x + \int_1^2 w'z' + \varepsilon (z')^2 \, \mathrm{d}x + bz(1) \\
&= \int_0^1 -v''z + \varepsilon (z')^2 \, \mathrm{d}x + \int_1^2 -w''z + \varepsilon (z')^2 \, \mathrm{d}x + \left[v'z\right]_0^1 + \left[w'z\right]_1^2 + bz(1) \\
&= \varepsilon \int_0^2 (z')^2 \, \mathrm{d}x + \int_0^1 -v''z \, \mathrm{d}x + \int_1^2 -w''z \, \mathrm{d}x + z(1)\left[v'(1) - w'(1) + b\right],
\end{aligned} \tag{1000}$$

where the third equality holds via integration by parts and the final line holds since $z(0) = z(2) = 0$. Letting $\varepsilon \to 0^+$, we see the Gâteaux derivative is given by

$$\begin{aligned}
\delta E(u,z) &= \lim_{\varepsilon\to 0^+} \frac{E(u+\varepsilon z)-E(u)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \varepsilon \int_0^2 (z')^2 \, \mathrm{d}x + \int_0^1 -v''z \, \mathrm{d}x + \int_1^2 -w''z \, \mathrm{d}x + z(1)\left[v'(1) - w'(1) + b\right] \\
&= \int_0^1 -v''z \, \mathrm{d}x + \int_1^2 -w''z \, \mathrm{d}x + z(1)\left[v'(1) - w'(1) + b\right].
\end{aligned} \tag{1001}$$

We shall use this expression for $\delta E(u,z)$ and the fact every extremizer of E over K satisfies $\delta E(u,z) = 0$ for all $z \in S$.

Suppose there exists an extremizer u of E over K. Then (1001) implies, by the arbitrariness of z,

$$v'' = 0 \text{ in } (0,1), \quad w'' = 0 \text{ in } (1,2), \quad w'(1) = v'(1) + b. \tag{1002}$$

Consequently,

$$u(x) = \begin{cases} c_1 x + c_2 & \text{if } x \in [0,1), \\ d_1 x + d_2 & \text{if } x \in (1,2]. \end{cases} \tag{1003}$$

Then (1001) also implies $d_1 = c_1 + b$ and the condition $u(2) = 0$ implies $d_2 = -2(c_1 + b)$. The condition $u(0) = 0$ implies $c_2 = 0$. Thus

$$u(x) = \begin{cases} c_1 x & \text{if } x \in [0,1), \\ (c_1 + b)(x-2) & \text{if } x \in (1,2]. \end{cases} \tag{1004}$$

Then

$$a = w(1) - v(1) = (c_1 + b)(1-2) - c_1 \cdot 1 = -2c_1 - b \quad \Longrightarrow \quad c_1 = -\frac{a+b}{2}. \tag{1005}$$

Whence

$$u(x) = \begin{cases} -\dfrac{a+b}{2} x & \text{if } x \in [0,1), \\ \dfrac{b-a}{2}(x-2) & \text{if } x \in (1,2]. \end{cases} \tag{1006}$$

Note u defined by (1006) satisfies $u \in K$. And, since $\delta E(u,z) = 0$ for u defined by (1001) and all $z \in S$, we deduce u is an extremizer of E over K. Because E is convex, it follows that u is a minmizer of E over K. Thus we have established the existence of a minimizer u of E over K and given an explicit expression for u, and we are done. □

F15.8. Let $u(x,t)$ be the entropy solution of the Burgers' equation

$$u_t + f(u)_x = 0 \text{ in } \mathbb{R}\times(0,\infty), \tag{1007}$$

with $f(u) = u^2/2$ and initial data

$$u(x,0) = \begin{cases} (x+1)^2 & \text{if } x \in [-1,0], \\ (x-1)^2 & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases} \tag{1008}$$

Find the time $T > 0$ when u becomes discontinuous for the first time.

Solution:

We proceed by using the method of characteristics. Set $F(p,q,z,x,t) := q + zp$. Then taking $q = u_t$, $p = u_x$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = 0 - p^2, & p(0) = g'(x^0), \\ \dot{q}(s) = -F_t - F_z q = 0 - pq, & q(0) = q^0, \\ \dot{x}(s) = F_p = z, & x(0) = x^0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0, & z(0) = g(x^0), \end{cases} \tag{1009}$$

where we set $g(x) := u(x,0)$. This implies $s = t$ and z is constant along characteristics. Additionally, using separation of variables reveals

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p^2 \quad\Longrightarrow\quad -p^{-2}\,\mathrm{d}p = \mathrm{d}t \quad\Longrightarrow\quad p^{-1} = t + C \quad\Longrightarrow\quad p = \frac{1}{t+C}, \tag{1010}$$

for some scalar $C \in \mathbb{R}$. Using the initial condition yields

$$g'(x^0) = p(0) = \frac{1}{0+C} \quad\Longrightarrow\quad C = \frac{1}{g'(x^0)} \quad\Longrightarrow\quad p = \frac{g'(x^0)}{tg'(x^0)+1}. \tag{1011}$$

Note $g'(x^0) = 0$ for $x \notin (-1,1)$. If $x^0 \in (-1,0)$, then $g'(x^0) = 2(x^0+1) > 0$. So, p will *not* blow up for characteristics originating outside of $(0,1)$. However, if $x^0 \in (0,1)$, then

$$g'(x^0) = 2(x^0 - 1) < 0 \quad \Longrightarrow \quad \lim_{t \to \phi(x^0)^-} p(t) = \lim_{t \to \phi(x^0)^-} \frac{-2(1-x^0)}{-2t(1-x^0)+1} = -\infty, \tag{1012}$$

where $\phi(x^0) := 1/2(1-x^0)$. Since $p = u_x$, this implies u becomes discontinuous at time $\phi(x^0)$. We are interested in the smallest time $T > 0$ such that u becomes discontinuous. Since

$$\phi'(x^0) = \frac{(1-x^0)^{-2}}{2} > 0, \tag{1013}$$

we see ϕ is increasing. Therefore, letting $x^0 \to 0^+$, we see $\phi(x^0) \to \frac{1}{2}$, and so the smallest time at which u becomes discontinuous is $T = \frac{1}{2}$. $\square$

2015 Spring

S15.1. Consider the damped conservation law

$$\begin{cases} u_t + f(u)_x = -u & \text{in } \mathbb{R}\times[0,\infty), \\ u = u^0 & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1014}$$

where $u^0(x)$ has compact support. Define the notion of integral solution and derive the jump (Rankine-Hugonoit) condition for a discontinuity (u^-, u^+) in an integral solution.

Solution:

Let u be a smooth solution to the given PDE and assume

$$v : \mathbb{R}\times(0,\infty)\to\mathbb{R} \ \text{ is smooth with compact support.} \tag{1015}$$

Then multiplying the PDE by v and integrating by parts yields

$$\begin{aligned} 0 &= \int_{\mathbb{R}}\int_0^\infty (u_t + \partial_x f(u) + u)v \ \mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} [uv]_{t=0}^\infty \ \mathrm{d}x + \underbrace{\int_0^\infty [uv]_{x=-\infty}^\infty \ \mathrm{d}t}_{\to 0} \\ &= \int_{\mathbb{R}}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}} u^0 v\big|_{t=0} \ \mathrm{d}x. \end{aligned} \tag{1016}$$

This implies

$$0 = \int_{\mathbb{R}}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}} u^0 v\big|_{t=0} \ \mathrm{d}x, \tag{1017}$$

which has meaning even if u is only bounded. Whence we say $u\in L^\infty(\mathbb{R}\times(0,\infty))$ is an integral solution of the PDE provided (1017) holds for each test function v satisfying (1015).

To derive the Rankine-Hugonoit condition, let $V\subset\mathbb{R}\times(0,\infty)$ be open and assume u is smooth on either side of the curve C of the discontinuity. Let V_ℓ and V_r be the portions of V to the left and right of C, respectively. We assume u is an integral solution of the PDE and has uniformly continuous first derivatives in V_ℓ and in V_r. For any test function v with compact support in V_ℓ we see

$$0 = \int_{\mathbb{R}}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}}\int_0^\infty (u_t + \partial_x f(u) + u)v \ \mathrm{d}x\mathrm{d}t, \tag{1018}$$

where the integration by parts is justified since u is C^1 in V_ℓ and v vanishes near the boundary of V_ℓ. Since this result holds for all v with compact support in V_ℓ, we deduce

$$u_t + \partial_x f(u) + u = 0 \text{ in } V_\ell. \tag{1019}$$

Likewise,

$$u_t + \partial_x f(u) + u = 0 \text{ in } V_r. \tag{1020}$$

Now consider v with compact support in V, but which does not necessarily vanish along the curve C. Using (1017), we discover

$$\begin{aligned} 0 &= \int_{\mathbb{R}}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t \\ &= \iint_{V_\ell}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t + \iint_{V_r}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{1021}$$

Since v has compact support in V,

$$\iint_{V_\ell}\int_0^\infty -uv_t - f(u)v_x + uv \ \mathrm{d}x\mathrm{d}t = -\cancelto{0}{\iint_{V_\ell} [u_t + \partial_x f(u) + u]v \ \mathrm{d}x\mathrm{d}t} + \int_C (u_-\nu^2 + f(u_-)\nu^1)v \ \mathrm{d}s, \tag{1022}$$

where $\nu = (\nu^1, \nu^2)$ is the outward normal along V_ℓ, and the subscript "-" denotes the limit from the left. Similarly,

$$0 = \int_C -(u_+\nu^2 + f(u_+)\nu^1)v \ \mathrm{d}s, \tag{1023}$$

where the negative sign is added since $-\nu$ forms the outward normal from V_r and the subscript "+" denotes the limit from the right. Combining our results, we see

$$\begin{aligned}0 &= \iint_{V_\ell} \int_0^\infty -uv_t - f(u)v_x + uv \; \mathrm{d}x\mathrm{d}t + \iint_{V_r} \int_0^\infty -uv_t - f(u)v_x + uv \; \mathrm{d}x\mathrm{d}t \\ &= \int_C (u_-\nu^2 + f(u_-)\nu^1)v \; \mathrm{d}s - \int_C (u_-\nu^2 + f(u_-)\nu^1)v \; \mathrm{d}s \\ &= \int_C \left[(u_- - u_r)\nu^2 + (f(u_-) - f(u_r))\nu^2\right] v \; \mathrm{d}s.\end{aligned} \tag{1024}$$

This holds for all test functions v as above, and so

$$(u_- - u_+)\nu^2 + (f(u_-) - f(u_+))\nu^1 = 0 \quad \text{along } C. \tag{1025}$$

Lastly, suppose C is represented parametrically as $C = \{(x,t) : x = s(t)\}$ for some smooth function $s : [0,\infty) \to \mathbb{R}$. Differentiating yields $(\dot{s}, 1)$, and so $\nu = (-1, \dot{s})/\|(-1,\dot{s})\|$. Thus (1025) implies

$$f(u_-) - f(u_+) = \dot{s}(u_- - u_+) \tag{1026}$$

in V, along the curve C. This result (1026) gives the Rankine-Hugonoit condition. □

S15.2. Show that there is at most one solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} + u_t = 0 & \text{in } \mathbb{R}\times[0,\infty), \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t = \psi & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1027}$$

where ϕ and ψ are smooth and there exists[38] $C > 0$ such that $|c'(x)| \leq C$ for all $x \in \mathbb{R}$.

Solution:

We claim solutions to the PDE are compactly support.[39] Let u and v be two solutions to the PDE and set $w := u - v$. Then w satisfies

$$\begin{cases} w_{tt} - c^2 w_{xx} + w_t = 0 & \text{in } \mathbb{R}\times[0,\infty), \\ w = w_t = 0 & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1028}$$

and so it suffices to verify $w = 0$ in $\mathbb{R}\times(0,\infty)$. To this end, define the energy $E : [0,\infty) \to \mathbb{R}$ via

$$E(t) := \frac{1}{2}\int_{\mathbb{R}} w_t^2 + c^2 w_x^2 \ \mathrm{d}x. \tag{1029}$$

Then the initial conditions in (1028) imply

$$E(0) = \frac{1}{2}\int_{\mathbb{R}} 0^2 + c^2 0^2 \ \mathrm{d}x = 0. \tag{1030}$$

[38] I think there was a typo in the original prompt.

[39] The solution to this problem becomes quite long if this is also to be verified... I'm not sure what the writers intended.

Moreover, differentiating in time reveals

$$\begin{aligned}
\dot{E}(t) &= \int_{\mathbb{R}} w_t w_{tt} + c^2 w_x w_{xt} \ \mathrm{d}x \\
&= \int_{\mathbb{R}} w_t \left(w_{tt} - \partial_x[c^2 w_x] \right) \ \mathrm{d}x \\
&= \int_{\mathbb{R}} w_t \left(w_{tt} - c^2 w_{xx} \right) \ \mathrm{d}x - 2 \int_{\mathbb{R}} w_t c c' w_x \ \mathrm{d}x \\
&\leq - \int_{\mathbb{R}} w w_t \ \mathrm{d}x + 2C \int_{\mathbb{R}} w_t c w_x \ \mathrm{d}x \\
&\leq - \int_{\mathbb{R}} w_t^2 \ \mathrm{d}x + C \int_{\mathbb{R}} w_t^2 + c^2 w_x^2 \ \mathrm{d}x \\
&\leq C E(t).
\end{aligned} \tag{1031}$$

The second line holds via integration by parts, noting the boundary terms vanish since w has compact support. The fourth line holds by our hypothesis regarding c'. The fifth line holds since

$$0 \leq (a-b)^2 = a^2 + b^2 - 2ab \quad \Longrightarrow \quad ab \leq \frac{1}{2}(a^2 + b^2), \quad \text{for all } a, b \in \mathbb{R}. \tag{1032}$$

The final inequality holds since the first term on the fifth line is nonpositive. From Grownwall's inequality, it follows that

$$E(t) \leq \exp\left(\int_0^t C \ \mathrm{d}\tau \right) E(0) = \exp(Ct) E(0) = 0. \tag{1033}$$

And, because the integrand of $E(t)$ contains only nonnegative terms, $E(t) \geq 0$. Combining our results, we deduce $E(t) = 0$ for all $t \in [0, \infty)$. This implies $w_t = 0$ in $\mathbb{R} \times (0, \infty)$, i.e., $w(x, \cdot)$ is constant. Therefore,

$$w(\tilde{x}, \tilde{t}) = w(\tilde{x}, 0) + \int_0^{\tilde{t}} w_t(\tilde{x}, \tau) \ \mathrm{d}\tau = 0 + \int_0^{\tilde{t}} 0 \ \mathrm{d}\tau = 0, \quad \text{for all } (\tilde{x}, \tilde{t}) \in \mathbb{R} \times (0, \infty), \tag{1034}$$

from which we conclude $w = 0$ in $\mathbb{R} \times (0, \infty)$, as desired. □

S15.5. Consider the PDE

$$u_t - \Delta u + u^2 = 0 \quad \text{in } \mathbb{R}^n \times [0,T]. \tag{1035}$$

Suppose u and v are bounded solutions of the above problem with $|u|, |v| \leq M$ and $|u|, |v| \to 0$ as $|x| \to \infty$ and

$$|u(x,0) - v(x,0)| < \varepsilon. \tag{1036}$$

Show

$$|u(x,t) - v(x,t)| < \varepsilon \exp(2Mt) \quad \text{in } \mathbb{R}^n \times (0,T]. \tag{1037}$$

Solution:

By hypothesis, there exists $R > 0$ such that $|u|, |v| \leq \varepsilon/4$ for $|x| \geq R$. Then define $\Omega := B(0,R)$, set $\Omega_T := \Omega \times (0,T]$, and set Γ_T to be the parabolic interior of Ω_T. Then define $w := u - v - \varepsilon\exp(2Mt)$, note $w < \varepsilon - \varepsilon = 0$ on $\Omega \times \{t = 0\}$, and observe

$$w = u - v - \varepsilon\exp(2Mt) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} - \varepsilon\exp(2Mt) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} - \varepsilon = -\frac{\varepsilon}{2} \quad \text{in } (\mathbb{R}^n - B(0,R)) \times [0,T]. \tag{1038}$$

These two observations reveal $w < 0$ in $(\mathbb{R}^n \times [0,T]) - \Omega_T$. By way of contradiction, now suppose there exists a point in Ω_T at which $w = 0$. Since w is continuous and negative at time $t = 0$, there exists a first time $\tilde{t}$ at which this occurs. So, let $(\tilde{x}, \tilde{t}) \in \Omega_T$ be such that $w(\tilde{x}, \tilde{t}) = 0$. Then $w_t(\tilde{x}, \tilde{t}) \geq 0$ and, as $\tilde{x}$ is a local max of $w(\cdot, \tilde{t})$, we see $\Delta w(\tilde{x}, \tilde{t}) \leq 0$. Thus, at $(\tilde{x}, \tilde{t})$,

$$\begin{aligned}
0 &\leq w_t - \Delta w \\
&= (u-v)_t - \Delta(u-v) - 2M\varepsilon\exp(2Mt) \\
&= v^2 - u^2 - 2M\varepsilon\exp(2Mt) \\
&= v^2 - [v + \varepsilon\exp(2Mt)]^2 - 2M\varepsilon\exp(2Mt) \\
&= -\varepsilon^2\exp(4Mt) + 2[-v - M]\varepsilon\exp(2Mt) \\
&\leq -\varepsilon^2\exp(4Mt) + 2[|v| - M]\varepsilon\exp(2Mt) \\
&\leq -\varepsilon^2\exp(4Mt) \\
&< 0,
\end{aligned} \tag{1039}$$

a contradiction. Consequently, $w < 0$ in Ω_T. Together our results demonstrate $w < 0$ in $\mathbb{R}^n \times [0, T]$, i.e.,

$$u - v < \varepsilon \exp(2Mt) \quad \text{in } \mathbb{R}^n \times [0, T]. \tag{1040}$$

Reversing the roles of u and v in the definition of w and repeating an analogous argument reveals

$$v - u < \varepsilon \exp(2Mt) \quad \text{in } \mathbb{R}^n \times [0, T], \tag{1041}$$

and the result follows from (1040) and (1041). □

S15.8. a) Sketch the phase plane for the dynamical system

$$\dot{x} = x(1-y^2), \quad \dot{y} = y(1-x^2). \tag{1042}$$

Include behavior of trajectories that start near the equilibrium points [it is sufficient to determine what the type of each equilibrium point is; you do not need to calculate the eigenvectors], and of any trajectories that connect equilibrium points, along with the asymptotic form of the trajectories for large x and y.

b) Suppose instead the dynamical system was slightly modified to read

$$\dot{x} = x(1-y^2), \quad \dot{y} = y^2(1-x^2). \tag{1043}$$

Prove the equilibrium point $(x, y) = (1, -1)$ is stable.

Solution:

a) The equilibrium points are $(0,0)$, $(\pm 1, \pm 1)$, and $(\mp 1, \pm 1)$. The Jacobian $J(x,y)$ for the system is

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 1-y^2 & -2xy \\ -2xy & 1-x^2 \end{pmatrix}. \tag{1044}$$

This implies

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1045}$$

which has the repeated eigenvalue 1. Thus, $(0,0)$ forms a source. Additionally,

$$J(\pm 1, \pm 1) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \quad \text{and} \quad J(\mp 1, \pm 1) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{1046}$$

and so for all the Jacobian matrices in (1046) the eigenvalues are $\lambda = \pm 2$. This shows $(\pm 1, \pm 1)$ and $(\mp 1, \pm 1)$ form saddle points. Nullclines occur along $y = 1$, $x = 0$, $x = 1$, and $y = 0$. With this, we obtain the following plot.

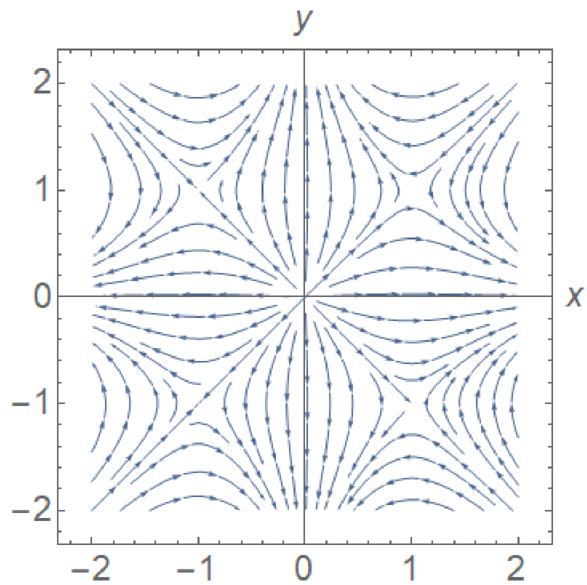

Figure 25: ODE Phase Plot for S15.8.

We note that along $y = x$ we there is a trajectory from $(0, 0)$ to $(1, 1)$ and from $(0, 0)$ to $(-1, -1)$. Similarly, along $y = -x$ there is a trajectory from $(0, 0)$ to $(1, -1)$ and from $(0, 0)$ to $(-1, 1)$.

Now, for large x and y, we see

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} \sim \frac{-yx^2}{-xy^2} = \frac{x}{y} \quad \Longrightarrow \quad y\,\mathrm{d}y = x\,\mathrm{d}x \quad \Longrightarrow \quad y^2 - x^2 = C, \tag{1047}$$

for some $C \in \mathbb{R}$. This shows the asymptotic form of trajectories for large x and y is that of a hyperbola.

b) We proceed by applying Lyapnov's second method. We must show there exists a Lyapunov function $V(x, y)$ such that $V(1, -1) = 0$, $V(x, y) > 0$ in a neighborhood of $(1, -1)$, and $\dot{V} \leq 0$. Manipulating our ODE reveals

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{y^2(1 - x^2)}{x(1 - y^2)} \quad \Longrightarrow \quad \left(y^{-2} - 1\right)\mathrm{d}y = \left(x^{-1} - x\right)\mathrm{d}x, \tag{1048}$$

and upon integration we find there exists $\alpha \in \mathbb{R}$ such that

$$0 = y^{-1} + y + \ln(x) - \frac{x^2}{2} + \alpha, \tag{1049}$$

for some $\alpha \in \mathbb{R}$. Define the Lyapunov function $V(x,y)$ via

$$V(x,y) := -y^{-1} - y - \ln(x) + \frac{x^2}{2} - \frac{5}{2}. \tag{1050}$$

Note $V(1,-1) = 0$ and

$$\begin{aligned}\dot{V} &= \left(\frac{1}{y^2} - 1\right)\dot{y} + \left(x - \frac{1}{x}\right)\dot{x} \\ &= \frac{1}{y^2}\left(1 - y^2\right)\dot{y} + \frac{1}{x}\left(x^2 - 1\right)\dot{x} \\ &= (1-y^2)(1-x^2) + (x^2-1)(1-y^2) \\ &= 0.\end{aligned} \tag{1051}$$

Thus, $\dot{V} \leq 0$. Furthermore, rearrangement reveals

$$\begin{aligned}V &= -\frac{1}{y}\left(y^2+1\right) + (x - \ln(x)) + \frac{1}{2}\left(x^2 - 2x\right) - \frac{5}{2} \\ &= -\frac{1}{y}\left(y^2 + 2y + 1\right) + (x - 1 - \ln(x)) + \frac{1}{2}\left(x^2 - 2x + 1\right) \\ &= -\frac{(y+1)^2}{y} + (x - 1 - \ln(x)) + \frac{(x-1)^2}{2}.\end{aligned} \tag{1052}$$

For $q \in (0,\infty)$ and $f(x) := \ln(x)$, Taylor's theorem asserts there is ξ between 1 and q such that

$$f(q) = f(1)f'(1)(q-1) + f''(\xi)(q-1)^2, \tag{1053}$$

which implies

$$q - 1 - \ln(q) = f'(1)(q-1) - \ln(q) = -f(1) - f''(\xi)(q-1)^2 = 0 + \frac{1}{\xi^2}(q-1)^2 > 0. \tag{1054}$$

Consequently, the second term on the final line of (1052) is positive whenever $x \in (0,\infty) - \{1\}$. The first term is positive whenever $y < 0$. Thus, $V(x,y) > 0$ whenever $x > 1$ and $y < 0$, and , in particular, in a neighborhood about $(1,-1)$. Thus, we conclude from Lyapunov's theorem that $(1,-1)$ is stable.

□

2014 Fall

F14.1. An ecosystem contains two species. At time t, there are x individuals of species 1 and y individuals of species 2. The dynamics of the two populations are described by the Lotka-Volterra equations

$$\dot{x} = 2x - x^2 - xy, \quad \dot{y} = y - xy. \tag{1055}$$

a) Describe in words what the terms in the equations might represent.

b) Sketch the possible trajectories of the ecosystem in the (x, y) plane. Your sketch should include any equilibrium points, null-clines, and the behavior of trajectories if x and y are both large.

Solution:

a) First we consider $\dot{x}$. The $2x$ term could represent the reproductive growth, meaning that as the population of species 1 grows more individuals of species 1 will be born. The x^2 term could model a restriction on growth due to limited resources as x increases. The xy term could represent a competition between the two species. Similarly, for $\dot{y}$, the y terms represents population growth and the xy term corresponds to a reduction in population due to competition between the species.

b) We first analyze the ODE system. The equilibrium points are $(0, 0)$, $(1, 1)$, and $(2, 0)$. The Jacobian for the system is given by

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 2 - 2x - y & -x \\ -y & 1 - x \end{pmatrix}. \tag{1056}$$

This implies

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1057}$$

which has eigenvalues 2 and 1, thereby implying $(0, 0)$ is an unstable source. Then

$$J(1, 1) = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \tag{1058}$$

which has eigenvalues $\lambda = (-1 \pm 3)/2 = -2, 1$. Thus, $(1,1)$ forms a saddle. Lastly,

$$J(2,0) = \begin{pmatrix} 0 & -2 \\ 0 & -1 \end{pmatrix}, \tag{1059}$$

which has eigenvalues $\lambda = 0, -1$. This implies $(2,0)$ is a sink. The null-clines are $x = 0$ and $y = 2 - x$, at which $\dot{x} = 0$, and $y = 0$ and $x = 1$, at which $\dot{y} = 0$. The trajectories are given in the figure below.

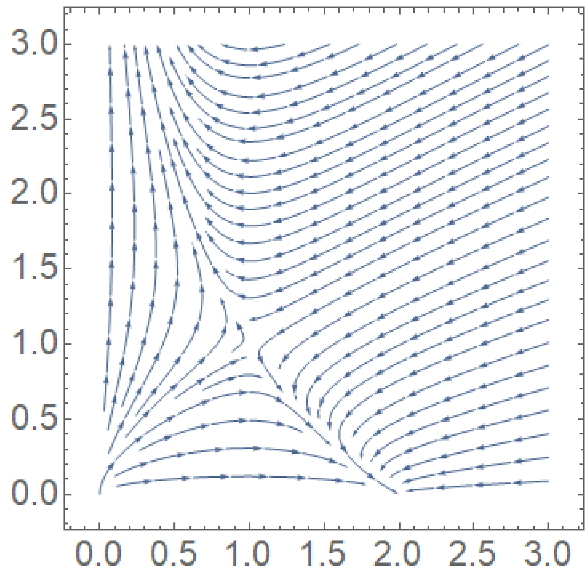

Figure 26: Trajectories plot for F14.1.

Lastly, for x and y large, we see

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{y - xy}{2x - x^2 - xy} \sim \frac{-xy}{-x(x+y)} = \frac{y}{x+y} \qquad \Longrightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = 1 + \frac{x}{y}. \tag{1060}$$

Writing this in standard form yields

$$\frac{\mathrm{d}x}{\mathrm{d}y} + P(y)x = 1, \tag{1061}$$

where $P(y) = -y^{-1}$. Multiplying by the integrating factor

$$\mu(y) = \exp\left(\int P(y)\ \mathrm{d}y\right) = \exp\left(-\ln(y)\right) = \frac{1}{y} \tag{1062}$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{x}{y}\right] = \frac{1}{y}\frac{\mathrm{d}x}{\mathrm{d}y} - \frac{x}{y^2} = \frac{1}{y} \quad \Longrightarrow \quad x = y\ln(y) + Cy, \tag{1063}$$

for some scalar $C \in \mathbb{R}$. This gives the form of trajectories when x and y are large.

$\square$

F14.2.

a) Find the eigenfunctions and eigenvalues for $1 \leq x \leq 2$ of the homogeneous ODE

$$x^2 y'' + xy' + \lambda y = 0, \quad \text{where} \;\; y(1) = y(2) = 0. \tag{1064}$$

b) By expanding in these eigenfunctions or otherwise, solve the inhomogeneous ODE

$$x^2 y'' + xy' + 3y = x\ln(x), \quad \text{where} \;\; y(1) = y(2) = 0. \tag{1065}$$

Solution:

a) This Cauchy-Euler equation admits solutions of the form x^m. Plugging this in yields

$$0 = x^m\left(m(m-1) + m + \lambda\right) = x^m\left(m^2 + \lambda\right) \quad \text{for all } x \in (1,2), \tag{1066}$$

which implies $m = \pm\sqrt{-\lambda}$. Therefore, the general solution to the ODE is of the form

$$y = c_1 x^{\sqrt{-\lambda}} + c_2 x^{-\sqrt{-\lambda}}. \tag{1067}$$

We now consider the three possible cases to determine the form of the eigenfunctions.

Case 1: Suppose $\lambda = 0$. This would imply that $y = \alpha$ for some constant α. The boundary condition then implies $\alpha = 0$, making the eigenfunction y identically zero, which contradicts the fact that eigenfunctions are nonzero functions.

Case 2: Suppose $\lambda < 0$. Let $\mu := \sqrt{-\lambda}$. Then observe the first boundary condition implies

$$0 = y(1) = \left[c_1 x^{\mu} + c_2 x^{-\mu}\right]_{x=1} = c_1 + c_2 \quad \Longrightarrow \quad \frac{c_1}{c_2} = -1. \tag{1068}$$

Similarly,

$$0 = y(2) = \left[c_1 x^{\mu} + c_2 x^{-\mu}\right]_{x=2} = c_1 2^{\mu} + c_2 2^{-\mu} \quad \Longrightarrow \quad \frac{c_1}{c_2} = -2^{-2\mu}. \tag{1069}$$

Combining these two results reveals

$$-1 = \frac{c_1}{c_2} = -2^{-2\mu} \quad \Longrightarrow \quad -2\mu = 0 \quad \Longrightarrow \quad \mu = 0, \tag{1070}$$

a contradiction.

Case 3: Lastly, suppose $\lambda > 0$. Set $\alpha := \sqrt{\lambda}$ so that we may write

$$\begin{aligned} y = c_1 x^{i\alpha} + c_2 x^{-i\alpha} &= c_1 \exp\left(\ln(x^{i\alpha})\right) + c_2 \exp\left(\ln(x^{-i\alpha})\right) \\ &= c_1 \exp\left(i\alpha \ln(x)\right) + c_2 \exp(-i\alpha \ln(x)) \\ &= d_1 \sin(\alpha \ln(x)) + d_2 \cos(\alpha \ln(x)), \end{aligned} \tag{1071}$$

for some scalars d_1 and d_2, where we recall the relation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Then observe

$$0 = y(1) = d_1 \sin(\alpha(\ln(1))) + d_2 \cos(\alpha \ln(1)) = d_1 \sin(0) + d_2 \cos(0) = d_2. \tag{1072}$$

Since eigenfunctions are not identically zero, it follows that $d_1 \neq 0$. The second condition then implies

$$0 = y(2) = d_1 \sin(\alpha \ln(2)) \quad \Longrightarrow \quad \alpha \ln(2) = k\pi, \tag{1073}$$

for some $k \in \mathbb{Z}$. Combined with the fact $\alpha = \sqrt{\lambda} > 0$, we see the eigenvalues are given by

$$\lambda_k = \left(\frac{k\pi}{\ln(2)}\right)^2, \quad \text{for all } k \in \mathbb{Z}^+, \tag{1074}$$

and the resulting eigenfunctions are given, up to a scalar constant, by

$$y_k = \sin\left(\frac{k\pi}{\ln(2)} \cdot \ln(x)\right), \quad \text{for all } k \in \mathbb{Z}^+. \tag{1075}$$

b) We proceed by employing Sturm-Liouville theory. Letting $p = x$, $q = 0$, and $r = 1/x$, the ODE may be rewritten as

$$\begin{cases} (py')' + qy = -\lambda yr \quad \text{in } (0,1), \\ 1 \cdot y(1) + 0 \cdot y'(1) = 0, \\ 1 \cdot y(2) + 0 \cdot y'(2) = 0, \end{cases} \tag{1076}$$

which is in Sturm-Liouville form. Therefore, the operator $Ly := ((py')' + qy)/r = x(xy')'$ is self-adjoint with respect to the scalar product

$$\langle f, g \rangle := \int_1^2 f(x)g(x)r(x)\ \mathrm{d}x. \tag{1077}$$

This implies the set of eigenfunctions $\{y_k\}$ is orthogonal since for $m \neq n$ we have $\lambda_m \neq \lambda_n$ and

$$-\lambda_m \langle y_m, y_n \rangle = \langle Ly_m, y_n \rangle = \langle y_m, Ly_n \rangle = -\lambda_n \langle y_m, y_n \rangle \quad \Longrightarrow \quad 0 = (\lambda_m - \lambda_n) \langle y_m, y_n \rangle, \tag{1078}$$

and so $\langle y_m, y_n \rangle = 0$. Consequently, the set of eigenfunctions $\{y^k\}$ form an orthogonal basis for the space $L^2(1,2)$ equipped with the scalar product $\langle \cdot, \cdot \rangle$. So, let v be a solution to the given ODE in 2b) and set $g(x) := x\ln(x)$. Then there exists scalars $\{\alpha_k\}_{k=1}^\infty \subset \mathbb{R}$ and $\{\beta_k\}_{k=1}^\infty \subset \mathbb{R}$ such that

$$v = \sum_{k=1}^\infty \alpha_k y_k \quad \text{and} \quad g = \sum_{k=1}^\infty \beta_k y_k, \tag{1079}$$

since g and v are continuous and, thus, in $L^2(1,2)$. Observe

$$\langle g, y_n \rangle = \langle \sum_{k=1}^\infty \beta_k y_k, y_n \rangle = \sum_{k=1}^\infty \beta_k \langle y_k, y_n \rangle = \beta_n \langle y_n, y_n \rangle \quad \Longrightarrow \quad \beta_n = \frac{\langle g, y_n \rangle}{\langle y_n, y_n \rangle}, \quad \text{for all } n \in \mathbb{Z}^+. \tag{1080}$$

Expanding our series reveals

$$\sum_{k=1}^\infty \beta_k y_k = g = (L+3)v = (L+3)\sum_{k=1}^\infty \alpha_k y_k = \sum_{k=1}^\infty \alpha_k(-\lambda_k + 3)y_k. \tag{1081}$$

Since the $\{y^k\}$ forms an orthogonal set, we may then equate coefficients to deduce

$$\alpha_k = \frac{\beta_k}{3 - \lambda_k}, \quad \text{for all } k \in \mathbb{Z}^+, \tag{1082}$$

where we note the division is well-defined since our work in a) implies there are no eigenfunctions with eigenvalue 3. Together, (1079), (1080), and (1082) yields the solution v to the given inhomogeneous ODE.

□

F14.3. Consider Burgers' equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = \cos(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1083}$$

a) Derive an implicit form of the solution $u(x,t)$ in terms of the initial data $u(x,0)$.

b) What is $\max_{x\in\mathbb{R}} u(x,t)$? You will need to use an implicit expression in terms of the initial data.

Solution:

a) We proceed by using the method of characteristics. Let $F(p,q,z,x,t) = q + zp$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the ODE system of characteristic ODE

$$\begin{cases} \dot{x}(s) = F_p = z, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0, \quad z(0) = \cos(x_0). \end{cases} \tag{1084}$$

This implies $t = s$ and z is constant along characteristics. Thus,

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t z(\tau)\ \mathrm{d}\tau = x_0 + t\cos(x_0). \tag{1085}$$

Consequently,

$$u(x,t) = z(t) = z(0) = \cos(x_0), \tag{1086}$$

where x_0 is defined implicitly via (1085).

b) We claim that for each t where $u(\cdot,t)$ is defined, $\max_x u(x,t) = 1$. This note is important since, although not noted in any way in the prompt, $|u_x|$ blows up by time $t = 1$. Note $u(x,t)$ is bounded above by unity for all time since the cosine function is bounded above by unity, i.e., $\cos(x_0) \leq 1$ for all $x_0 \in \mathbb{R}$. For the characteristic $(\tilde{x}(t),t)$ starting at the origin $(x_0,0) = (0,0)$, we see $\cos(x_0) = 1$. Whence

$$1 \geq \max_x u(x,t) \geq u(\tilde{x},t) = \cos(x_0) = \cos(0) = 1 \quad \Longrightarrow \quad \max_x u(x,t) = 1, \tag{1087}$$

as desired. □

F14.5. Given $\phi \in H^1(0,1)$ with $\phi(0) = 0$, define the energy

$$e(\phi) = \int_0^1 \psi(\phi_x) \, \mathrm{d}x - T\phi(1), \quad \psi(F) := (F^2 - 1)^2. \tag{1088}$$

a) Derive a differential equation (and associated boundary conditions) satisfied by the extrema of this energy.

b) Are the extrema unique?

Solution:

a) First define the admissibility class $\mathcal{A} := \{y \in H^1(0,1) : y(0) = 0\}$ and note $\mathcal{A}$ is a closed subset of $H^1(0,1)$. We seek to identify a PDE satisfied by extrema of $e : \mathcal{A} \to \mathbb{R}$. Let $u \in \mathcal{A}$ be an extrema of e. For all nonzero $\varepsilon \in \mathbb{R}$ and $v \in \mathcal{A}$ we see $u + \varepsilon v \in \mathcal{A}$ and

$$\frac{e(u+\varepsilon v) - e(u)}{\varepsilon} = \int_0^1 \frac{\psi(u_x + \varepsilon v_x) - \psi(v_x)}{\varepsilon} \, \mathrm{d}x - Tv(1). \tag{1089}$$

We shall compute the first variation, but first justify its existence. Note ψ may be expanded and written as a polynomial, i.e., there exists scalars $c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$\psi(\alpha) = \sum_{k=0}^{4} c_k \alpha^k, \tag{1090}$$

which implies

$$\begin{aligned} \psi(u_x + \varepsilon v_x) - \psi(u_x) &= \sum_{k=0}^{4} c_k \left((u_x + \varepsilon v_x)^k - u_x^k \right) \\ &= \sum_{k=0}^{4} c_k \left(\left[\sum_{\ell=0}^{k} \binom{k}{\ell} u_x^{k-\ell} (\varepsilon v_x)^\ell \right] - u^k \right) \\ &= \sum_{k=1}^{n} \sum_{\ell=1}^{k} c_k \binom{k}{\ell} u_x^{k-\ell} (\varepsilon v_x)^\ell. \end{aligned} \tag{1091}$$

For each nonzero $\varepsilon \in (-1,1)$, it follows that

$$\left| \frac{\psi(u_x + \varepsilon v_x) - \psi(u_x)}{\varepsilon} \right| = \left| \sum_{k=1}^{n} \sum_{\ell=1}^{k} c_k \binom{k}{\ell} u_x^{k-\ell} v_x^\ell \varepsilon^{\ell-1} \right| \leq \sum_{k=1}^{n} \sum_{\ell=1}^{k} |c_k| \binom{k}{\ell} \left| u_x^{k-\ell} v_x^\ell \right|. \tag{1092}$$

Note the right hand side is contained in $L^1(0,1)$ due to the fact $u_x, v_x \in L^2(0,1)$ and the embedding of L^p spaces. Whence the left hand side is dominated by an integrable function, and the dominated convergence theorem implies

$$\begin{aligned} \delta e(u,v) &= \lim_{\varepsilon\to 0} \frac{e(u+\varepsilon v) - e(u)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \int_0^1 \frac{\psi(u_x+\varepsilon v_x) - \psi(v_x)}{\varepsilon}\ \mathrm{d}x - Tv(1) \\ &= \int_0^1 \lim_{\varepsilon\to 0} \frac{\psi(u_x+\varepsilon v_x) - \psi(v_x)}{\varepsilon}\ \mathrm{d}x - Tv(1) \\ &= \int_0^1 \psi'(u_x) v_x\ \mathrm{d}x - Tv(1). \end{aligned} \tag{1093}$$

Integrating by parts reveals

$$\begin{aligned} \delta e(u,v) &= \int_0^1 -\partial_x \left[\psi'(u_x)\right] v\ \mathrm{d}x + \left[\psi'(u_x) v\right]_0^1 - Tv(1) \\ &= \int_0^1 -\partial_x \left[\psi'(u_x)\right] v\ \mathrm{d}x + v(1)\left[\psi'(u_x(1)) - T\right]. \end{aligned} \tag{1094}$$

Since v was chosen arbitrarily and the fact u is an extrema implies $\delta e(u,v) = 0$ for all $v \in \mathcal{A}$, we deduce u satisfies

$$\begin{cases} \partial_x[\psi'(u_x)] = 0 & \text{in } (0,1), \\ \psi'(u_x) = T & \text{on } \{x=1\}, \\ u = 0 & \text{on } \{x=0\}. \end{cases} \tag{1095}$$

b) We claim the solutions are not necessarily unique. For example, suppose $T = 0$. Then observe $u = 0$ and $u = x$ are distinct and both satisfy $u(0) = 0$ and

$$\left[\psi'(u_x)\right]_{x=1} = \left[4u_x(u_x^2 - 1)\right]_{x=1} = 0 = T, \tag{1096}$$

and

$$\partial_x\left[\psi'(u_x)\right] = \partial_x\left[4u_x(u_x^2-1)\right] = \partial_x 0 = 0 \ \text{ in } (0,1). \tag{1097}$$

This shows the solutions to (1095) are *not* necessarily unique.

□

2014 Spring

S14.6. The function $y(t)$ satisfies the ODE

$$\ddot{y} = -y(1-y)^2. \tag{1098}$$

a) Determine the stability of any stationary points (justify your answers).

b) Sketch the solution orbits in the (y, y') phase plane.

c) Now suppose that damping is added to the system to yield

$$\ddot{y} + |y|\dot{y} = -y(1-y)^2. \tag{1099}$$

Prove the point $(y, y') = (0, 0)$ is asymptotically stable.

Solution:

a) Let $x = \dot{y}$ so that the ODE may be rewritten as the system

$$\dot{y} = x, \quad \dot{x} = -y(1-y)^2. \tag{1100}$$

The equilibrium points are $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$. Since

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial x}\left[-y(1-y)^2\right] + \frac{\partial}{\partial y}[x] = 0, \tag{1101}$$

the system is Hamiltonian. This implies each equilibrium point is either a center or a saddle. The Jacobian $J(x, y)$ for the system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & -1+4y-3y^2 \\ 1 & 0 \end{pmatrix}. \tag{1102}$$

Thus,

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1103}$$

which has eigenvalues $\lambda = \pm i$. Therefore, $(0,0)$ is a center, which is stable. Also,

$$J(0,1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1104}$$

which has 0 as its repeated eigenvalue, and this does not reveal whether $(0,1)$ is a stable or not. However, that just to the above $(x,y) = (0,1)$ we see $\dot{x} < 0$ and just below $(0,1)$ we see $\dot{x} < 0$ also. And, just to the left of $(0,1)$ we see $\dot{y} < 0$ while to the right of $(0,1)$ we see $\dot{y} > 0$. So, $(0,1)$ must be an unstable saddle.

b) We sketch the solution in the figure below.

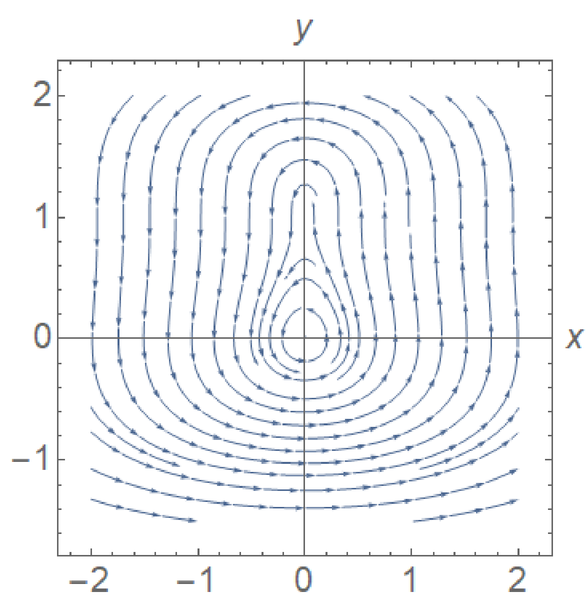

Figure 27: ODE plot for S14.6.

c) We proceed by applying Lasalle's theorem. To show $(0,0)$ is asymptotically stable, it suffices to define a function $V(x,y)$ for which $V(0,0) = 0$, $\dot{V}(x,y) \leq 0$ everywhere, $V(x,y) > 0$ in a neighborhood of $(0,0)$, and $(0,0)$ is the unique fixed point for which $\dot{V}(x,y) = 0$. We first derive such a function V, using the undamped version of the ODE system $(\tilde{x}, \tilde{y})$. Observe[40]

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}} = \frac{\tilde{x}}{-\tilde{y}(1-\tilde{y})^2} \quad \Longrightarrow \quad \left(-\tilde{y} + 2\tilde{y}^2 - \tilde{y}^3\right) \mathrm{d}\tilde{y} = \tilde{x}\ \mathrm{d}\tilde{x}. \tag{1105}$$

[40] Alternatively, we could take the approach of integrating to find the Hamiltonian, as done on other exams.

Integrating reveals there exists $\alpha \in \mathbb{R}$ such that

$$0 = \frac{\tilde{y}^2}{2} - \frac{2\tilde{y}^3}{3} + \frac{\tilde{y}^4}{4} + \frac{\tilde{x}^2}{2} + \alpha. \tag{1106}$$

Define the Lyapunov function $V(x, y)$ via

$$V(x, y) := \frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} + \frac{x^2}{2}. \tag{1107}$$

Then $V(0, 0) = 0$. Furthermore, for the damped ODE we see

$$\dot{y} = x, \quad \dot{x} = -y(1-y)^2 - |y|x, \tag{1108}$$

and so

$$\begin{aligned} \dot{V} &= \left(y - 2y + y^3\right)\dot{y} + x\dot{x} \\ &= y(1-y)^2\dot{y} + x\dot{x} \\ &= y(1-y)^2 x + x\left[-y(1-y)^2 - |y|x\right] \\ &= -|y|x^2 \\ &\leq 0. \end{aligned} \tag{1109}$$

Furthermore, as $y \to 0$, the y^2 term in $V(x, y)$ dominates, and so

$$V(x, y) \sim \frac{x^2 + y^2}{2} \quad \text{as } y \to 0, \tag{1110}$$

revealing $V(x, y) > 0$ in a neighborhood of $(0, 0)$. Lastly, note (1109) shows $(0, 0)$ is the only fixed point of V for which $\dot{V} = 0$. Thus, we conclude $(0, 0)$ is asymptotically stable.

□

S14.7. Let $u \in C^2(\Omega)$ with $u(x) + \nabla u \cdot n(x) = 0$ for $x \in \partial\Omega$ where $n(x)$ is the outward normal for $x \in \partial\Omega$. Consider $r : C^2(\Omega) \to \mathbb{R}$ defined as the scalar $r(u)$ such that

$$E(r(u)) \leq E(\alpha), \quad \text{for all } \alpha \in \mathbb{R}, \tag{1111}$$

where

$$E(\alpha) = \int_\Omega (\Delta u + \alpha u)^2 \text{ d}x. \tag{1112}$$

a) Show that

$$r(u) = \frac{\int_\Omega |\nabla u|^2 \text{ d}x + \int_{\partial\Omega} u^2 \text{ d}\sigma}{\int_\Omega u^2 \text{ d}x}. \tag{1113}$$

b) Show that if v minimizes r (over functions that satisfy $v + \nabla v \cdot n = 0$) that

$$-\Delta v = r(v)v. \tag{1114}$$

Solution:

a) Define $\phi(\varepsilon) := E(r(u) + \varepsilon)$. Then

$$\begin{aligned}
\phi'(\varepsilon) &= \frac{\text{d}}{\text{d}\varepsilon}\left[\int_\Omega (\Delta u + (r(u) + \varepsilon)u)^2 \text{ d}x\right] \\
&= \int_\Omega \frac{\partial}{\partial\varepsilon}\left[(\Delta u + (r(u) + \varepsilon)u)^2\right] \text{ d}x \\
&= 2\int_\Omega (\Delta u + (r(u) + \varepsilon)u) \cdot u \text{ d}x \\
&= 2\int_\Omega -|\nabla u|^2 + (r(u) + \varepsilon)u^2 \text{ d}x + 2\int_{\partial\Omega} (\nabla u \cdot n)u \text{ d}\sigma \\
&= 2\int_\Omega -|\nabla u|^2 + (r(u) + \varepsilon)u^2 \text{ d}x - 2\int_{\partial\Omega} u^2 \text{ d}\sigma.
\end{aligned} \tag{1115}$$

By our hypothesis (1111), $\phi'(0) = 0$, which implies

$$0 = \int_\Omega -|\nabla u|^2 + r(u)u^2 \text{ d}x - \int_{\partial\Omega} u^2 \text{ d}\sigma = -\left[\int_\Omega |\nabla u|^2 \text{ d}x + \int_{\partial\Omega} u^2 \text{ d}\sigma\right] + r(u)\int_\Omega u^2 \text{ d}x, \tag{1116}$$

upon which rearranging yields the desired equality (1113), where we note $r(u)$ is a constant with respect to the integration and can, thus, be pulled outside the integral.

b) First note $r(u)$ is scale invariant, i.e., $r(cu) = r(u)$ for all $c \neq 0$. This follows from the linearity of the integral and (1113). Consequently, the minimizer v of r is also a solution to

$$\min_{u\in\mathcal{A}} J(u) := \int_\Omega |\nabla u|^2 \ \mathrm{d}x + \int_{\partial\Omega} u^2 \ \mathrm{d}\sigma, \tag{1117}$$

where

$$\mathcal{A} := \{u \in C^2(\Omega) : u + \nabla u \cdot n = 0,\ G(u) = 1\}, \quad G(u) := \int_\Omega u^2 \ \mathrm{d}x. \tag{1118}$$

Lagrange's theorem for multipliers asserts there exists $\lambda \in \mathbb{R}$ such that v satisfies

$$\delta J(v, w) = \lambda \delta G(v, w), \quad \text{for all } w \in \mathcal{A}. \tag{1119}$$

Through direct computation, we see

$$\begin{aligned}
\delta J(u, w) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[J(u + \varepsilon w)\right]_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_\Omega |\nabla u + \varepsilon \nabla w|^2 \ \mathrm{d}x + \int_{\partial\Omega} (u + \varepsilon w)^2 \ \mathrm{d}\sigma\right]_{\varepsilon=0} \\
&= \int_\Omega 2\nabla u \cdot \nabla w \ \mathrm{d}x + \int_{\partial\Omega} 2uw \ \mathrm{d}\sigma \\
&= \int_\Omega -2w\Delta u \ \mathrm{d}x + \underbrace{\int_{\partial\Omega} 2(u + \nabla u \cdot n)w \ \mathrm{d}\sigma}_{=0} \\
&= \int_\Omega -2w\Delta u \ \mathrm{d}x.
\end{aligned} \tag{1120}$$

Similarly,

$$\delta G(u, w) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[G(u + \varepsilon w)\right]_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_\Omega (u + \varepsilon w)^2 \ \mathrm{d}x\right]_{\varepsilon=0} = \int_\Omega 2uw \ \mathrm{d}x. \tag{1121}$$

Compiling our results yields

$$0 = \int_\Omega (\Delta v + \lambda v)w \ \mathrm{d}x \quad \implies \quad -\Delta v = \lambda v, \tag{1122}$$

where the implication holds since the first equality holds for all $w \in \mathcal{A}$. However,

$$\begin{aligned} r(v) &= \int_\Omega |\nabla v|^2 \ \mathrm{d}x + \int_{\partial\Omega} v^2 \ \mathrm{d}\sigma \\ &= \int_\Omega -v\Delta v \ \mathrm{d}x + \underbrace{\int_{\partial\Omega} (v + \nabla v \cdot n) v \ \mathrm{d}\sigma}_{=0} \\ &= \int_\Omega \lambda v^2 \ \mathrm{d}x \\ &= \lambda. \end{aligned} \tag{1123}$$

Whence (1122) and (1123) imply (1114) holds, as desired.

□

S14.8. Consider Burgers' equation

$$u_t + f(u)_x = 0 \tag{1124}$$

for $u(x,t)$ over the periodic domain $(0,1)\times(0,\infty)$, where $f(u) = \frac{1}{2}u^2$. Solve the periodic initial value problem for $u(x,t)$ with initial data

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < \alpha, \\ 0 & \text{if } \alpha < x < 1, \end{cases} \tag{1125}$$

for an arbitrary $\alpha \in (0,1)$.

Solution:

We proceed by using the method of characteristics. Let $F(p,q,z,x,t) = q + f'(z)p = q + zp$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the system of characteristic ODE

$$\begin{cases} \dot{x}(s) = F_p = z, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0, \quad z(0) = 0. \end{cases} \tag{1126}$$

This implies $t = s$ and z is constant along characteristics. Thus,

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t z(\tau)\ \mathrm{d}\tau = x_0 + g(x_0)t = \begin{cases} x_0 + t, & \text{if } x_0 \in (0,\alpha), \\ x_0, & \text{if } x_0 \in (\alpha, 1), \end{cases} \tag{1127}$$

where we set $g(x) := u(x,0)$. This shows the characteristics crash immediately at $(\alpha, 0)$. Now we must consider two separate cases:

Case 1: $\alpha \geq 1/2$. Applying the Rankine-Hugoniot (RH) condition yields that the shock curve, parameterized as $(s(t),t)$, satisfies $s(0) = \alpha$ and

$$\dot{s}(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot 0^2}{1-0} = \frac{1}{2} \quad \Longrightarrow \quad s(t) = \alpha + \frac{t}{2}. \tag{1128}$$

A time $t^\star = 2(1-\alpha) \leq 1$, the shock curve hits $(s(t^\star), t^\star) = (1, t^\star)$, at which we must again use the RH condition to determine the future behavior of the shock curve. In $(0,1)\times(0,t^\star)$ we have

$$u(x,t)=\begin{cases} x/t & \text{if } 0<x<t,\\ 1 & \text{if } t\leq x\leq s(t),\\ 0 & \text{if } s(t)\leq x<1,\end{cases} \tag{1129}$$

where the value for $0<x/t<1$ is determined by assuming u is of the form $u(x,t)=v(x/t)$ and plugging this into our PDE to find

$$0=u_t+f'(u)u_x=v'\cdot-\frac{x}{t^2}+vv'\cdot\frac{x}{t}=\frac{v}{t}\left[v-\frac{x}{t}\right]\quad\Longrightarrow\quad v=\frac{x}{t}, \tag{1130}$$

assuming v is nonzero.

For time $t>1$, the RH condition for the shock curve reveals

$$\dot{s}(t)=\frac{f(u_\ell)-f(u_r)}{u_\ell-u_r}=\frac{\frac{1}{2}\cdot(x/t)^2+\frac{1}{2}\cdot 0^2}{x/t-0}=\frac{x}{2t}. \tag{1131}$$

Using separation of variables with $x=s(t)$ gives

$$\int\frac{\mathrm{d}\tilde{s}}{\tilde{s}}=\int\frac{\mathrm{d}\tilde{t}}{2\tilde{t}}\quad\Longrightarrow\quad s(t)=Ct^{1/2}, \tag{1132}$$

where C is such that $s(t^\star)=1$.

(Return and complete. This is MESSY.) □

2013 Fall

F13.1. Consider the ODE system

$$\dot{x} = v, \quad \dot{v} = -\frac{\mathrm{d}\psi}{\mathrm{d}x}(x) - \alpha v, \tag{1133}$$

for a given function $\psi(x) \in C^2(\mathbb{R})$.

a) Find all stationary points and analyze their type when $\psi(x) = \frac{1}{2}(x^2-1)^2$.

b) Sketch the phase plane for $\psi(x) = \frac{1}{2}(x^2-1)^2$.

c) Show that $H(x,v) = \frac{v^2}{2} + \psi(x)$ is nonincreasing in time.

Solution:

a) Since $\psi'(x) = 2x(x^2-1)$, we see the stationary points are $(x,v) = (\pm 1, 0)$ and $(x,v) = (0,0)$. The Jacobian for the system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\psi''(x) & -\alpha \end{pmatrix}. \tag{1134}$$

Then observe

$$\psi''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\cdot 2(x^2-1)^1 \cdot 2x\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[2x^3 - 2x\right] = 6x^2 - 2. \tag{1135}$$

From this, we see

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 2 & -\alpha \end{pmatrix}, \tag{1136}$$

which has eigenvalues

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2+8}}{2}, \tag{1137}$$

and so $(0,0)$ is a saddle point. In similar fashion,

$$J(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -(6\cdot 1 - 2) & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -\alpha \end{pmatrix}, \tag{1138}$$

which has eigenvalues satisfying the characteristic equation

$$0 = \lambda(\lambda + \alpha) + 4 = \lambda^2 + \lambda\alpha + 4 \quad \Longrightarrow \quad \lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 16}}{2}. \tag{1139}$$

We assume α corresponds to a damping term and can be taken to be positive. In this case, we see the real portion of the eigenvalues of $J(\pm 1, 0)$ is negative. Then $(\pm 1, 0)$ can be characterized according to the following cases. If $\alpha \in (0, 4)$, then the eigenvalues are complex-valued and $(\pm 1, 0)$ correspond to inward pointing stable spirals. If $\alpha = 4$, then $(\pm 1, 0)$ correspond to nodes. And, if $\alpha > 4$, then $(\pm 1, 0)$ form sinks.

b) The phase plane for $\alpha = 1$ is given below.

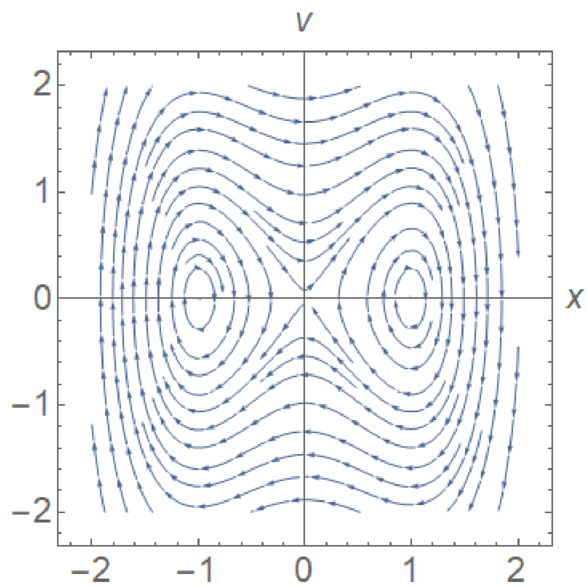

Figure 28: Phase plane for F13.1.

c) Differentiating in time reveals

$$\dot{H} = v\dot{v} + \psi'(x)\dot{x} = v\left(-\psi'(x) - \alpha v\right) + \psi'(x)v = -\alpha v^2 \leq 0. \tag{1140}$$

This shows $\dot{H} \leq 0$, and so we conclude H is nonincreasing, as desired.

$\square$

F13.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \Omega \to \mathbb{R}$ for $\Omega \subset \mathbb{R}^2$. Let $E(u)$ be defined as

$$E(u) = \int_\Omega f(u_x, u_y) + gu \ \mathrm{d}A + \frac{1}{2}\int_{\partial\Omega} u^2 \ \mathrm{d}\sigma, \tag{1141}$$

for $u \in C^2(\Omega)$.

a) Suppose $f(v,w) = \frac{1}{2}(v^2 + w^2)$ and $g \in L^2(\Omega)$. Show that the minimizer exists in $H^1(\Omega)$.

b) What differential equation with what boundary equations does the minimizer of E over $u \in C^2(\Omega)$ satisfy? Assume $f \in C^2(\mathbb{R}^2)$ and $g \in L^2(\Omega)$.

Solution:

a) Letting $z = (x,y)$, note the functional may be rewritten as

$$E(u) = \int_\Omega \frac{1}{2}|Du|^2 + gu \ \mathrm{d}z + \frac{1}{2}\int_{\partial\Omega} u^2 \ \mathrm{d}\sigma. \tag{1142}$$

For notational convenience, set $H := H^1(\Omega)$ and let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_H$ be the usual scalar product and norm on H, respectively. We claim there exists a unique $\overline{u} \in H$ such that

$$B[\overline{u}, v] = \ell(v), \quad \text{for all } v \in H, \tag{1143}$$

where we define the bilinear form B and the linear form ℓ via

$$B[u,v] := \int_\Omega Du \cdot Dv \ \mathrm{d}z + \int_{\partial\Omega} Tuv \ \mathrm{d}\sigma \quad \text{and} \quad \ell(v) := -\int_\Omega gv \ \mathrm{d}z, \tag{1144}$$

with T denoting the trace (which we shall omit writing). This implies, for all $v \in H$ and $\varepsilon \in \mathbb{R}$,

$$\begin{aligned} E(\overline{u} + \varepsilon v) &= \int_\Omega \frac{1}{2}|D\overline{u} + \varepsilon Dv|^2 + g(\overline{u} + \varepsilon v) \ \mathrm{d}z + \frac{1}{2}\int_{\partial\Omega} (\overline{u} + \varepsilon v)^2 \ \mathrm{d}\sigma \\ &= \int_\Omega \frac{1}{2}|D\overline{u}|^2 + \varepsilon D\overline{u} \cdot Dv + \frac{\varepsilon^2}{2}|Dv|^2 + g\overline{u} + \varepsilon gv \ \mathrm{d}z + \frac{1}{2}\int_{\partial\Omega} \overline{u}^2 + 2\varepsilon\overline{u}v + \varepsilon^2 v^2 \ \mathrm{d}\sigma \\ &= E(\overline{u}) + \varepsilon\left(B[\overline{u}, v] - \ell(v)\right] + \varepsilon^2 \left[\frac{1}{2}\int_\Omega |Dv|^2 \ \mathrm{d}z + \frac{1}{2}\int_{\partial\Omega} v^2 \ \mathrm{d}\sigma\right] \\ &= E(\overline{u}) + \varepsilon^2 \left[\frac{1}{2}\int_\Omega |Dv|^2 \ \mathrm{d}z + \frac{1}{2}\int_{\partial\Omega} v^2 \ \mathrm{d}\sigma\right]. \end{aligned} \tag{1145}$$

Consequently, for all $v \in H$,

$$\delta E(\overline{u}, v) = \lim_{\varepsilon \to 0^+} \frac{E(\overline{u} + \varepsilon v) - E(\overline{u})}{\varepsilon} = \lim_{\varepsilon \to 0^+} \varepsilon \left[\frac{1}{2} \int_\Omega |Dv|^2 \text{ d}z + \frac{1}{2} \int_{\partial\Omega} v^2 \text{ d}\sigma \right] = 0. \tag{1146}$$

Therefore, $\overline{u}$ is a critical point of E. Furthermore, E is convex since, for all $u, v \in H$ and $\lambda \in (0, 1)$,

$$\begin{aligned} E(\lambda u + (1 - \lambda v)) &= \frac{1}{2}\|\lambda Du + (1-\lambda)Dv\|^2_{L^2(\Omega)} + \frac{1}{2}\|\lambda u + (1-\lambda)v\|^2_{L^2(\partial\Omega)} + \int_\Omega g[\lambda u + (1-\lambda)v] \text{ d}z \\ &\leq \lambda \left[\frac{1}{2}\|Du\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\partial\Omega)} + \int_\Omega gu \text{ d}z \right] \\ &+ (1-\lambda) \left[\frac{1}{2}\|Dv\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\partial\Omega)} + \int_\Omega gv \text{ d}z \right] \\ &= \lambda E(u) + (1-\lambda)E(v), \end{aligned} \tag{1147}$$

where we note norms are convex, compositions of convex functions are convex, and $\phi(\alpha) := \alpha^2$ is convex since $\phi''(\alpha) = 2 > 0$. Therefore, all critical points of E are minimizers of E, from which we conclude $\overline{u} \in H$ is a minimizer of E.

All that remains is to verify (1143), which we do by applying the Lax-Milgram theorem. It suffices to show $B[\cdot, \cdot]$ is bounded and coercive and $\ell(\cdot)$ is bounded. Assuming Ω is bounded and $\partial\Omega$ is C^1, the trace theorem implies there exists $C > 0$, dependent only on Ω, such that

$$\begin{aligned} |B[u, v]| &\leq \|Du \cdot Dv\|_{L^1(\Omega)} + \|uv\|_{L^1(\Omega)} \\ &\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} \\ &\leq \|u\|_H\|v\|_H + C\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq (1 + C)\|u\|_H\|v\|_H, \end{aligned} \tag{1148}$$

where we have made repeated use of Hölder's inequality above. Observe ℓ is bounded since

$$|\ell(v)| \leq \|gv\|_{L^1(\Omega)} \leq \|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}\|v\|_H, \quad \text{for all } v \in H. \tag{1149}$$

Lastly, by way of contradiction, suppose $B[\cdot, \cdot]$ is not coercive, i.e., there does *not* exist a scalar $\beta > 0$ such that

$$\beta\|u\|^2_H \leq B[u, u], \quad \text{for all } u \in H. \tag{1150}$$

This implies there exists a sequence of nonzero functions $\{u^k\} \subset H$ such that

$$\lim_{k\to\infty} B[u^k, u^k] < \frac{1}{k}\|u^k\|_H^2, \quad \text{for all } k \in \mathbb{N}. \tag{1151}$$

Setting $v^k := u^k/\|u^k\|_H$ yields $\|v^k\|_H = 1$, for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \|Dv^k\|_{L^2(\Omega)}^2 + \|Tv^k\|_{L^2(\partial\Omega)}^2 = \lim_{k\to\infty} B[v^k, v^k] = 0. \tag{1152}$$

Assuming $\partial\Omega$ is C^1 and Ω is bounded, the Rellich-Kondrachov compactness theorem[41] asserts H is compactly embedded in $L^2(\Omega)$. Thus $\{v^k\}$ is precompact in $L^2(\Omega)$, i.e., there is a subsequence $\{v^{n_k}\} \subset \{v^k\}$ and $v^\star \in L^2(\Omega)$ such that

$$\lim_{k\to\infty} \|v^{n_k} - v^\star\|_{L^2(\Omega)} = 0. \tag{1153}$$

Let α be a multi-index with $|\alpha| = 1$ and $\phi \in C_c^\infty(\Omega)$. Then observe

$$\lim_{k\to\infty} \left|\int_\Omega v^{n_k}\partial^\alpha\phi \ \mathrm{d}x\right| = \lim_{k\to\infty} \left|-\int_\Omega \phi\partial^\alpha v^{n_k} \ \mathrm{d}x\right| \le \lim_{k\to\infty} \|\phi\partial^\alpha v^{n_k}\|_{L^1(\Omega)} \le \lim_{k\to\infty} \|\phi\|_{L^2(\Omega)}\|Dv^{n_k}\|_{L^2(\Omega)} = 0, \tag{1154}$$

where the final equality holds by (1152) and the second inequality is an application of Hölder's inequality. Similarly, (1153) implies

$$\lim_{k\to\infty} \left|\int_\Omega (v^\star - v^{n_k})\phi \ \mathrm{d}x\right| \le \lim_{k\to\infty} \|(v^\star - v^{n_k})\phi\|_{L^1(\Omega)} \le \lim_{k\to\infty} \|v^\star - v^{n_k}\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} = 0. \tag{1155}$$

These two results reveal

$$\int_\Omega v^\star\partial^\alpha\phi \ \mathrm{d}x = \lim_{k\to\infty}\int_\Omega (v^\star - v^{n_k} + v^{n_k})\partial^\alpha\phi \ \mathrm{d}x = \lim_{k\to\infty}\int_\Omega (v^\star - v^{n_k})\phi \ \mathrm{d}x + \int_\Omega v^{n_k}\partial^\alpha\phi \ \mathrm{d}x = 0. \tag{1156}$$

By the arbitrariness of α and ϕ, this implies $v^\star$ has a weak derivative $Dv^\star$ and $Dv^\star = 0$ a.e. in Ω,

[41] See Theorem 1 on page 288 of the PDE text by Evans.

i.e., $v^\star$ is constant a.e. in Ω. Again using the trace theorem, we see

$$\begin{aligned}\|v^\star\|_{L^2(\partial\Omega)} &\leq \lim_{k\to\infty} \|v^\star - v^{n_k}\|_{L^2(\partial\Omega)} + \|v^{n_k}\|_{L^2(\partial\Omega)} \\ &\leq \lim_{k\to\infty} C\|v^\star - v^{n_k}\|_{L^2(\Omega)} + \|v^{n_k}\|_{L^2(\partial\Omega)} \\ &= C0 + 0 \\ &= 0,\end{aligned} \tag{1157}$$

where the third line follows from (1152) and (1153). Since $v^\star$ is constant a.e., this reveals $v^\star = 0$ a.e. in Ω. Therefore,

$$1 = \lim_{k\to\infty} \|v^{n_k}\|_H = \lim_{k\to\infty} \|v^{n_k}\|^2_{L^2(\Omega)} + \|Dv^{n_k}\|^2_{L^2(\Omega)} = \lim_{k\to\infty} \|v^{n_k} - v^\star\|^2_{L^2(\Omega)} + \|Dv^{n_k}\|^2_{L^2(\Omega)} = 0, \tag{1158}$$

which implies $1 = 0$, a contradiction. Whence the initial assumption was false, and the result follows.

b) Using a general $f \in C^2$, we compute the first variation of E,

$$\begin{aligned}\delta E(u,v) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[E(u+\varepsilon v)\right]_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_\Omega f(Du + \varepsilon Dv) + g(u+\varepsilon v)\ \mathrm{d}z + \frac{1}{2}\int_{\partial\Omega}(u+\varepsilon v)^2\ \mathrm{d}\sigma\right]_{\varepsilon=0} \\ &= \left[\int_\Omega \nabla_q f(Du+\varepsilon Dv)\cdot Dv + gv\ \mathrm{d}z + \int_\Omega (u+\varepsilon v)v\ \mathrm{d}\sigma\right]_{\varepsilon=0} \\ &= \int_\Omega \nabla_q f(Du)\cdot Dv + gv\ \mathrm{d}z + \int_{\partial\Omega} uv\ \mathrm{d}\sigma,\end{aligned} \tag{1159}$$

where we let $q = (u, v)$ denote the input argument of f. Note this generalizes the form of $B[u,v] - \ell(v)$ in a) where there $f(Du) = \frac{1}{2}|Du|^2$ and $\nabla_q f(Du) = Du$. If $\overline{u} \in C^2(\Omega)$ is a minimizer of E, then $\delta E(u,v) = 0$ for all $v \in H$, which implies

$$\begin{aligned}0 &= \int_\Omega \nabla_q f(D\overline{u})\cdot Dv + gv\ \mathrm{d}z + \int_{\partial\Omega} \overline{u}v\ \mathrm{d}\sigma \\ &= \int_\Omega \left(-\nabla_z\cdot[\nabla_q f(D\overline{u})] + g\right)v\ \mathrm{d}z + \int_{\partial\Omega}\left(\nabla_q f(D\overline{u})\cdot n + \overline{u}\right)v\ \mathrm{d}\sigma,\end{aligned} \tag{1160}$$

where n is the outward normal along $\partial\Omega$. By the arbitrariness of v, it follows that $\overline{u}$ satisfies

$$\begin{cases} -\nabla_z \cdot [\nabla_q f(D\overline{u})] + g = 0 & \text{in } \Omega, \\ \nabla_q f(D\overline{u}) \cdot n + \overline{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1161}$$

$\square$

F13.3. Consider the initial value problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1162}$$

Assume there exists $\theta > 0$ such that $f''(u) \geq \theta$ for all u. Show that if $\phi(x) = -x$, then $|u_x| \to \infty$ in finite time.

Solution:

We proceed by using the method of characteristics. Define $F(p,q,z,x,t) = q + f'(z)p$. Taking $q = u_t$, $p = u_x$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = 0 - f''(z)p^2, & p(0) = -1, \\ \dot{q}(s) = -F_t - F_z q = 0 - f''(z)pq, & q(0) = -f'(z(0))p(0), \\ \dot{x}(s) = F_p = f'(z), & x(0) = x_0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = f'(z)p + q = 0, & z(0) = -x_0. \end{cases} \tag{1163}$$

This implies $t = s$ and z is constant along characteristics. Additionally,

$$\dot{p}(s) = -f''(z(s))p^2(s) \leq -\theta p^2(s) \leq 0, \tag{1164}$$

which shows $p(s)$ is nonincreasing. Combined with the fact $p(0) = -1 < 0$, we deduce $p(s) \leq -1 < 0$ for all s. Moreover, using separation of variables reveals

$$\int_{p(0)}^{p(t)} \frac{\mathrm{d}\tilde{p}}{\tilde{p}^2} = \int_0^t -f''(z(\tau))\ \mathrm{d}\tau \leq \int_0^t -\theta\ \mathrm{d}\tau = -\theta t, \tag{1165}$$

which implies

$$-1 - \frac{1}{p(t)} = \frac{1}{p(0)} - \frac{1}{p(t)} \leq -\theta t \quad \Longrightarrow \quad 0 \leq \frac{1}{|p(t)|} = -\frac{1}{p(t)} \leq 1 - \theta t. \tag{1166}$$

Thus,

$$0 = \lim_{t\to(1/\theta)^-} 0 \leq \lim_{t\to(1/\theta)^-} \frac{1}{|p(t)|} \leq \lim_{t\to(1/\theta)^-} 1 - \theta t = 0, \tag{1167}$$

and so the squeeze lemma asserts $1/|p(t)| \to 0$ as $t \to (1/\theta)^-$. However, this is the case if and only if $|p(t)|$ diverges by the time $t = 1/\theta$. Consequently, we conclude $|u_x| = |p(t)| \to \infty$ in finite time. $\square$

F13.4. Consider the PDE[42]

$$\begin{cases} u_{tt} + c^2 u_{xxxx} + au_t = 0, & \text{in } \mathbb{R} \times (0,\infty), \\ u = \phi & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t = \psi & \text{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{1168}$$

where $a > 0$ and ϕ and ψ have compact support.

a) For solutions with compact support, define an associated energy $E(t)$ that is nonincreasing in time.

b) Use this to show that solutions to (1168) are unique.

Solution:

a) Define the energy by

$$E(t) := \frac{1}{2} \int_{\mathbb{R}} u_t^2 + c^2 u_{xx}^2 \ \mathrm{d}x. \tag{1169}$$

For solutions with compact support, this choice of energy $E(t)$ is well-defined. Differentiating in time reveals

$$\begin{aligned} \dot{E}(t) &= \int_{\mathbb{R}} u_t u_{tt} + c^2 u_{xx} u_{xxt} \ \mathrm{d}x \\ &= \int_{\mathbb{R}} u_t u_{tt} - c^2 u_{xxx} u_{xt} \ \mathrm{d}x \\ &= \int_{\mathbb{R}} u_t \left(u_{tt} + c^2 u_{xxxx} \right) \ \mathrm{d}x \\ &= \int_{\mathbb{R}} -a u_t^2 \ \mathrm{d}x \\ &\leq 0. \end{aligned} \tag{1170}$$

The second and third lines hold via integration by parts, noting the boundary terms vanish since u is assumed to have compact support. This shows $\dot{E}(t) \leq 0$ for all $t \in [0,\infty)$. Whence $E(t)$ is nonincreasing.

b) Suppose u and v are solutions to the PDE (1168) with compact support. Then set $w := u - v$, which implies

$$\begin{cases} w_{tt} - c^2 w_{xxxx} + aw_t = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ w = w_t = 0 & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{1171}$$

[42] We have changed the PDE to have a "+" rather than a "-" following u_{tt} as there appears to have been a typo in the original prompt.

Thus, it suffices to show $w = 0$ in $\mathbb{R} \times (0, \infty)$. Now consider the energy $E(t)$ defined as in (1169), but with w in place of u. By (1171), we know $w_t = w_{xx} = 0$ on $\mathbb{R} \times \{t = 0\}$, and so $E(0) = 0$. And, by our result in a), we know $E(t)$ is nonincreasing. However, since the integrand in the definition of $E(t)$ is nonnegative, we deduce $0 \le E(t) \le E(0) = 0$, and so $E(t) = 0$ for all $t \in [0, \infty)$. Assuming $c \neq 0$, this implies $w_t = w_{xx} = 0$ in $\mathbb{R} \times (0, \infty)$. Thus, w is constant in time and linear in x, i.e., of the form

$$w(x, t) = \alpha_1 x + \alpha_2, \tag{1172}$$

for some constants $\alpha_1, \alpha_2 \in \mathbb{R}$. Because $w(0, 0) = 0$, we know $\alpha_2 = 0$. And, since $\alpha_1 x = w(x, 0) = 0$ for all $x \in \mathbb{R}$, $\alpha_1 = 0$ also. (One could also argue from the compact support of w, arising from the compact support of u and v.) Therefore, $w = 0$ in $\mathbb{R} \times (0, \infty)$, as desired.

$\square$

F13.5. Let $u(x,t)$ be the solution of the Cauchy problem for the heat equation

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}\times(0,T), \\ u = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1173}$$

Prove that if there exists scalars C and a such that

$$|u_t| \le Ce^{ax^2} \quad \text{and} \quad |\partial^\beta u| \le Ce^{ax^2} \quad \text{in } \mathbb{R}\times(0,T), \tag{1174}$$

where β is any multi-index with $|\beta| \le 2$, then $u \equiv 0$.

Solution:

(See Theorem 6 on page 57 in §2.3 of the PDE text by Evans.) □

F13.6. Consider the Cauchy problem

$$\begin{cases} u_t - u_x + u^2 = 0 & \text{in } \mathbb{R}\times(0,T), \\ u = \psi & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1175}$$

where ψ is smooth with compact support. Prove the existence and uniqueness of a smooth solution when T is small.

Solution:

We begin by analyzing our PDE via the method of characteristics. Define $F(p,q,z,x,t) := q - p + z^2$. Taking $q = u_t$, $p = u_x$, and $z = u$, we see $F = 0$ and obtain the system of ODE

$$\begin{cases} \dot{x}(s) = F_p = -1, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = -p + q = -z^2, \quad z(0) = \psi(x_0). \end{cases} \tag{1176}$$

This implies $t = s$, $x = x_0 - t$, and

$$z(t) = z(0) + \int_0^t \dot{z}(\tau)\ \mathrm{d}\tau = \psi(x+t) - \int_0^t z^2(\tau)\ \mathrm{d}\tau. \tag{1177}$$

Let $T \in (0,\infty)$ be chosen sufficiently small to ensure

$$4T\|\psi\|_\infty \leq \frac{1}{2}, \tag{1178}$$

where $\|\cdot\|_\infty$ denotes the sup norm and $\psi \in L^\infty(\mathbb{R})$ since ψ is smooth with compact support. Thus, (1177) reveals u is a solution to the PDE if and only if u is a fixed point of the operator $\varphi : V \to V$ defined by

$$\varphi(u)(x,t) := \psi(x+t) - \int_0^t u^2(x-\tau,\tau)\ \mathrm{d}\tau, \tag{1179}$$

where we note our above results show the characteristics are linear, originating at $(x,0)$ and proceeding in the direction $(\dot{x},\dot{t}) = (-1,1)$, and we set $V := B(\mathbb{R}\times(0,T)\to\mathbb{R})$ to be the space of bounded continuous functions mapping $\mathbb{R}\times[0,T)$ into $\mathbb{R}$. Observe V is a complete metric space when equipped with the sup norm $\|\cdot\|_\infty$.

We shall now obtain our result by applying the Banach fixed point theorem. Let $W := \{w \in V : \|w\|_\infty \leq 2\|\psi\|_\infty\}$. We claim W is closed, and so $W \subseteq V$ is complete also. To apply the theorem, it suffices to show the restriction of φ to W maps into W and that φ is a contraction. Indeed, by our choice of W, the definition of φ, and the inequality (1178),

$$\|\varphi(w)\|_\infty \leq \|\psi\|_\infty + T\|w\|^2 \leq \|\psi\|_\infty + 4T\|\psi\|_\infty^2 \leq \|\psi\|_\infty + \frac{1}{2}\cdot\|\psi\|_\infty \leq 2\|\psi\|_\infty \quad \implies \quad \varphi(w) \in W. \tag{1180}$$

Next, for all $u, v \in W$ and $(x,t) \in \mathbb{R}\times[0,T)$, we see

$$\begin{aligned} |\varphi(u)(x,t) - \varphi(v)(x,t)| &= \left| -\int_0^t u^2(x-\tau,\tau) - v^2(x-\tau,\tau)\ \mathrm{d}\tau \right| \\ &\leq \int_0^T \|u-v\|_\infty \|u+v\|_\infty\ \mathrm{d}\tau \\ &= T\|u-v\|_\infty\|u+v\|_\infty \\ &\leq 4T\|\psi\|\|u-v\|_\infty \\ &\leq \frac{1}{2}\|u-v\|_\infty. \end{aligned} \tag{1181}$$

Since the right hand side above forms an upper bound among all $(x,t) \in \mathbb{R}\times(0,T)$, it follows from the definition of the supremum that

$$\|\varphi(u) - \varphi(v)\|_\infty \leq \frac{1}{2}\|u-v\|_\infty, \tag{1182}$$

i.e., φ is a contraction on W. Whence φ has a unique fixed point in W. Since the fixed points of φ are solutions to the PDE (1175), the result follows.

All that remains is to verify W is closed. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be any convergent sequence in W with limit $\tilde{\varphi}$. It suffices to show $\tilde{\varphi}$ is necessarily in W. Since $\|\varphi_n\|_\infty \leq 2\|\psi\|_\infty$, for all $n \in \mathbb{N}$, it follows from elementary analysis and the continuity of the norm that

$$\|\tilde{\varphi}\|_\infty = \lim_{n\to\infty}\|\varphi_n\|_\infty \leq \lim_{n\to\infty} 2\|\psi\|_\infty = 2\|\psi\|_\infty \quad \implies \quad \tilde{\varphi} \in W, \tag{1183}$$

and the proof is complete. □

F13.8. Consider the Neumann problem in the half-plane

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u_y = f & \text{on } \mathbb{R}\times\{y=0\}. \end{cases} \tag{1184}$$

a) Show that if $f = 0$ and $u(x,y) \longrightarrow 0$ as $(x^2+y^2) \longrightarrow \infty$, then $u \equiv 0$.

b) Assume f has compact support and $\int_{\mathbb{R}} f(x)\ \mathrm{d}x = 0$. Prove there exists a solution to the PDE that tends to zero when $(x^2+y^2) \longrightarrow \infty$.

Solution:

a) Let $D := \mathbb{R}\times(0,\infty)$ and $z = (x,y)$. Then, multiplying the PDE by u, we see

$$\begin{aligned} 0 &= \int_D u\Delta u\ \mathrm{d}z \\ &= -\int_D |\nabla u|^2\ \mathrm{d}z + \int_{\partial D} u\frac{\partial u}{\partial n}\ \mathrm{d}\sigma \\ &= -\int_D |\nabla u|^2\ \mathrm{d}z + \underbrace{\int_{-\infty}^{\infty} u(x,0)u_y(x,0)\ \mathrm{d}x}_{=0} \\ &= -\int_D |\nabla u|^2\ \mathrm{d}z. \end{aligned} \tag{1185}$$

This implies $\nabla u = 0$ in D, i.e., u is constant. Since $u \longrightarrow 0$ as $|z| \longrightarrow \infty$, it then follows that $u \equiv 0$.

b) Since f has compact support, f is a Schwarz function and its Fourier transform exists. Thus, taking the Fourier transform of the PDE, in the variable x, reveals

$$\begin{cases} -4\pi^2|\xi|^2\hat{u} + \hat{u}_{yy} = 0 & \text{in } D, \\ \hat{u}_y = \hat{f} & \text{on } \partial D. \end{cases} \tag{1186}$$

Thus,

$$\hat{u}(\xi,y) = C_1(\xi)\exp(2\pi|\xi|y) + C_2(y)\exp(-2\pi|\xi|y) \quad \text{in } D, \tag{1187}$$

for some functions $C_1(\xi)$ and $C_2(\xi)$. Since $y \geq 0$ and we are interested in solutions that tend to zero as $y \to +\infty$, we take $C_1(\xi) = 0$. Then the boundary condition implies

$$\hat{u}(\xi, y) = -\frac{\hat{f}(\xi)}{2\pi|\xi|} \exp\left(-2\pi|\xi|y\right). \tag{1188}$$

We then take the inverse Fourier transform to find an expression describing $u(x, y)$. Observe

$$\begin{aligned}
u(x,y) &= \int_{-\infty}^{\infty} -\frac{\hat{f}(\xi)}{2\pi|\xi|} \exp\left(-2\pi|\xi|y + 2\pi i x\xi\right)\ \mathrm{d}\xi \\
&= \int_{-\infty}^{0} \frac{\hat{f}(\xi)}{2\pi\xi} \exp\left(2\pi(-y+ix)\xi\right)\ \mathrm{d}\xi + \int_{-\infty}^{0} -\frac{\hat{f}(\xi)}{2\pi\xi} \exp\left(2\pi(y+ix)\xi\right)\ \mathrm{d}\xi \\
&= \int_{-\infty}^{0} \frac{\hat{f}(\xi)}{2\pi\xi} \cdot \frac{1}{2\pi(y+ix)} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\exp\left(2\pi(y+ix)\xi\right)\right]\ \mathrm{d}\xi \\
&+ \int_{0}^{\infty} -\frac{\hat{f}(\xi)}{2\pi\xi} \cdot \frac{1}{2\pi(-y+ix)} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\exp\left(2\pi(-y+ix)\xi\right)\right]\ \mathrm{d}\xi \\
&= \left[\frac{\hat{f}(\xi)}{2\pi\xi} \cdot \frac{\exp(2\pi(y+ix)\xi)}{2\pi(y+ix)\xi}\right]_{-\infty}^{0} - \frac{1}{2\pi(y+ix)} \int_{-\infty}^{0} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\hat{f}(\xi)}{2\pi\xi}\right] \exp\left(2\pi(y+ix)\xi\right)\ \mathrm{d}x \\
&+ \left[-\frac{\hat{f}(\xi)}{2\pi\xi} \cdot \frac{\exp(2\pi(-y+ix)\xi)}{2\pi(-y+ix)\xi}\right]_{-\infty}^{0} - \frac{1}{2\pi(-y+ix)} \int_{0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[-\frac{\hat{f}(\xi)}{2\pi\xi}\right] \exp\left(2\pi(-y+ix)\xi\right)\ \mathrm{d}x.
\end{aligned} \tag{1189}$$

The other condition on f implies $\hat{f}(0) = \hat{f}'(0) = 0$. With the fact that $\hat{f} \to 0$ as $|\xi| \to \infty$,

$$\left[-\frac{\hat{f}(\xi)}{2\pi\xi} \exp(2\pi i\xi(-y+ix))\right]_{-\infty}^{0} = \lim_{\xi\to 0^-} -\frac{\hat{f}(\xi)}{2\pi\xi} \exp(2\pi i(-y+ix)) = 0. \tag{1190}$$

Similarly, the other boundary condition terms vanish. Letting α_1 and α_2 be the remaining integrals following the final equality in (1189), which are finite since the inverse Fourier transform exists, we see

$$u(x,y) = \frac{\alpha_1}{2\pi(y+ix)} + \frac{\alpha_2}{2\pi(-y+ix)} \tag{1191}$$

which implies

$$|u(x,y)| \leq \frac{|\alpha_1| + |\alpha_2|}{2\pi\sqrt{x^2+y^2}} \to 0 \quad \text{as } x^2+y^2 \to \infty. \tag{1192}$$

Therefore, the choice of u in (1191) is a solution to the PDE that satisfies the required condition, and the result follows.

□

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S13.2. Let u be a solution to

$$\begin{cases} u_{tt} - u_{xx} + cu = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = \varphi & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t = \psi & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{1193}$$

Assume $c(x,t)$, $\varphi(x)$, and $\psi(x)$ are smooth functions equal to zero for $|x| > R$. Prove that $u(x,t) = 0$ for $|x| > R + t$.

Solution:

Let $t_0 > 0$, and then fix any $x_0 \in \mathbb{R} - B(0, R + t_0)$. It suffices[43] to show $u(x_0, t_0) = 0$. Define the energy $e : [0, t_0) \to \mathbb{R}$ via

$$e(t) := \frac{1}{2} \int_{S(t)} u_t^2 + u_x^2 \, \mathrm{d}x, \tag{1194}$$

where for each $t \in [0, t_0)$ we define $S(t) := B(0, R + (t_0 - t))$. By our hypothesis, $\varphi = \psi = 0$ in $S(0)$, which implies $e(0) = 0$. Differentiating in time further reveals

$$\begin{aligned} \dot{e}(t) &= \int_{S(t)} u_t u_{tt} + u_x u_{xt} \, \mathrm{d}x + \int_{\partial S(t)} \frac{1}{2} \left(u_t^2 + u_x^2\right) v \cdot n \, \mathrm{d}\sigma \\ &= \int_{S(t)} u_t \left(u_{tt} - u_{xx}\right) \, \mathrm{d}x + \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} + \frac{1}{2} \left(u_t^2 + u_x^2\right) v \cdot n \, \mathrm{d}\sigma \\ &= \underbrace{\int_{S(t)} -cu \, \mathrm{d}x}_{=0} + \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} - \frac{1}{2} \left(u_t^2 + u_x^2\right) \, \mathrm{d}\sigma, \end{aligned} \tag{1195}$$

where $v = -n$ is the Eulerian velocity of the boundary, n is the outward normal along the boundary $\partial S(t)$, and the first integral on the final line vanishes since $S(t)$ does not intersect with the support of c. For all $a, b \in \mathbb{R}$ we have

$$0 \le (a-b)^2 = a^2 + b^2 - 2ab \quad \Longrightarrow \quad ab \le \frac{1}{2}(a^2 + b^2). \tag{1196}$$

[43] From our experiences with grading on the ADE exam, we assume that the graders would not be happy if we merely applied Duhamel's principle with d'Alembbert's formula to assert a formula for u, even though this seems perfectly reasonable to do.

Combined with the Cauchy-Schwarz inequality, this implies

$$\left|u_t \frac{\partial u}{\partial n}\right| = |u_t|\,|u_x \cdot n| \leq |u_t||u_x| \leq \frac{1}{2}(u_t^2 + u_x^2), \tag{1197}$$

and so

$$\dot{e}(t) = \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} - \frac{1}{2}\left(u_t^2 + u_x^2\right)\ \mathrm{d}\sigma \leq \int_{\partial S(t)} \frac{1}{2}(u_t^2 + u_x^2) - \frac{1}{2}(u_t^2 + u_x^2)\ \mathrm{d}\sigma = 0. \tag{1198}$$

This implies $e(t)$ is nonincreasing. Since the integrand in the definition of $e(t)$ is nonnegative, we see that $0 \leq e(t) \leq e(0) = 0$ for all $t \in [0, t_0)$. Consequently, $u_t = u_x = 0$ in the cone

$$K(x_0, t_0) := \{(x,t) : t \in [0, t_0),\ x \in B(x_0, t - t_0)\}, \tag{1199}$$

and so u is constant therein. Combined with the fact $u = 0$ on $S(0) \times \{t = 0\}$, we see $u = 0$ everywhere in $K(x_0, t_0)$. In particular, this shows $u(x_0, t) = 0$ for all $t \in [0, t_0)$. By the continuity of the solution u, it follows that

$$u(x_0, t_0) = \lim_{t \to t_0^-} u(x_0, t) = \lim_{t \to t_0^-} 0 = 0, \tag{1200}$$

as desired. □

S13.4.

a) Let

$$\Delta u - qu = 0 \quad \text{in } \mathbb{R}^n, \tag{1201}$$

where $q(x) \geq 0$ is bounded. suppose $u(x) \to 0$ uniformly as $|x| \to \infty$. Prove that $u \equiv 0$.

b) Find a nontrivial solution of

$$\Delta u + u = 0 \quad \text{in } \mathbb{R}^3 \tag{1202}$$

such that $u(x) \to 0$ as $|x| \to \infty$.

Hint: Consider radial solutions.

Solution:

a) Let $\varepsilon > 0$ be given. Then choose $R > 0$ sufficiently large that $|u| \leq \varepsilon$ for $|x| \geq R$, which is possibly by our hypothesis. Now let $\delta > 0$ and define

$$w := u + \delta\left(|x|^2 - 2R^2\right). \tag{1203}$$

Since w is continuous and $\overline{B(0,R)}$ is compact, w attains its supremum over $\overline{B(0,R)}$. If this occurs at an interior point $\overline{x} \in B(0,R)$, then

$$0 \geq \Delta w(\overline{x}) = \Delta u(\overline{x}) + 2n\delta = q(\overline{x})u(\overline{x}) + 2n\delta = q(\overline{x})w(\overline{x}) + \underbrace{2n\delta + q(\overline{x})\left(2R^2 - |\overline{x}|^2\right)}_{>0}, \tag{1204}$$

which implies

$$q(\overline{x})w(\overline{x}) \leq -2n\delta < 0 \quad \Longrightarrow \quad w(\overline{x}) < 0, \tag{1205}$$

where the implication holds since q is nonnegative. Alternatively, if the supremum is obtained at $\overline{x} \in \partial B(0,R)$, then

$$w(\overline{x}) = u(\overline{x}) + \delta\left(|\overline{x}|^2 - 2R^2\right) \leq \varepsilon + \delta\left(R^2 - 2R^2\right) = \varepsilon - \delta R^2. \tag{1206}$$

In either case, we see

$$\max_{B(0,R)} u - 2\delta R^2 \leq \max_{B(0,R)} u + \delta(|x|^2 - 2R^2) = \max_{B(0,R)} w \leq \max\{0, \varepsilon - \delta R^2\}, \tag{1207}$$

and so

$$\max_{B(0,R)} u \leq \max\{2\delta R^2, \varepsilon + \delta R^2\}. \tag{1208}$$

Since δ was arbitrarily chosen, we may let $\delta \to 0^+$ to deduce

$$\max_{B(0,R)} u \leq \varepsilon. \tag{1209}$$

Consequently, with our initial choice of R, we see

$$\sup_{\mathbb{R}^n} u \leq \max\{\sup_{\mathbb{R}^n - \overline{B(0,R)}} u, \sup_{\overline{B(0,R)}} u\} \leq \max\{\varepsilon, \varepsilon\} = \varepsilon. \tag{1210}$$

Since ε was arbitrary, we may let $\varepsilon \to 0^+$ to see $u \leq 0$ in $\mathbb{R}^n$. By an analogous argument with inifmums and a choice of w with the sign of the second term flipped, we deduce $u \geq 0$ in $\mathbb{R}^n$, from which the result follows.

b) Assume u is radial so that $u(x) = v(r)$, where $r = |x|$. For each index i,

$$v_{x_i} = v'(r) r_{x_i} = v'(r) \cdot \frac{x_i}{r} \quad \Longrightarrow \quad v_{x_i x_i} = v''(r) \cdot \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right), \tag{1211}$$

which implies

$$\Delta v = \sum_{i=1}^{3} v''(r) \cdot \frac{x_i^2}{r^2} + v'(r) \cdot \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) = v''(r) + \frac{2v'(r)}{r}. \tag{1212}$$

Therefore,

$$0 = \Delta u + u = v'' + \frac{2v'}{r} + v, \tag{1213}$$

and so

$$0 = rv'' + 2v' + rv = (rv' - v)' + 2v' + rv = (rv' + v)' + rv = (rv)'' + rv = w'' + w, \tag{1214}$$

where we take $w := rv$. Since the solution of the ODE for w is well-known to be of the form

$$w(r) = c_1 \cos(r) + c_2 \sin(r), \tag{1215}$$

we see

$$v(r) = \frac{c_1 \cos(r)}{r} + \frac{c_2 \sin(r)}{r}. \tag{1216}$$

Since we want v to be defined for all $r \in [0, \infty)$, we take $c_1 = 0$ and $c_2 = 1$, noting $\sin(r)/r \longrightarrow 1$ as $r \longrightarrow 0$. Thus,

$$u(x) := \begin{cases} \frac{\sin(|x|)}{|x|}, & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \tag{1217}$$

forms a nontrivial solution to the PDE a.e. in $\mathbb{R}^3$ that vanishes as $|x| \longrightarrow \infty$.

$\square$

S13.3a. Let D be a domain in $\mathbb{R}^n$ with smooth boundaries.[44] Let u be a H^1 solution of

$$\begin{cases} -\Delta u + u^{1/3} = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1218}$$

Prove $u \equiv 0$ in D.

Solution:

Let $\varepsilon > 0$ be given. Since $u \in H_0^1(D)$ and $H_0^1(D)$ is the closure of all $C_c^\infty(D)$ functions in $H^1(D)$, there exists $v \in C_c^\infty(D) \cap H^1(D)$ such that

$$\|u - v\|_{H^1(D)} < \varepsilon. \tag{1219}$$

Let us momentarily assume $u \in C_c^\infty(D)$. Then we see

$$\int_D f \tag{1220}$$

□

[44] We omit S13.3b since it appears to have been in error. See Peter and Zane's notes on this.

S13.5. Consider the autonomous ODE

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_1 + (1 - y_1^2 - y_2^2)y_2. \tag{1221}$$

Show that any solution $x(t) = (x_1(t), x_2(t))$ of the above system converges to $(\sin(t + c), \cos(t + c))$ as $t \to \infty$ for some constant c.

Solution:

The only fixed point of the system is at the origin. So, if $x(0)$ is at the origin, then $x(t)$ remains at the origin for all time. Now assume $x(0)$ is not the origin. We proceed by making the change of variables to a variant of polar coordinates to write

$$x = (x_1, x_2) = (r \sin\theta, r\cos\theta). \tag{1222}$$

Differentiating reveals

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r^2\right] = 2r\dot{r} = 2r \cdot \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} = 2\left(x_1(-x_2) + x_2\left[-x_1 + (1-r^2)x_2\right]\right) = 2(1-r^2)x_2^2, \tag{1223}$$

and so

$$\dot{r} = \frac{1-r^2}{r} \cdot x_2^2 = r(1-r^2)\cos^2\theta. \tag{1224}$$

Since $r(0) > 0$ and (1224) implies $\dot{r} \geq 0$ for $r \in (0,1)$, we see $r > 0$ for all time. Then observe

$$r\cos\theta = x_2 = \dot{x}_1 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \qquad \Longrightarrow \qquad r(1-\dot{\theta})\cos\theta = \dot{r}\sin\theta \tag{1225}$$

Rearranging reveals

$$\dot{\theta} = 1 - (1-r^2)\sin\theta\cos\theta = 1 - \frac{1-r^2}{2} \cdot \sin 2\theta. \tag{1226}$$

Define $\varphi(t) := |r-1|^2$ and note

$$\dot{\varphi} = 2(r-1)\dot{r} = 2(r-1)r(1-r^2)\sin^2\theta = -2(1-r)^2(1+r)r\sin^2\theta = -2\varphi(1+r)r\sin^2\theta \leq 0. \tag{1227}$$

And, $\varphi \geq 0$. The monotone convergence theorem then asserts $\varphi(t)$ converges to a limit $\varphi^\star = |1 - r^\star|^2$.

Consequently, $\dot{\varphi} \longrightarrow 0$, and so

$$\begin{aligned} 0 &= \liminf_{t\to\infty} \dot{\varphi} \\ &= \liminf_{t\to\infty} -2\varphi(1+r)\sin^2\theta \\ &= -2\cdot\limsup_{t\to\infty}\varphi(1+r)\sin^2\theta \\ &= -2\varphi^\star(1+r^\star)\cdot\limsup_{t\to\infty}\sin^2\theta. \end{aligned} \tag{1228}$$

By way of contradiction, suppose the limit on the final line above is nonzero. This would require there to exist $k\in\mathbb{Z}$ such that

$$\lim_{t\to\infty}\theta = k\pi, \tag{1229}$$

which would then imply $\dot{\theta}\longrightarrow 0$. However, if (1229) holds, then (1226) implies

$$\lim_{t\to\infty}\dot{\theta} = \lim_{t\to\infty} 1 - \frac{1-r^2}{2}\cdot\sin 2\theta = 1-0, \tag{1230}$$

a contradiction. Therefore, the limit in the final line of (1228) must be positive, making the equation hold precisely when

$$0 = \varphi^\star = |1-r^\star|^2 \quad\Longleftrightarrow\quad r^\star = 1. \tag{1231}$$

Thus, as $t\longrightarrow\infty$, we see $r\longrightarrow 1$ and $\dot{\theta}\longrightarrow 1$, therey implying $\theta\longrightarrow c+t$ for some $c\in\mathbb{R}$, from which the result follows. □

S13.6. Draw the phase space for the competing species system

$$\dot{x} = x(2 - x - y), \quad \dot{y} = y(3 - 2x - y). \tag{1232}$$

How likely is it that both species survive?

Solution:

First observe[45] the equilibrium points are $(0,0)$, $(0,3)$, $(2,0)$, and $(\tilde{x}, \tilde{y})$ such that

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \tag{1233}$$

i.e., $(\tilde{x}, \tilde{y}) = (1,1)$. The Jacobian for the system is given by

$$J(x,y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2x - 2y \end{pmatrix}. \tag{1234}$$

This implies

$$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \tag{1235}$$

which has eigenvalues $\lambda = 2, 3$, and so $(0,0)$ forms an unstable node. Then note

$$J(0,3) = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}, \tag{1236}$$

which has eigenvalues $\lambda = -1, -3$, and so $(0,3)$ corresponds to a sink. Similarly,

$$J(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}, \tag{1237}$$

[45] This problem is the example at the beginning of Section 6.4 (with the roles of x and y swapped) in Strogatz's text *Nonlinear Dynamics and Chaos* and it follows closely to Problems 6.4.1, 6.4.2, and 6.4.3.

which has eigenvalues $\lambda = -2, -1$, and so $(2, 0)$ corresponds to a sink. Lastly,[46]

$$J(1,1) = \begin{pmatrix} (2-\tilde{x}-\tilde{x})-\tilde{y} & -\tilde{x} \\ -2\tilde{y} & (3-2\tilde{x}-\tilde{y})-\tilde{y} \end{pmatrix} = \begin{pmatrix} -\tilde{y} & -\tilde{x} \\ -2\tilde{y} & -\tilde{y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}, \tag{1238}$$

which has the eigenvalues $\lambda = -1 \pm \sqrt{2}$, and so $(1, 1)$ is a saddle point. Along the null-clines $x = 0$ and $y = 2 - x$ we have $\dot{x} = 0$, and along the null-clines $y = 0$ and $y = 3 - 2x$ we have $\dot{y} = 0$. With these facts, we obtain the following phase plane diagram.

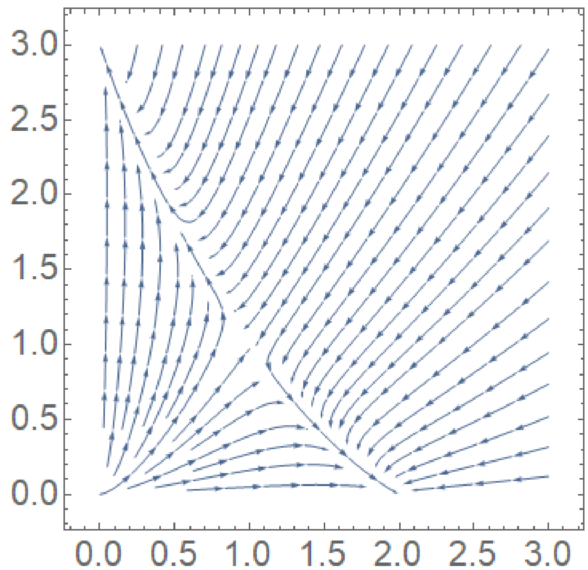

Figure 29: ODE phase plane for S13.6.

Based on this diagram and the fact that $(1, 1)$ is a saddle point and the only equilibrium point as which both species survive, the likelihood that both species survive is negligible. □

[46]We write this out in this odd way to shed long on situations where we may not know $\tilde{x}$ and $\tilde{y}$ explicitly, as occurs on some of the more recent exams.

S13.7. Let Ω be a connected, bounded domain in $\mathbb{R}^n$ with smooth boundary, and let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth. Show there is at most one smooth solution of the PDE

$$\begin{cases} u_t - \Delta u + |\nabla u|^2 = 0 & \text{in } \Omega \times (0,\infty), \\ u = g & \text{on } \partial\Omega \times (0,\infty), \\ u = f & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{1239}$$

Solution:

Let u and v be two solutions to the PDE. Define $w := u - v$. It suffices to show $w = 0$ in $\overline{\Omega} \times [0,\infty)$. Using the definition of w, we see it satisfies

$$\begin{cases} w_t - \Delta w = |Dv|^2 - |Du|^2 & \text{in } \Omega \times (0,\infty), \\ w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ w = 0 & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{1240}$$

Fix $T > 0$ and define $\Omega_T := \Omega \times (0, T]$. Let Γ_T be the parabolic boundary of Ω_T. Then fix $\delta > 0$ and define $\tilde{w} = w + \delta e^t$. This implies

$$\begin{cases} \tilde{w}_t - \Delta \tilde{w} = |Dv|^2 - |Du|^2 + \delta e^t & \text{in } \Omega_T, \\ \tilde{w} = \delta e^t & \text{on } \Gamma_T. \end{cases} \tag{1241}$$

Now let $\varepsilon > 0$. Because $\overline{\Omega}_T$ is compact and $\tilde{w}$ is smooth, $\tilde{w}$ attains its infimum and supremum over $\overline{\Omega}_T$. By way of contradiction, suppose

$$\inf_{\overline{\Omega}_T} \tilde{w} \leq -\varepsilon. \tag{1242}$$

Since

$$\tilde{w} = \delta e^t \geq \delta > 0 > -\varepsilon \quad \text{on } \Gamma_T, \tag{1243}$$

it follows that any minimizer $(\overline{x}, \overline{t})$ of $\tilde{w}$ over $\overline{\Omega}_T$ is in Ω_T. Consequently, $\tilde{w}_t(\overline{x}, \overline{t}) \leq 0$ and, because $\overline{x}$ is a local minimizer of $\tilde{w}(\cdot, \overline{t})$, we see $\Delta \tilde{w}(\overline{x}, \overline{t}) \geq 0$ and

$$0 = D\tilde{w}(\overline{x}, \overline{t}) = Du(\overline{x}, \overline{t}) - Dv(\overline{x}, \overline{t}) \quad \Longrightarrow \quad Du(\overline{x}, \overline{t}) = Dv(\overline{x}, \overline{t}). \tag{1244}$$

Thus, at $(\overline{x}, \overline{t})$,

$$0 \geq \tilde{w}_t - \Delta \overline{w} = \underbrace{|Dv|^2 - |Du|^2}_{=0} + \delta e^{\overline{t}} = \delta e^{\overline{t}} > 0, \tag{1245}$$

which implies $0 > 0$, a contradiction. Therefore, the assumption that (1242) holds was false, i.e.,

$$\inf_{\overline{\Omega}_T} \tilde{w} > -\varepsilon \tag{1246}$$

holds. Because this holds for arbitrary $\varepsilon > 0$, we may let $\varepsilon \longrightarrow 0^+$ to deduce

$$\tilde{w} \geq 0 \quad \text{in } \overline{\Omega}_T. \tag{1247}$$

This implies

$$w = \tilde{w} - \delta e^t \geq -\delta e^t \geq -\delta e^T \quad \text{in } \overline{\Omega}_T. \tag{1248}$$

Similarly, because (1248) holds for arbitrary $\delta > 0$, we may let $\delta \longrightarrow 0^+$ to find

$$w \geq 0 \quad \text{in } \overline{\Omega}_T. \tag{1249}$$

Since (1249) holds for arbitrary $T > 0$, we may let $T \longrightarrow \infty$ to discover

$$w \geq 0 \quad \text{in } \overline{\Omega} \times [0, \infty). \tag{1250}$$

Because u and v were arbitrary, we may repeat an analogous argument with their roles swapped to deduce $w \leq 0$ in $\overline{\Omega} \times [0, \infty)$. Whence w is identically zero and the result follows. □

S13.8. Show that[47]

$$u(x,t) := \begin{cases} -\frac{2}{3}\left(t+\sqrt{3x+t^2}\right) & \text{if } 4x+t^2>0, \\ 0 & \text{otherwise} \end{cases} \tag{1251}$$

is an entropy solution of the equation $u_t + uu_x = 0$.

Solution:

Let $f(u) := \frac{1}{2}u^2$ so that the PDE may be expressed as the conservation law

$$u_t + f(u)_x = 0. \tag{1252}$$

We proceed in the following manner. Let C be the curve in $\mathbb{R}\times(0,\infty)$ parameterized by $(s(t),t) = (-t^2/4, t)$. We show u satisfies the PDE to the left and right of C. Then we must verify the Rankine-Hugoniot (RH) condition holds along C and that the entropy conditions

$$f'(u_\ell) > \dot{s}(t) > f'(u_r) \quad \text{along } C \tag{1253}$$

hold.

To the left of C, i.e., where $x < -t^2/4$, we have $u = 0$. In this region, it immediately follows that

$$u_t + f(u)_x = 0 + f(0) = 0. \tag{1254}$$

To the right of C, we see

$$u_t = -\frac{2}{3}\left(1+\frac{1}{2}(3x+t^2)^{-1/2}\cdot 2t\right) = -\frac{2}{3}\left(1+t(3x+t^2)^{-1/2}\right), \tag{1255}$$

and

$$u_x = -\frac{2}{3}\cdot\frac{1}{2}\cdot\left(3x+t^2\right)^{-1/2}\cdot 3 = -\left(3x+t^2\right)^{-1/2}, \tag{1256}$$

[47] This is Evans Problem 3.17.

which implies

$$\begin{aligned} u_t + f(u)_x &= -\frac{2}{3}\left(1 + t(3x+t^2)^{-1/2}\right) + \frac{2}{3}\left(t + (3x+t^2)^{1/2}\right)\left(3x+t^2\right)^{-1/2} \\ &= -\frac{2}{3}\left(1 + t(3x+t^2)^{-1/2}\right) + \frac{2}{3}\left(t(3x+t^2)^{-1/2} + 1\right) \\ &= 0, \end{aligned} \tag{1257}$$

as desired.

We now verify the RH condition holds. Since the limiting value of u from the right of C, denote u_r, is

$$\begin{aligned} u_r &= \lim_{x\to(-t^2/4)^+} u(x,t) \\ &= \lim_{x\to(-t^2/4)^+} -\frac{2}{3}\left(t + (3x+t^2)^{1/2}\right) \\ &= -\frac{2}{3}\left(t + \left(3\left(-\frac{t^2}{4}\right) + t^2\right)^{1/2}\right) \\ &= -\frac{2}{3}\left(t + \frac{t}{2}\right) \\ &= -t, \end{aligned} \tag{1258}$$

and the limiting value of u from the left is $u_\ell = 0$, we see

$$\frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}\cdot u_\ell^2 - \frac{1}{2}\cdot 0^2}{u_\ell - 0} = \frac{1}{2}u_\ell = -\frac{t}{2} = \dot{s}(t), \tag{1259}$$

i.e., the RH condition holds. Lastly, the entropy conditions hold since

$$f'(u_\ell) = u_\ell = 0 > -\frac{t}{2} = \dot{s}(t) > -t = u_r = f'(u_r). \tag{1260}$$

This completes the proof. □

2012 Fall

F12.1. Show the PDE

$$\begin{cases} -\Delta u = -1 & \text{for } |x| < 1,\ |y| < 1, \\ u = 0 & \text{for } |x| = 1, \\ u_x - u_y = 0 & \text{for } |y| = 1, \end{cases} \tag{1261}$$

has at most one solution[48] in $|x| < 1$, $|y| < 1$.

Solution:

Define $\Omega := (-1,1) \times (1,1)$, $\Gamma_1 := [-1,1] \times \{-1,1\}$ and $\Gamma_2 := \{-1,1\} \cup (-1,1)$. Then $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let u and v be two solutions to the PDE. Define $w := u - v$. It suffices to show w is identically zero. Observe

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_2, \\ w_y = w_x & \text{on } \Gamma_1. \end{cases} \tag{1262}$$

We first show w attains its supremum along the boundary $\partial\Omega$. Let $\delta > 0$ and set $\tilde{w} := w + \delta|z|^2$, where $z = (x, y)$. Because $\tilde{w}$ is smooth and $\overline{\Omega}$ is compact, $\tilde{w}$ attains it supremum over $\overline{\Omega}$. By way of contradiction, suppose the supremum is obtained at a point $z^\star \in \Omega$. Then

$$0 \geq \Delta\tilde{w}(z^\star) = \Delta w(z^\star) + 2n\delta = 2n\delta > 0, \tag{1263}$$

which implies $0 > 0$, a contradiction. Thus,

$$\sup_{\overline{\Omega}} \tilde{w} = \sup_{\partial\Omega} \tilde{w}. \tag{1264}$$

Observe

$$\sup_{\overline{\Omega}} w \leq \sup_{\overline{\Omega}} w + \delta|z|^2 = \sup_{\overline{\Omega}} \tilde{w} = \sup_{\partial\Omega} \tilde{w} = \sup_{\partial\Omega} w + \delta|z|^2 \leq \sup_{\partial\Omega} w + \delta\sqrt{2}. \tag{1265}$$

Combined with the fact $\partial\Omega \subseteq \Omega$, we see

$$\sup_{\partial\Omega} w \leq \sup_{\overline{\Omega}} w \leq \sup_{\partial\Omega} w + \delta\sqrt{2}. \tag{1266}$$

[48] We presume a typo was made in the original prompt and have attempted to correct it.

Because this holds for arbitrary $\delta > 0$, we may let $\delta \to 0^+$ to deduce

$$\sup_{\partial\Omega} w = \sup_{\overline{\Omega}} w. \tag{1267}$$

Multiplying our PDE by w and integrating reveals

$$0 = \int_\Omega w\Delta w \text{ d}z = -\int_\Omega |Dw|^2 \text{ d}z + \int_{\partial\Omega} w\frac{\partial w}{\partial n} \text{ d}\sigma. \tag{1268}$$

Then

$$\begin{aligned}
\int_{\Gamma_1} w\frac{\partial w}{\partial n} \text{ d}\sigma &= \int_{\Gamma_1} ww_y \text{ d}\sigma \\
&= \int_{\Gamma_1} ww_x \text{ d}\sigma \\
&= \int_{\Gamma_1} \frac{\text{d}}{\text{d}x}\left[\frac{w^2}{2}\right] \text{ d}\sigma \\
&= \int_{-1}^1 \frac{\text{d}}{\text{d}x}\left[\frac{w^2(x,1)}{2} + \frac{w^2(x,-1)}{2}\right] \text{ d}x \\
&= \frac{w(1,1)^2 + w(1,-1)}{2} - \frac{w(-1,1)^2 + w(-1,-1)^2}{2} \\
&= 0,
\end{aligned} \tag{1269}$$

where the final equality holds since $w = 0$ on Γ_2. The fact $w = 0$ on Γ_2 also implies

$$\int_{\Gamma_2} w\frac{\partial w}{\partial n} \text{ d}\sigma. = 0 \tag{1270}$$

Thus, compiling our results reveals

$$\int_\Omega |Dw|^2 \text{ d}z = \int_{\partial\Omega} w\frac{\partial w}{\partial n} \text{ d}\sigma = \int_{\Gamma_1} w\frac{\partial w}{\partial n} \text{ d}\sigma + \int_{\Gamma_2} w\frac{\partial w}{\partial n} \text{ d}\sigma = 0. \tag{1271}$$

Hence $Dw = 0$ in Ω. Together with the fact Ω is connected, this implies w is constant in Ω. And, because $w = 0$ on Γ_2, it follows that $w = 0$ in $\overline{\Omega}$. This completes the proof. □

F12.2. Consider the equation

$$\rho_t - \Delta(\rho^2) - \nabla \cdot (2x\rho) = 0 \quad \text{in } \mathbb{R}^2 \times [0,\infty), \tag{1272}$$

where the initial data $\rho_0(x) \geq 0$ is compactly supported and $\int \rho_0 = 1$. Let us assume $\rho(\cdot,t)$ stays nonnegative and compactly supported for all times $t > 0$. Using formal calculations, show the following.

a) $\int \rho(\cdot,t)\ \mathrm{d}x = 1$ for all $t > 0$.

b) The energy

$$\int \rho^2 + \rho|x|^2 + C\rho\ \mathrm{d}x \tag{1273}$$

decreases in time for any C.

c) Using a) and b), show that ρ converges to $(C_0 - |x|^2/2)_+$ for an appropriate C_0.

Solution:

a) Define $e : \mathbb{R} \to \mathbb{R}$ by

$$e(t) := \int_{\mathbb{R}^2} \rho(x,t)\ \mathrm{d}x. \tag{1274}$$

We are given that $e(0) = 0$. Then observe for $t \in (0,\infty)$

$$\begin{aligned} \dot{e}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \rho(x,t)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \rho_t\ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \Delta(\rho^2) + \nabla \cdot (2x\rho)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} (2\rho\Delta\rho + 2|D\rho|^2) + \nabla \cdot (2x\rho)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} (-2|D\rho|^2 + 2|D\rho|^2) + (2n\rho + 2x \cdot D\rho)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} 0 + (2n\rho - 2n\rho)\ \mathrm{d}x \\ &= 0. \end{aligned} \tag{1275}$$

The second equality holds by hypothesis, the fourth through integration by parts and the fact ρ

has compact support, and the fifth through integration by parts on the last term.

b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := \int_{\mathbb{R}^2} \rho(x,t)^2 + \rho(x,t)|x|^2 + C\rho(x,t) \ \mathrm{d}x = \int_{\mathbb{R}^2} \rho^2 + \rho|x|^2 \ \mathrm{d}x + Ce(t). \tag{1276}$$

Then

$$\begin{aligned} \dot{f}(t) &= \int_{\mathbb{R}^2} \partial_t \left(\rho^2 + \rho|x|^2\right) \ \mathrm{d}x + C\dot{e}(t) \\ &= \int_{\mathbb{R}^2} \rho_t \left(2\rho + |x|^2\right) \ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left(\Delta(\rho^2) + \nabla \cdot (2x\rho)\right) \left(2\rho + |x|^2\right) \ \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \nabla \cdot (2\rho\nabla\rho + 2x\rho)) \left(2\rho + |x|^2\right) \ \mathrm{d}x \\ &= -\int_{\mathbb{R}^2} 2\rho(\nabla\rho + x) \cdot (2\nabla\rho + 2x) \ \mathrm{d}x \\ &= -4\int_{\mathbb{R}^2} \rho|D\rho + x|^2 \ \mathrm{d}x \\ &\leq 0. \end{aligned} \tag{1277}$$

The third line holds by the PDE ρ solves and the fifth line from integration by parts and the fact ρ has compact support. Since $\dot{f}(t) \leq 0$, we conclude f is nonincreasing in time.

c) Note $f(t) \geq 0$ for all $t \in (0, \infty)$ since we assume $\rho \geq 0$. Since f is monotonically decreasing and is bounded below as t increases, it follows from the monotone convergence theorem that $\lim_{t\to\infty} f(t)$ exists. Consequently,

$$0 = \lim_{t\to\infty} \dot{f}(t) = -4 \lim_{t\to\infty} \int_{\mathbb{R}^2} \rho|D\rho + x|^2 \ \mathrm{d}x. \tag{1278}$$

This implies either ρ approaches the zero function or $|D\rho + x|$ approaches the zero function. However, the fact $e(t) = 1$ for all $t \in [0, \infty)$ implies ρ cannot go to zero everywhere. Let ρ^∞ denote the limit of ρ. Whence, wherever $\rho^\infty \neq 0$,

$$\lim_{t\to\infty} |D\rho + x| = 0 \quad \Longrightarrow \quad D\rho^\infty = \lim_{t\to\infty} D\rho = -x \quad \Longrightarrow \quad \rho^\infty = \lim_{t\to\infty} \rho = C - \frac{|x|^2}{2}. \tag{1279}$$

Because we assume ρ is nonnegative (which follows from the nonnegativity of ρ for all times), it follows that

$$\rho^\infty = \left(C - \frac{|x|^2}{2}\right)_+, \tag{1280}$$

where C is yet to be determined. Observe

$$\begin{aligned} 1 = \int_{\mathbb{R}^2} \rho^\infty \ \mathrm{d}x &= \int_{B(0,\sqrt{2C})} C - \frac{|x|^2}{2} \ \mathrm{d}x \\ &= \int_0^{\sqrt{2C}} \int_0^{2\pi} \left(C - \frac{r^2}{2}\right) \ r\mathrm{d}\phi\mathrm{d}r \\ &= \pi \int_0^{\sqrt{2C}} 2Cr - r^3 \ \mathrm{d}r \\ &= \pi \left[Cr^2 - \frac{r^4}{4}\right]_0^{\sqrt{2C}} \\ &= \pi \left[2C^2 - \frac{4C^2}{4}\right] \\ &= \pi C^2. \end{aligned} \tag{1281}$$

This implies $C = 1/\sqrt{\pi}$ and

$$\rho^\infty = \left(\frac{1}{\sqrt{\pi}} - \frac{|x|^2}{2}\right)_+. \tag{1282}$$

□

F12.3. Consider the ODE

$$u'' + f(u) + \lambda u' = 0 \tag{1283}$$

for $u \in C^2(\mathbb{R})$, $f \in C^\infty(\mathbb{R})$, and $\lambda > 0$. Prove no periodic solutions exist other than a stationary equilibrium solution.

Solution:

We first rewrite the ODE as a system via

$$\dot{u} = y, \quad \dot{y} = -f(u) - \lambda y, \tag{1284}$$

where the argument is taken to be time t. For notational convenience, let $z := (u, y)$. Dulac's Criterion asserts no closed orbits exist provided there exists a real-valued function $g(z)$ such that $\nabla \cdot (g\dot{z})$ is single-signed everywhere. Then observe, for each smooth function $g : \mathbb{R}^2 \to \mathbb{R}$,

$$\nabla \cdot (g\dot{z}) = Dg \cdot \dot{z} + g(\nabla \cdot \dot{z}) = Dg \cdot \dot{z} - \lambda g. \tag{1285}$$

If g is a constant function, e.g., $g(z) = 1$, then we see

$$\nabla \cdot (g\dot{z}) = Dg \cdot \dot{z} - \lambda g = 0 - \lambda < 0. \tag{1286}$$

Dulac's Criterion thus asserts no closed orbits exist. This implies the only periodic solutions are fixed points, which are of the form $(\overline{u}, 0)$, where $f(\overline{u}) = 0$. At such a point, we see $\dot{\overline{u}} = 0$, and so the only periodic solutions to the original ODE are constant solutions satisfying $f(u) = 0$. $\square$

Remark: This problem is something that may be derived from Section 7.2 of Strogatz's *Nonlinear Dynamics and Chaos.* There the author also comments that candidates for g that usually work are $g = 1$, $g = 1/(x^a y^b)$, $g = e^{ax}$, and $g = e^{ay}$. $\triangle$

F12.4. We say that u is a weak solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = f & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1287}$$

if for all $v \in C_0^\infty(\mathbb{R} \times [0,\infty))$, u satisfies

$$\int_0^\infty \int_{\mathbb{R}} u[v_{tt} - v_{xx}] \ \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} f(x) v_t(x,0) \ \mathrm{d}x - \int_{\mathbb{R}} g(x) v(x,0) \ \mathrm{d}x = 0. \tag{1288}$$

Let $f(x)$ be a piecewise continuous function with a jump at $x = x_0$. Show that $u(x,t) = f(x+t)$ is a weak solution of the wave equation.

Solution:

We assume g is the weak derivative of f. Then note

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty u v_{tt} \ \mathrm{d}x\mathrm{d}t &= \int_{-\infty}^\infty \int_0^\infty f(x+t) v_{tt} \ \mathrm{d}t\mathrm{d}x \\ &= \int_{-\infty}^{x_0} \int_0^\infty f(x+t) v_{tt} \ \mathrm{d}t\mathrm{d}x + \int_{x_0}^\infty \int_0^\infty f(x+t) v_{tt} \ \mathrm{d}x\mathrm{d}t \\ &= \int_{-\infty}^{x_0} \left[\int_0^{x_0 - x} f(x+t) v_{tt} \ \mathrm{d}t + \int_{x_0-x}^\infty f(x+t) v_{tt} \ \mathrm{d}t \right] \mathrm{d}x + \int_{x_0}^\infty \int_0^\infty f(x+t) v_{tt} \ \mathrm{d}t\mathrm{d}x. \end{aligned} \tag{1289}$$

Integrating by parts reveals

$$\int_0^{x_0-x} f(x+t) v_{tt} \ \mathrm{d}t = f(x_0^-) v_t(x, x_0 - x) - f(x) v_t(x,0) - \int_0^{x_0-x} g(x+t) v_t \ \mathrm{d}t, \tag{1290}$$

and

$$\int_{x_0-x}^\infty f(x+t) v_{tt} \ \mathrm{d}t = \underbrace{\left[\lim_{t\to\infty} f(x+t) v_t \right]}_{=0} - f(x_0^+) v_t(x, x_0 - x) - \int_{x_0-x}^\infty g(x+t) v_t \ \mathrm{d}t, \tag{1291}$$

and

$$\int_0^\infty f(x+t) v_{tt} \ \mathrm{d}t = \underbrace{\left[\lim_{t\to\infty} f(x+t) v_t(x,t) \right]}_{=0} - f(x) v_t(x,0) - \int_0^\infty g(x+t) v_{tt} \ \mathrm{d}t, \tag{1292}$$

where we have utilized the compact support of v when evaluating the limits. Combining our results,

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty uv_{tt}\ \mathrm{d}x\mathrm{d}t &= \int_{-\infty}^{x_0}\left[f(x_0^-)v_t(x,x_0-x) - f(x)v_t(x,0) - \int_0^{x_0-x} g(x+t)v_t\ \mathrm{d}t\right]\ \mathrm{d}x \\ &+ \int_{-\infty}^{x_0}\left[-f(x_0^+)v_t(x,x_0-x) - \int_{x_0-x}^\infty g(x+t)v_t\mathrm{d}t\right]\ \mathrm{d}x \\ &+ \int_{x_0}^\infty\left[-f(x)v_t(x,0) - \int_0^\infty g(x+t)v_{tt}\ \mathrm{d}t\right]\ \mathrm{d}x \\ &= \left[f(x_0^-) - f(x_0^+)\right]\int_{-\infty}^{x_0} v_t(x,x_0-x)\ \mathrm{d}x - \int_{-\infty}^\infty f(x)v_t(x,0)\ \mathrm{d}x - \int_{-\infty}^\infty\int_0^\infty g(x+t)v_t\ \mathrm{d}t\mathrm{d}x.\end{aligned} \tag{1293}$$

In similar fashion to above, observe

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty uv_{xx}\ \mathrm{d}x\mathrm{d}t &= \int_0^\infty\left(\int_{-\infty}^{x_0-t} f(x+t)v_{xx}\ \mathrm{d}x + \int_{x_0-t}^\infty f(x+t)v_{xx}\ \mathrm{d}x\right)\mathrm{d}t \\ &= \int_0^\infty\left[f(x_0^-)v_x(x_0-t,t) - \int_{-\infty}^{x_0-t} g(x+t)v_x\ \mathrm{d}x\right]\mathrm{d}t \\ &+ \int_0^\infty\left[-f(x_0^+)v_x(x_0-t,t) - \int_{x_0-t}^\infty g(x+t)v_x\ \mathrm{d}x\right]\mathrm{d}t \\ &= \left[f(x_0^-) - f(x_0^+)\right]\int_0^\infty v_x(x_0-t,t)\ \mathrm{d}t - \int_0^\infty\int_{-\infty}^\infty g(x+t)v_x\ \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{1294}$$

where

$$\int_{-\infty}^{x_0-t} f(x+t)v_{xx}\ \mathrm{d}x = f(x_0^-)v_x(x_0-t,t) - \underbrace{\left[\lim_{x\to-\infty} f(x+t)v_x(x,t)\right]}_{=0} - \int_{-\infty}^{x_0-t} g(x+t)v_x\ \mathrm{d}x, \tag{1295}$$

and

$$\int_{x_0-t}^\infty f(x+t)v_{xx}\ \mathrm{d}x = \underbrace{\left[\lim_{x\to\infty} f(x+t)v_x(x,t)\right]}_{=0} - f(x_0^+)v_x(x_0-t,t) - \int_{x_0-t}^\infty g(x+t)v_x\ \mathrm{d}x. \tag{1296}$$

Due to the compact support of v, repeated integration by parts and a change of variables $x = x_0 - t$ reveals

$$\begin{aligned}
\int_{-\infty}^{x_0} v_t(x, x_0 - x)\ \mathrm{d}x &= \underbrace{[xv_t(x, x_0 - x)]_{x=-\infty}^{x_0}}_{=0} - \int_{-\infty}^{x_0} xv_{xt}(x, x_0 - x)\ \mathrm{d}x \\
&= \int_0^{\infty} (x_0 - t)v_{xt}(x_0 - t, t)\ \mathrm{d}t \\
&= -\int_0^{\infty} tv_{xt}(x_0 - t, t)\ \mathrm{d}t \\
&= \int_0^{\infty} v_x(x_0 - t, t)\ \mathrm{d}t - \underbrace{[t \cdot v_x(x_0 - t, t)]_{t=0}^{\infty}}_{=0} \\
&= \int_0^{\infty} v_x(x_0 - t, t)\ \mathrm{d}t.
\end{aligned} \tag{1297}$$

Therefore, (1293), (1294), and (1297) together imply

$$\int_0^{\infty} \int_{-\infty}^{\infty} u(v_{tt} - v_{xx})\ \mathrm{d}x\mathrm{d}t = -\int_{-\infty}^{\infty} f(x)v_t(x, 0)\ \mathrm{d}x - \int_0^{\infty} \int_{-\infty}^{\infty} g(x + t)(v_t - v_x)\ \mathrm{d}x\mathrm{d}t. \tag{1298}$$

Since

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_0^{\infty} g(x + t)v_t\ \mathrm{d}t\mathrm{d}x &= \int_0^{\infty} \int_{-\infty}^{\infty} g(x + t)v_t\ \mathrm{d}x\mathrm{d}t \\
&= \int_0^{\infty} \left[\underbrace{[f(x + t)v_t]_{x=-\infty}{}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f(x + t)v_{tx}\ \mathrm{d}x \right] \mathrm{d}t \\
&= -\int_0^{\infty} \int_{-\infty}^{\infty} f(x + t)v_{tx}\ \mathrm{d}x\mathrm{d}t \\
&= -\int_{-\infty}^{\infty} \left[\underbrace{\left[\lim_{t\to\infty} f(x + t)v_x(x, t) \right]}_{=0} - f(x)v_x(x, 0) - \int_0^{\infty} g(x + t)v_x\ \mathrm{d}t \right] \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x + t)v_x(x, 0)\ \mathrm{d}x + \int_0^{\infty} \int_{-\infty}^{\infty} g(x + t)v_x\ \mathrm{d}x\mathrm{d}t \\
&= -\int_{-\infty}^{\infty} g(x + t)v(x, 0)\ \mathrm{d}x + \underbrace{[f(x + t)v(x, 0)]_{x=-\infty}{}^{\infty}}_{=0} + \int_0^{\infty} \int_{-\infty}^{\infty} g(x + t)v_x\ \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{1299}$$

we see

$$\int_0^\infty \int_{-\infty}^\infty g(x+t)(v_t - v_x)\ \mathrm{d}x\mathrm{d}t = \int_{-\infty}^\infty g(x+t)v(x,0)\ \mathrm{d}x. \tag{1300}$$

Together (1298) and (1300) yield the desired result. □

F12.8. Give the entropy satisfying weak solution to Burgers' equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{1301}$$

on the periodic domain $[0,4]$ with initial data

$$u_0(x) = \begin{cases} 2 & \text{if } x \in (0,2), \\ 0 & \text{if } x \in (2,4). \end{cases} \tag{1302}$$

Show the slope of the solution is $1/t$ almost everywhere for $t > 2$.

Solution:

We proceed by using the method of characteristics. Let $F(p,q,z,x,t) = q + zp$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the system of characteristic ODE

$$\begin{cases} \dot{x}(s) = F_p = z, & x(0) = x_0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0, & z(0) = u_0(x_0). \end{cases} \tag{1303}$$

This implies $t = s$ and z is constant along characteristics. Thus,

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t z(\tau)\ \mathrm{d}\tau = x_0 + tz(0) = \begin{cases} x_0 + 2t & \text{if } x_0 \in (0,2), \\ x_0 & \text{if } x_0 \in (2,4). \end{cases} \tag{1304}$$

The characteristics collide immediately as $(2,0)$. Applying the Rankine-Huogoniot (RH) condition along the shock curve, parameterized by $(\tilde{x}(t), t)$, reveals

$$\dot{\tilde{x}}(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}\cdot 2^2 - \frac{1}{2}\cdot 0^2}{2 - 0} = 1, \tag{1305}$$

where $f(u) := \frac{u^2}{2}$ is determined from our PDE, writing it as a conservation law $u_t + f(u)_x = 0$. Combined with the fact $\tilde{x}(0) = 2$, we deduce

$$\tilde{x}(t) = t + 2, \quad \text{for all } t \in [0,2], \tag{1306}$$

with the restriction on the domain since $\tilde{x}(2) = 4$, beyond which requires further analysis. We claim, in $(0,4) \times (0,2)$,

$$u(x,t) = \begin{cases} x/t & \text{if } 0 < x < 2t, \\ 2 & \text{if } 2t < x < \tilde{x}(t), \\ 0 & \text{if } \tilde{x}(t) < x < 4, \end{cases} \tag{1307}$$

where the rarefaction wave $g(x/t) = x/t$ satisfies our PDE since

$$0 = u_t + uu_x = g' \cdot -\frac{x}{t^2} + gg' \cdot \frac{1}{t} = \frac{g'}{t}\left[g - \frac{x}{t}\right] = 0, \quad \text{for } 0 < x < 2t. \tag{1308}$$

Note (1307) implies $|u_x| \leq 1/t$ a.e. in $(0,4) \times (0,2)$. So, let $(x,t) \in (0,4) \times (0,2)$ and $z > 0$ such that $(x+2) \in (0,4)$. Then the mean value theorem asserts there exists $\xi_z \in (0,z)$ such that

$$\frac{u(x+z,t) - u(x,t)}{z - 0} = u_x(\xi_z, t) \leq \frac{1}{t} \quad \Longrightarrow \quad u(x+z,t) - u(x,t) \leq \frac{z}{t}, \tag{1309}$$

with the above equations holding a.e., and so u satisfies the entropy condition for $t \in (0,2)$. Whence the choice of u in (1307) gives the unique entropy satisfying weak solution. For $t > 2$, the shock curve satisfies $s(2) = 2$ and

$$\dot{s}(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}\left(\frac{x}{t}\right)^2 - \frac{1}{2}\left(\frac{x-4}{t}\right)^2}{\frac{x}{t} - \frac{x-4}{t}} = \frac{x-2}{t}. \tag{1310}$$

Using $x = s(t)$ and the initial condition yields the shock curve $s(t)$ for $t > 2$. Note the periodicity of u implies these results are repeated periodically. For these latter times $t > 2$, for each curve $s_k(t)$ satisfying the appropriate RH condition (analogous to above) with $s_k(2) = 4(k+1)$ and $k \in \mathbb{Z}$, immediately to the left of the curve we have $u(x,t) = (x - 4k)/t$ and to the right of the curve we have $u(x,t) = (x - 4(k+1))/t$. And, like above, such choice of u satisfies the entropy condition. Consequently, differentiating (except along the shock curves, which are of measure zero) u yields $u_x = 1/t$ a.e. in $\mathbb{R} \times (2, \infty)$, as desired. $\square$

2012 Spring

S12.2.

a) Consider the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - u_{xx} = f & \text{in } \mathbb{R} \times (0, \infty), \\ u = u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1311}$$

where $f(x, t)$ is smooth and $f(x, t) = 0$ for $t < 0$. Find an explicit solution of this Cauchy problem.

b) (Return and complete.)

Solution:

a) We proceed by applying Duhamel's principle. For each $s \in [0, \infty)$, let $\tilde{u}(x, t; s)$ be the solution to

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (s, \infty), \\ \tilde{u}_t = f(\cdot, s) & \text{on } \mathbb{R} \times \{t = s\}, \\ \tilde{u} = 0 & \text{on } \mathbb{R} \times \{t = s\}. \end{cases} \tag{1312}$$

Then D'Alembert's formula yields

$$\tilde{u}(x, t; s) = \int_{x-(t-s)}^{x+(t-s)} f(z, s) \, \mathrm{d}z. \tag{1313}$$

and Duhamel's principle states

$$u(x, t) = \int_0^t \tilde{u}(x, t; s) \, \mathrm{d}s. \tag{1314}$$

Therefore,

$$u(x, t) = \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(z, s) \, \mathrm{d}z\mathrm{d}s. \tag{1315}$$

□

S12.4. Consider the Hemholtz equation in $\mathbb{R}^3$

$$\Delta u(x) + k^2 u(x) = 0, \quad k > 0. \tag{1316}$$

a) Check that $E(x) = e^{ik|x|}/4\pi|x|$ is a fundamental solution.

b) Prove the Green's formula

$$u(x_0) = \int_{\partial B(0,R)} \left(\frac{\partial u}{\partial \nu}(y) E(x_0 - y) - u \frac{\partial E}{\partial \nu}(x_0 - y) \right) \text{ d}s(y), \tag{1317}$$

where $x_0 \in \mathbb{R}^3$ is arbitrary, $R > |x_0|$, $B(0,R)$ is the ball of radius R centered at 0, and ν is the outward normal to $B(0,R)$.

c) Use b) to prove that if $(\Delta + k^2)u = 0$ in $\mathbb{R}^3$ and if $u(x) = \mathcal{O}(1/r)$ and if $u_r - iku = o(1/r)$, where $r = |x|$, then $u(x) = 0$.

Solution:

a) Let $f \in C_c^2(\mathbb{R}^3)$ and L be the operator defined by $L := \Delta + k^2$. We must show if u is defined by

$$u(x) := \int_{\mathbb{R}^3} E(y) f(x - y) \text{ d}y, \tag{1318}$$

then

$$Lu = -f \quad \text{in } \mathbb{R}^3. \tag{1319}$$

(Show integral can be brought inside.) Since E blows up at the origin, we must proceed delicately in our integration there. Fix $\varepsilon > 0$. Then observe

$$Lu(x) = \underbrace{\int_{B(0,\varepsilon)} E(y) Lf(x-y) \text{ d}y}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^3 - B(0,\varepsilon)} E(y) Lf(x-y) \text{ d}y}_{J_\varepsilon} = I_\varepsilon + J_\varepsilon, \tag{1320}$$

where I_ε and J_ε are the underbraced quantities. Since

$$\begin{aligned} |Lf(x-y)| &= \left| \Delta_x f(x-y) + k^2 f(x-y) \right| \\ &\leq \|D^2 f\|_{L^\infty(B(x,\varepsilon))} + k^2 \|f\|_{L^\infty(B(x,\varepsilon))} \quad \text{for all } y \in B(0,\varepsilon) \end{aligned} \tag{1321}$$

Consequently, $Lf \in L^\infty(B(x,\varepsilon))$. This implies

$$\begin{aligned} |I_\varepsilon| &\le \|Lf\|_{B(x,\varepsilon)} \int_{B(0,\varepsilon)} |E(y)| \ \mathrm{d}y \\ &= \|Lf\|_{B(x,\varepsilon)} \int_{B(0,\varepsilon)} \frac{1}{4\pi|x|} \ \mathrm{d}x \\ &= \|Lf\|_{B(x,\varepsilon)} \int_0^\varepsilon \frac{1}{r}\ r^2 \ \mathrm{d}r \\ &= \|Lf\|_{B(x,\varepsilon)} \frac{\varepsilon^2}{2}, \end{aligned} \tag{1322}$$

where we have employed the use of spherical coordinates. Thus, by the squeeze lemma, we see $\lim_{\varepsilon\to 0^+} I_\varepsilon = 0$. Using integration by parts,

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^3 - B(0,\varepsilon)} E(y)\left[\Delta_y f(x-y) + k^2 f(x-y)\right] \ \mathrm{d}y \\ &= \int_{\mathbb{R}^3 - B(0,\varepsilon)} -D_y E(y)\cdot D_y f(x-y) + k^2 E(y) f(x-y) \ \mathrm{d}y + \int_{\partial B(0,\varepsilon)} E(y)\frac{\partial f}{\partial \nu}(x-y) \ \mathrm{d}\sigma(y). \end{aligned} \tag{1323}$$

However, for $y \neq 0$, we claim $(\Delta_y + k^2)E(y) = 0$. Thus integrating by parts once more reveals

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^3 - B(0,\varepsilon)} (\Delta_y + k^2)E(y) f(x-y) \ \mathrm{d}y + \int_{\partial B(0,\varepsilon)} E(y)\frac{\partial f}{\partial \nu}(x-y) - \frac{\partial E}{\partial \nu}(y) f(x-y) \ \mathrm{d}\sigma(y) \\ &= \int_{\partial B(0,\varepsilon)} E(y)\frac{\partial f}{\partial \nu}(x-y) - \frac{\partial E}{\partial \nu}(y) f(x-y) \ \mathrm{d}\sigma(y) \\ &= L_\varepsilon - M_\varepsilon, \end{aligned} \tag{1324}$$

where L_ε and M_ε are the integrals for the first and second terms in the second line. By (??), we know

$$\frac{\partial E}{\partial \nu} = \frac{e^{ik|x|}}{4\pi}\left(\frac{ik}{|x|^2} - \frac{2}{|x|^3}\right) x \cdot -\frac{x}{|x|} = \frac{e^{ik|x|}}{4\pi}\left(\frac{2}{|x|^2} - \frac{ik}{|x|}\right) = \frac{e^{ik\varepsilon}}{4\pi}\left(\frac{1}{\varepsilon^2} - \frac{ik}{\varepsilon}\right) \quad \text{along } \partial(\mathbb{R}^3 - B(0,\varepsilon)), \tag{1325}$$

noting $\nu = -x/|x|$. So,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} M_\varepsilon &= \lim_{\varepsilon\to 0^+} \int_{\partial B(0,\varepsilon)} \frac{\partial E}{\partial \nu}(y) f(x-y)\ \mathrm{d}\sigma(y) \\
&= \lim_{\varepsilon\to 0^+} \left(\fint_{\partial B(0,\varepsilon)} f(x-y)\ \mathrm{d}y - ik\varepsilon \fint_{\partial B(0,\varepsilon)} f(x-y)\ \mathrm{d}y \right) \\
&= f(x) - ik0\cdot f(x) \\
&= f(x).
\end{aligned} \tag{1326}$$

Additionally,

$$|L_\varepsilon| \le \|Df\|_{L^\infty(B(x,\varepsilon))} \int_{\partial B(0,\varepsilon)} \frac{1}{4\pi|y|}\ \mathrm{d}y = \|Df\|_{L^\infty(B(x,\varepsilon))} \int_0^\varepsilon \frac{1}{r}\cdot r^2\ \mathrm{d}r = \|Df\|_{L^\infty(B(x,\varepsilon))} \cdot \frac{\varepsilon^2}{2}. \tag{1327}$$

Again by the squeeze lemma, we see $\lim_{\varepsilon\to 0^+} L_\varepsilon = 0$. Compiling our results, we see

$$Lu(x) = \lim_{\varepsilon\to 0^+} I_\varepsilon + J_\varepsilon = \lim_{\varepsilon\to 0^+} I_\varepsilon + L_\varepsilon - M_\varepsilon = 0 + 0 - f(x), \tag{1328}$$

as desired.

All that remains is to verify our claim.

$$\partial_{x_i} E(x) = \left(\frac{ik}{4\pi|x|^2} - \frac{1}{4\pi|x|^{-3}} \right) x_i e^{ik|x|} \quad \text{for all } i \in \{1,2,\dots,n\}. \tag{1329}$$

Differentiating once more reveals

$$\begin{aligned}
\partial_{x_i x_i} E(x) &= \left(-\frac{2ikx_i}{|x|^3} + \frac{3x_i}{4\pi|x|^{-4}} \right) x_i e^{ik|x|} + \left(\frac{ik}{4\pi|x|^2} - \frac{1}{4\pi|x|^{-3}} \right) \left(1 + \frac{ikx_i}{|x|} \right) e^{ik|x|} \\
&= \left(\frac{ik}{|x|^2} \right) e^{ik|x|}
\end{aligned} \tag{1330}$$

□

S12.6. Consider Burgers' equation

$$\begin{cases} u_t + (u^2/2)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = u_0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1331}$$

a) Derive the classical/strong solution with initial data $u_0(x) = x^2$. This will only be defined for some x and t, which you should specify.

b) Show where the magnitude of the derivative of the strong solution becomes infinite.

c) Consider the entropy satisfying weak solution arising from the piecewise smooth initial data

$$u_0(x) = \begin{cases} 1/4 & \text{if } x < -1/2 \text{ or } x > 1/2, \\ x^2 & \text{if } -1/2 < x < 1/2. \end{cases} \tag{1332}$$

Specify when and where the first entropy satisfying shock will appear. Write the ODE that describes the trajectory of the shock (you do not need to solve the ODE).

Solution:

a) We proceed by using the method of characteristics. Let $F(p, q, z, x, t) = q + zp$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0, \quad z(0) = u_0(x_0). \end{cases} \tag{1333}$$

This implies $t = s$ and z is constant along characteristics. Thus,

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t z(\tau)\ \mathrm{d}\tau = x_0 + tu_0(x_0) = x_0 + tx_0^2. \tag{1334}$$

Using the quadratic formula yields

$$x_0 = \frac{-1 \pm \sqrt{1 + 4xt}}{2t}. \tag{1335}$$

We take the expression with the "+" since

$$\lim_{t\to 0^+} \frac{-1 \pm \sqrt{1+4xt}}{2t} = \lim_{t\to 0^+} \pm \frac{\frac{1}{2}(1+4xt)^{-1/2}\cdot 4x}{2} = \pm x, \tag{1336}$$

and we require $x \to x_0$ as $t \to 0^+$. Note also $x_0 = x_0(x,t)$ is well-defined in $\mathbb{R}\times(0,\infty)$ when $x \geq -1/4t$. Whence the strong solution u is given by

$$u(x,t) = z(t) = u_0(x_0) = \left(\frac{-1+\sqrt{1+4xt}}{2t}\right)^2, \quad \text{for } (x,t)\in\mathbb{R}\times(0,\infty) \text{ with } x \geq -1/4t. \tag{1337}$$

b) Differentiating u reveals

$$u_x(x,t) = 2\left(\frac{-1+\sqrt{1+4xt}}{2t}\right)\left(\frac{1}{2t}\cdot\frac{1}{2}\cdot(1+4xt)^{-1/2}\cdot 4t\right) = \frac{1}{t}\left(-\frac{1}{\sqrt{1+4xt}}+1\right). \tag{1338}$$

Consequently, we see $|u_x| \to +\infty$ as x approaches $-1/4t$ from the right.

c) By our system of characteristic ODE in (1333), with the initial data swapped, we see

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + tu_0(x_0) = \begin{cases} x_0 + \frac{t}{4}, & \text{if } x_0 < -\frac{1}{2} \text{ or } x_0 > \frac{1}{2}, \\ x_0 + tx_0^2, & \text{if } -\frac{1}{2} < x_0 < \frac{1}{2}. \end{cases} \tag{1339}$$

This shows the characteristics are linear in t. Just to the left of $(1/2,0)$ we see the characteristics have slope $\dot{x}(t) = x_0^2 < 1/4$ while just to the right the characteristics have slop $\dot{x}(t) = 1/4$. So, the characteristics do not cross there. Similarly, the characteristics do not crash at $(-1/2,0)$. All that remains is to investigate the characteristics originating in the interval $(-1/2,1/2)$. Let $a,b \in (-1/2,1/2)$ and suppose characteristics originating at these points collide at $(\tilde{x},\tilde{t})$. Then

$$a + \tilde{t}a^2 = \tilde{x} = b + \tilde{t}b^2 \quad\Longrightarrow\quad \tilde{t} = \frac{a-b}{b^2-a^2} = -\frac{1}{a+b} \quad\Longrightarrow\quad \tilde{x} = a - \frac{a^2}{a+b} = \frac{ab}{a+b}. \tag{1340}$$

Letting $b \to a$ we see

$$\tilde{x} = \lim_{b\to a}\frac{ab}{a+b} = \frac{a}{2} \quad \text{and} \quad \tilde{t} = \lim_{b\to a} -\frac{1}{a+b} = -\frac{1}{2a}, \tag{1341}$$

which implies

$$\tilde{t} = -\frac{1}{4\tilde{x}} \quad \Longrightarrow \quad 1 + 4\tilde{x}\tilde{t} = 0. \tag{1342}$$

The previous two results show that only characteristics originating in $(-1/2, 0)$ will crash (since otherwise $\tilde{t} \leq 0$) and that characteristics originating in $(-1/2, 0)$ will crash along the curve $1+4xt = 0$. Since $x(t)$ is increasing in time along characteristics, to find the first time at which the characteristics crash, it suffices to find the most negative starting point such characteristics. The limiting point is at $(x_0, 0) = (-1/2, 0)$, and so

$$1 + 4tx = 0 \quad \text{and} \quad x - \frac{t}{4} = x_0 = -\frac{1}{2} \quad \Longrightarrow \quad t = 1 \quad \text{and} \quad x = -\frac{1}{4}. \tag{1343}$$

Therefore, the shock curve first occurs at $(-1/4, 1)$. The shock curve, parameterized as $(s(t), t)$, satisfies $s(1) = -1/4$ and the Rankine-Hugoniot condition

$$\dot{s}(t) = \sigma = \frac{\frac{u_\ell^2}{2} - \frac{u_r^2}{2}}{u_\ell - u_r}, \tag{1344}$$

where u_ℓ and u_r are the limiting values of u approaching the shock curve from the left and right, i.e.,

$$u_\ell = \frac{1}{4} \quad \text{and} \quad u_r = \left(\frac{-1 + \sqrt{1 + 4xt}}{2t}\right)^2. \tag{1345}$$

Note this is, indeed, the entropy satisfying solution since $u_\ell > u_r$ and $f''(u) = 2 > 0$, which implies the entropy condition $f'(u_\ell) > \sigma > f'(u_r)$ holds.

□

2011 Fall

F11.1. Let $a, b \in \mathbb{R}^2$ and consider a smooth function $U : (\mathbb{R}^2 - \{a, b\}) \to \mathbb{R}$, which satisfies

$$\lim_{|q|\to\infty} \sup |U(q)| = 1. \tag{1346}$$

Let us consider a system of ODEs for $(p(t), q(t)) \in \mathbb{R}^2 \times (R^2 - \{a, b\})$:

$$\begin{cases} \dot{p}(t) = \nabla U(q(t)), \\ \dot{q}(t) = p(t), \end{cases} \tag{1347}$$

with initial data $p(0) \in \mathbb{R}^2$ and $q(0) \in \mathbb{R}^2 - \{a, b\}$.

a) Show that if $T \in (0, \infty)$ and if $p(t)$ and $q(t)$ are defined on $[0, T)$, then

$$\sup_{t\in[0,T)} |p(t)|, |q(t)| < \infty. \tag{1348}$$

b) Let $[0, T)$ be the maximal interval of existence for $p(t)$ and $q(t)$ with $T < \infty$. Show that the limit of $q(t)$ exists as $t \longrightarrow T$ and, moreover,

$$\lim_{t\to T} q(t) = a \text{ or } b. \tag{1349}$$

Solution:

a) Set $f(x) := x^2/2$. Then observe

$$\begin{aligned} f(p(t)) - f(p(0)) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \left[f(p(\tau))\right] \, \mathrm{d}\tau \\ &= \int_0^t p(\tau) \cdot \dot{p}(\tau) \, \mathrm{d}\tau \\ &= \int_0^t \dot{q}(t) \cdot \nabla U(q(\tau)) \, \mathrm{d}\tau = U(q(t)) - U(q(0)), \end{aligned} \tag{1350}$$

where the final equality holds by the fundamental theorem of line integrals. This implies

$$|p(t)| = \left(|p(0)|^2 + 2\left[U(q(t)) - U(q(0))\right]\right)^{1/2}. \tag{1351}$$

We claim U is bounded by some $M > 0$, and so

$$\begin{aligned}\sup_{t\in[0,T)} |p(t)| &= \sup_{t\in[0,T)} \left(|p(0)|^2 + 2\left[U(q(t)) - U(q(0))\right]\right)^{1/2} \\ &\leq \sup_{t\in[0,T)} \left(|p(0)|^2 + 2M\right)^{1/2} \\ &= \left(|p(0)|^2 + 2M\right)^{1/2} \\ &< \infty.\end{aligned} \tag{1352}$$

Now let us verify that U is bounded. Let $\delta > 0$. By our hypothesis, there exists $r > 0$ such that

$$|U(q)| \leq 1 + \delta, \quad \text{whenever } |q| \geq r. \tag{1353}$$

Since U is smooth... I just realized U might not be bounded due to the hole... It could be like $U = 1/|q|$....

b) (Incomplete.)

□

F11.2. Let $v : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field. Let $\theta : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a smooth function solving the PDE

$$\theta_t = \Delta\left(\theta^2\right) + \nabla \cdot (v\theta), \tag{1354}$$

where $\theta(x, 0)$ is bounded from above and below.

a) Show that θ stays bounded, both from above and below, for all times $t \geq 0$ if $\nabla \cdot v = 0$ for all times.

b) Now suppose $|\nabla \cdot v| \leq M$ for all $x \in \mathbb{R}^n$. If $\theta(x, 0) \leq 1$, show that $\theta(x, t) \leq e^{Mt}$ for all $t > 0$.

Solution:

a) Let B be an upper bound for $|\theta(\cdot, 0)|$ so that $|\theta(x, 0)| \leq B$ for all $x \in \mathbb{R}$. Fix $R > 0$ and $T > 0$ and set $\Omega_T := B(0, R) \times (0, T]$. Let Γ_T be the parabolic boundary of Ω_T. Fix $\varepsilon > 0$ and define $u = \theta - B - \varepsilon e^t$. Because $\overline{\Omega}_T$ is compact and u is smooth, the supremum of u over $\overline{\Omega}_T$ is obtained. By way of contradiction, suppose this supremum is nonnegative. Combined with the fact

$$u(x^\star, 0) = \underbrace{\theta(x^\star, 0) - B}_{\leq 0} - \varepsilon \leq -\varepsilon < 0, \tag{1355}$$

the continuity of u implies there exists $(x^\star, t^\star) \in \overline{\Omega}_T$ such that $u(x^\star, t^\star) = 0$. Let $t^\star$ be the first time at which this occurs, and note $t^\star > 0$ by (1355). This implies, because $t^\star$ is the first time, $u_t(x^\star, t^\star) \geq 0$ and, since $x^\star$ is a local maximizer of $u(\cdot, t^\star)$, $\Delta u(x^\star, t^\star) \leq 0$. At the local maximizer we also have $Du = D\theta = 0$. Whence, at $(x^\star, t^\star)$,

$$\begin{aligned} 0 \leq u_t &= \theta_t - \varepsilon e^t \\ &= \Delta(\theta^2) + \nabla \cdot (v\theta) - \varepsilon e^t \\ &= 2\theta\Delta\theta + 2|D\theta|^2 + v \cdot D\theta - \varepsilon e^t \\ &= \underbrace{2(B + \varepsilon e^t)}_{\geq 0} \underbrace{\Delta u}_{\leq 0} + 2 \cdot 0^2 + v \cdot 0 - \varepsilon e^t \\ &\leq -\varepsilon e^t \\ &< 0, \end{aligned} \tag{1356}$$

a contradiction. Note the derivatives are well-defined everywhere in $\overline{\Omega}_T - (B(0, R) \times \{t = 0\})$ since our original PDE is defined over $\mathbb{R}^n \times (0,\infty)$. Our contradiction shows

$$\sup_{\overline{\Omega}_T} u \leq 0 \quad \Longrightarrow \quad \theta \leq B + \varepsilon e^t \leq B + \varepsilon e^T \ \text{ in } \Omega_T. \tag{1357}$$

Since this holds for arbitrary $\varepsilon > 0$, we may let $\varepsilon \longrightarrow 0^+$ to deduce

$$\theta \leq B \ \text{ in } \Omega_T. \tag{1358}$$

Because this result holds for arbitrary $T > 0$ and $R > 0$, we may let $T \longrightarrow \infty$ to write

$$\theta \leq B \ \text{ in } \overline{B(0,R)} \times [0,\infty), \tag{1359}$$

and then let $R \longrightarrow \infty$ to write

$$\theta \leq B \ \text{ in } \mathbb{R}^n \times [0,\infty), \tag{1360}$$

from which we deduce θ is bounded above. An analogous argument can be applied by instead using the definition $u := -\theta - B - \varepsilon e^t$ to deduce $-\theta \leq B$ in $\mathbb{R}^n \times [0,\infty)$. Together, these results prove θ remains bounded by B over $\mathbb{R}^n$ for all time.

b) This problem is similar to that of a). Fix $R > 0$ and $T > 0$ and $\varepsilon > 0$ and define $v := \theta - e^{Mt} - \varepsilon e^{2Mt}$. Set $\Omega_T := B(0,R) \times (0,T]$. Because $\overline{\Omega}_T$ is compact and v is smooth, v attains its supremum over $\overline{\Omega}_T$. By way of contradiction, suppose this supremum is nonnegative. Combined with the fact

$$v(x,0) = \theta(x,0) - e^{M0} - \varepsilon e^{2M0} \leq 1 - 1 - \varepsilon = -\varepsilon < 0, \tag{1361}$$

the continuity of v implies there exists $(\tilde{x},\tilde{t}) \in \overline{\Omega}_T$ such that $v(\tilde{x},\tilde{t}) = 0$, and $\tilde{t} > 0$ by (1361). Let $\tilde{t}$ be the first time at which this occurs so that $v_t(\tilde{x},\tilde{t}) \geq 0$. And, since $\tilde{x}$ is a local maximizer of $v(\cdot,\tilde{t})$, we see $Dv(\tilde{x},\tilde{t}) = D\theta(\tilde{x},\tilde{t}) = 0$ and $\Delta v(\tilde{x},\tilde{t}) = \Delta v(\tilde{x},\tilde{t}) \leq 0$. Consequently, at $(\tilde{x},\tilde{t})$,

$$\begin{aligned} 0 \leq v_t &= \theta_t - Me^{Mt} - 2M\varepsilon e^{2Mt} \\ &= 2\theta\Delta\theta + 2|D\theta|^2 + v\cdot D\theta + (\nabla\cdot v)\theta - Me^{Mt} - 2M\varepsilon e^{2Mt} \\ &= 2\underbrace{\left(e^{Mt} + \varepsilon e^{2Mt}\right)}_{\geq 0}\underbrace{\Delta\theta}_{\leq 0} + 2\cdot 0^2 + v\cdot 0 + (\nabla\cdot v)\left(e^{Mt}+\varepsilon e^{2Mt}\right) - Me^{Mt} - 2M\varepsilon e^{2Mt} \\ &\leq M\left(e^{Mt}+\varepsilon e^{2Mt}\right) - Me^{Mt} - 2M\varepsilon e^{2Mt} \\ &= -M\varepsilon e^{2Mt} \\ &< 0, \end{aligned} \tag{1362}$$

a contradiction. Therefore,

$$\sup_{\overline{\Omega}_T} v \leq 0 \quad \Longrightarrow \quad \theta \leq e^{Mt} + \varepsilon e^{2Mt} \ \text{ in } \overline{\Omega}_T. \tag{1363}$$

Because this holds for arbitrary $\varepsilon > 0$, we may send $\varepsilon \to 0^+$ to deduce

$$\theta \leq e^{Mt} \ \text{ in } \overline{\Omega}_T. \tag{1364}$$

As done in a), we may then send $T \to \infty$ followed by sending $R \to \infty$ to deduce

$$\theta \leq e^{Mt} \ \text{ in } \mathbb{R}^n \times [0,\infty), \tag{1365}$$

as desired.

□

F11.5. Consider the initial value problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1366}$$

Assume f is smooth and uniformly convex, i.e., $f'' \geq \theta > 0$ for some $\theta > 0$.

a) Show that if $\phi(x) = -x$, then there is a point at which $|u_x| \to \infty$ in finite time.

b) Consider the Riemann initial data

$$\phi(x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0. \end{cases} \tag{1367}$$

Compute the entropy solution and show that the entropy condition is satisfied. Consider both cases: $u^- > u^+$ and $u^- < u^+$.

Solution:

a) We proceed by using the method of characteristics. Let $F(p,q,z,x,t) = q + f'(z)p$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the system of characteristic ODE

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = -f''(z)p^2, & p(0) = -1, \\ \dot{q}(s) = -F_t - F_z q = -f''(z)pq, & q(0) = -f'(z(0))p(0), \\ \dot{x}(s) = F_p = f'(z), & x(0) = x_0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = f'(z)p + q = 0, & z(0) = -x_0. \end{cases} \tag{1368}$$

This implies $t = s$ and z is constant along characteristics. Additionally, $\dot{p} = -f''(z)p^2 \leq -\theta p^2 < 0$, and so $p(t) \leq p(0) = -1$ for all times $t \in (0,\infty)$. Using separation of variables reveals

$$\int_{p(0)}^{p(t)} \frac{\mathrm{d}\tilde{p}}{\tilde{p}^2} = \int_0^t -f''(z(\tau))\ \mathrm{d}\tau \quad \Longrightarrow \quad -1 - \frac{1}{p(t)} = \frac{1}{p(0)} - \frac{1}{p(t)} = -f''(-x_0)t. \tag{1369}$$

Rearranging reveals, for sufficiently small t,

$$p(t) = -\frac{1}{1 - f''(-x_0)t} \leq -\frac{1}{1-\theta t} \quad \Longrightarrow \quad |p(t)| \geq \frac{1}{1-\theta t}. \tag{1370}$$

Therefore,

$$\lim_{t\to(1/\theta)^-} |p(t)| \geq \lim_{t\to(1/\theta)^-} \frac{1}{1-\theta t} = +\infty, \tag{1371}$$

from which we conclude $|u_x| = |p| \to +\infty$ by the time $t = 1/\theta$.

b) We may proceed using (1368), with the exception that $z(0) = \phi(x^0)$ for different initial data. Integrating reveals

$$x(t) = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t f'(z(\tau))\ \mathrm{d}\tau = x_0 + tf'(\phi(x^0)). \tag{1372}$$

We now split the result into two cases:

Case 1: $u^- > u^+$.

Since f is strictly convex, $f'(u^-) > f'(u_\ell)$. Consequently, (1372) implies the characteristics crash immediately at the origin. Thus, the shock curve, parameterized by $(s(t), t)$ satisfies $s(0) = 0$ and the Rankine-Hugoniot (RH) condition

$$\dot{s}(t) = \underbrace{\frac{f(u^-) - f(u^+)}{u^- - u^+}}_{:=\sigma} = \sigma, \tag{1373}$$

where σ is set to be the underbraced quantity. Thus $s(t) = \sigma t$ and

$$u(x,t) = \begin{cases} u^- & \text{if } x < s(t) = \sigma t, \\ u^+ & \text{if } x > s(t) = \sigma t. \end{cases} \tag{1374}$$

We must also verify the entropy condition is satisfied. The fact that $u^- > u^+$ implies $u(\cdot, t)$ is nonincreasing for each t, and so

$$u(x+z,t) - u(x,t) \leq 0 \leq \frac{z}{t}, \quad \text{for all } (x,t) \in \mathbb{R} \times (0,\infty),\ \ z \in (0,\infty), \tag{1375}$$

and so the entropy condition holds. Since entropy solutions are unique (up to a set of measure zero) when f is smooth and convex, we conclude (1374) gives *the* unique entropy solution.

Case 2: $u^- < u^+$.

We claim

$$u(x,t) = \begin{cases} u^- & \text{if } x < tf'(u^-), \\ g(x/t) & \text{if } f'(u^-) < x/t < f'(u^+), \\ u^+ & \text{if } x > tf'(u^+), \end{cases} \tag{1376}$$

where $g := (f')^{-1}$, which is well-defined since f' is strictly increasing. Of course, in the regions where u is constant, the PDE is satisfied. In the remaining region, observe that if $u(x,t) = v(x/t)$ for some function v, then

$$0 = u_t + f'(u)u_x = v' \cdot -\frac{x}{t^2} + f'(v) \cdot v' \cdot \frac{1}{t} = \frac{v'}{t}\left[f'(v) - \frac{x}{t}\right] \quad \Longrightarrow \quad v = (f')^{-1}\left(\frac{x}{t}\right) = g\left(\frac{x}{t}\right), \tag{1377}$$

assuming v' never vanishes, which is indeed the case for $v = g$ since the fact $f'' \geq \theta > 0$ implies

$$v' = g' = \left((f')^{-1}\right)' = \frac{1}{(f')'} = \frac{1}{f''} > 0. \tag{1378}$$

Hence $g(x/t)$ solves the conservation law. Note also

$$\frac{x}{t} = f'(u^-) \quad \Longrightarrow \quad u^- = g\left(\frac{x}{t}\right) \quad \text{and} \quad \frac{x}{t} = f'(u^+) \quad \Longrightarrow \quad u^+ = g\left(\frac{x}{t}\right). \tag{1379}$$

Together with the continuity of g, this implies u, as defined in (1376), is continuous. All that remains is to verify the entropy condition. Following (1378), we see

$$0 < g' = \frac{1}{f''} \leq \frac{1}{\theta}, \tag{1380}$$

i.e., g' is positive and bounded above. Fix any $(x,t) \in (0,\infty)$. Note u is differentiable a.e. (i.e., in $\mathbb{R} - \{tf'(u^-), tf'(u^+)\}$). Thus, by the mean value theorem, for each $z > 0$ with $(x+z) \in$

$\mathbb{R} - \{tf'(u^-), tf'(u^+)\}$, there exists $\xi_z \in (0, z)$ such that

$$\frac{u(x+z,t) - u(x,t)}{z - 0} = u_x(\xi_z, t). \tag{1381}$$

Since

$$u_x(x,t) = \begin{cases} 0 & \text{if } x < tf'(u^-) \text{ or } x > tf'(u^+), \\ g'(x/t) \cdot \frac{1}{t} & \text{if } f'(u^-) < x/t < f'(u^+), \end{cases} \tag{1382}$$

we therefore deduce

$$\frac{u(x+z,t) - u(x,t)}{z - 0} \leq u_x(\xi_z, t) \leq g'\left(\frac{\xi_z}{t}\right) \cdot \frac{1}{t} \leq \frac{1}{\theta} \cdot \frac{1}{t} \quad \Longrightarrow \quad u(x+z,t) - u(x,t) \leq \frac{1}{\theta} \cdot \frac{z}{t}. \tag{1383}$$

This shows that for a.e. $x, z \in \mathbb{R}$ and $t, z > 0$,

$$u(x+z,t) - u(x,t) \leq \frac{1}{\theta} \cdot \frac{z}{t}, \tag{1384}$$

from which we conclude u satisfies the entropy condition. This completes the proof.

$\square$

Remark: In the solution above, one could potentially replace the entropy condition verification in Case 1 with the following variation:

By the mean value theorem, there exists $\tilde{u} \in (u^+, u^-)$ such that

$$f'(\tilde{u}) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \sigma. \tag{1385}$$

By the strict convexity of f, it then follows that

$$f'(u^-) < f'(\tilde{u}) = \sigma < f'(u^+), \tag{1386}$$

and so the entropy condition holds.

△

Fl1.6. Consider the initial value problem

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = \phi & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t = \psi & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1387}$$

a) Use energy methods to prove the value of the solution u at the point (x_0, t_0) depends at most on the values of the initial data in the interval $(x_0 - 3t_0, x_0 + t_0)$.

b) Use energy methods to prove uniqueness of solutions if the initial data has compact support.

Solution:

a) First note this PDE can be "factored" as

$$(\partial_t + 3\partial_x)(\partial_t - \partial_x)u = 0 \quad \text{in } \mathbb{R} \times (0,\infty). \tag{1388}$$

Let $(x_0, t_0) \in \mathbb{R} \times (0,\infty)$ and assume $u = u_t = 0$ on $(x_0 - 3t_0, x_0 + t_0) \times \{t = 0\}$. We shall prove this implies $u(x_0, t_0) = 0$, from which it follows that, at most, u depends on the initial data in the specified interval. The "factored" form of the PDE reveals there will be waves traveling to the left with speed three and to the right with speed unity. Consequently, let us define the energy

$$e(t) := \frac{1}{2} \int_{S(t)} u_t^2 + 3u_x^2 \ \mathrm{d}x, \tag{1389}$$

where, for each $t \in [0, t_0)$,

$$S(t) := (x_0 - 3(t_0 - t), x_0 + (t_0 - t)). \tag{1390}$$

By our assumption on the initial data,

$$e(0) = \frac{1}{2} \int_{S(0)} \underbrace{\psi^2 + 3\phi^2}_{=0} \ \mathrm{d}x = 0. \tag{1391}$$

Differentiating in time reveals

$$\begin{aligned}\dot{e}(t) &= \int_{S(t)} u_t u_{tt} + 3u_x u_{xt} \ \mathrm{d}x + \int_{\partial S(t)} \frac{1}{2}\left(u_t^2 + 3u_x^2\right) v\cdot n \ \mathrm{d}\sigma \\ &= \int_{S(t)} u_t\left(u_{tt} - 3u_{xt}\right)\ \mathrm{d}x + \int_{\partial S(t)} 3u_t\frac{\partial u}{\partial n} + \frac{1}{2}(u_t^2+3u_x^2)v\cdot n\ \mathrm{d}\sigma \\ &= \int_{S(t)} -2u_tu_{xt}\ \mathrm{d}x + \int_{\partial S(t)} 3u_t\frac{\partial u}{\partial n} + \frac{1}{2}(u_t^2+3u_x^2)v\cdot n\ \mathrm{d}\sigma,\end{aligned} \tag{1392}$$

where v is the Eulerian velocity of the boundary and n is the outward normal along $\partial S(t)$. Note

$$\int_{S(t)} -2u_tu_{xt}\ \mathrm{d}x = \int_{S(t)} 2u_tu_{xt}\ \mathrm{d}x - \int_{\partial S(t)} 2u_t^2 n\ \mathrm{d}\sigma \quad\Longrightarrow\quad \int_{S(t)} -2u_tu_{xt}\ \mathrm{d}x = \int_{\partial S(t)} -u_t^2 n\ \mathrm{d}\sigma. \tag{1393}$$

Therefore,

$$\begin{aligned}\dot{e}(t) &= \int_{\partial S(t)} -u_t^2 n + 3u_t\frac{\partial u}{\partial n} + \frac{1}{2}(u_t^2+3u_x^2)v\cdot n\ \mathrm{d}\sigma \\ &= \left[-u_t^2 + 3u_tu_x + \frac{1}{2}(u_t^2+3u_x^2)\cdot(-1)\right]_{x=x_0+(t_0-t)} \\ &\quad + \left[u_t^2 - 3u_tu_x + \frac{1}{2}(u_t^2+3u_x^2)\cdot(-3)\right]_{x=x_0-3(t-t_0)} \\ &= \left[-\frac{3}{2}(u_t+u_x)^2\right]_{x=x_0+(t_0-t)} + \left[-\frac{1}{2}(u_t+3u_x)^2\right]_{x=x_0+3(t_0-t)} \\ &\leq 0.\end{aligned} \tag{1394}$$

This implies $e(t)$ is nonincreasing. Since the integrand in (1389) is nonnegative, it follows that $e(t)\geq 0$, and so $0\leq e(t)\leq e(0)=0$. Thus $e(0)=0$ for all $t\in[0,t_0)$, which implies $u_t=u_x=0$ in $S(t)$ for each $t\in[0,t_0)$, i.e., u is constant therein. Combined with the fact $u=0$ on $S(0)\times\{t=0\}$, we deduce $u=0$ in in $S(t)$ for each $t\in[0,t_0)$. In particular, with the continuity of u, this reveals

$$u(x_0,t_0) = \lim_{t\to t_0^-} u(x_0,t) = \lim_{t\to t_0^-} 0 = 0, \tag{1395}$$

as desired. The result then follows.

b) Given that ϕ and ψ are compactly supported, we claim u is compactly supported for all times. Let $t\in(0,\infty)$. Let $r>0$ be sufficiently large that $\mathrm{spt}(u(\cdot,0))\subseteq B(0,r)$. Now let $x\in\mathbb{R}-B(0,r+3t+1)$. Then, by the choice of r, it follows that $u=0$ in $(x-3t,x+t)\subseteq B(x,3t)$. By our work in a), $u(x,t)=0$.

Since x was arbitrarily chosen in $\mathbb{R} - B(0, r+3t+1)$, it follows that $\text{spt}(u(\cdot,t)) \subseteq B(0, r+3t+1)$, i.e., $u(\cdot,t)$ is compactly supported. Thus, our claim follows.

Now let u and v be two solutions to the given PDE and set $w := u - v$. Then

$$\begin{cases} w_{tt} + 2w_{xt} - 3w_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ w = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \\ w_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1396}$$

So, it suffices to show $w = 0$ in $\mathbb{R} \times (0,\infty)$. Define the energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}} w_t^2 + 3w_x^2 \ \mathrm{d}x, \tag{1397}$$

and note this is well-defined for all time due to the compact support of w (which follows from the compact support of u and v). Then (1396) implies $E(0) = 0$. And, differentiating in time reveals

$$\begin{aligned} \dot{E}(t) &= \int_{\mathbb{R}} w_t w_{tt} + 3w_x w_{xt} \ \mathrm{d}x \\ &= \int_{\mathbb{R}} w_t (w_{tt} - 3w_{xx}) \ \mathrm{d}x \\ &= \int_{\mathbb{R}} w_t \cdot (-2w_{xt}) \ \mathrm{d}x \\ &= -2 \int_{\mathbb{R}} w_t w_{xt} \ \mathrm{d}x \\ &= 2 \int_{\mathbb{R}} w_{xt} w_t \ \mathrm{d}x \\ &= -\dot{E}(t). \end{aligned} \tag{1398}$$

The first equality holds by using integrating by parts, noting the boundary terms vanish. The fifth equality holds again via integration by parts, and the final equality holds since $\dot{E}(t)$ equals the quantity on the fifth line. This implies $\dot{E}(t) = 0$, and so $E(t) = E(0) = 0$ for all $t \in (0,\infty)$. Consequently, $w_x = w_t = 0$ in $\mathbb{R} \times (0,\infty)$, which reveals w is constant. Because $w = 0$ on $\mathbb{R} \times \{t = 0\}$, we then conclude $w = 0$ in $\mathbb{R} \times (0,\infty)$.

□

2011 Spring

S11.1. The equation of motion for a "nonlinear spring" is $\ddot{y} = -ky - ay^3$, where $k > 0$ is the constant in Hooke's Law. Rewrite this equation as an equivalent first order system, and analyze the phase plane for it. Indicate the differences between a hard ($a > 0$) and a soft ($a < 0$) spring. Also explain what differences you would see if a damping term were added to the equation.

Solution:

Observe this second-order ODE may be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -ky - ay^3 \\ x \end{pmatrix}, \tag{1399}$$

where we set $x := \dot{y}$. This system is Hamiltonian since

$$\partial_x \dot{x} + \partial_y \dot{y} = \partial_x(-ky - ay^3) + \partial_y(x) = 0. \tag{1400}$$

This implies every equilibrium point is either a center or a saddle. If $a \geq 0$, then the only equilibrium point is $(0, 0)$. If $a < 0$, then $(0, 0)$ and $(\pm\sqrt{-k/a}, 0)$ are the equilibrium points. The Jacobian matrix for the ODE system is

$$J(x, y) = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & -k - 3ay^2 \\ 1 & 0 \end{pmatrix}. \tag{1401}$$

Consequently,

$$J(0, 0) = \begin{pmatrix} 0 & -k \\ 1 & 0 \end{pmatrix}, \tag{1402}$$

which has eigenvalues $\lambda = \pm i\sqrt{k}$, and so $(0, 0)$ is a center. In the case that $a < 0$, we see

$$J\left(\pm\sqrt{-\frac{k}{a}}, 0\right) = \begin{pmatrix} 0 & 2k \\ 1 & 0 \end{pmatrix}, \tag{1403}$$

which has eigenvalues $\lambda = \pm\sqrt{2k}$, and so $(\pm\sqrt{-k/a}, 0)$ form saddle points.

Now suppose a damping term is added so that $\ddot{y} = -ky - ay^3 + b\dot{y}$ for some scalar $b \in \mathbb{R}$. Then the associated ODE system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -ky - ay^3 + bx \\ x \end{pmatrix}. \tag{1404}$$

The equilibrium points are the same as in the undamped case. However, the Jacobian becomes

$$J(x,y) = \begin{pmatrix} b & -k - 3ay^2 \\ 1 & 0 \end{pmatrix}. \tag{1405}$$

So, the eigenvalues of $J(0,0)$ satisfy

$$0 = \lambda(\lambda - b) + k = \lambda^2 - b\lambda + k \quad \Longrightarrow \quad \lambda = \frac{b \pm \sqrt{b^2 - 4k}}{2}. \tag{1406}$$

If $b^2 - 4k \geq 0$, then $(0,0)$ is an improper node (stable if $b < 0$ and unstable if $b > 0$). If $b^2 - 4k < 0$, then $(0,0)$ is a spiral (stable if $b < 0$ and unstable if $b > 0$). Additionally,

$$J\left(\pm\sqrt{-\frac{k}{a}}, 0\right) = \begin{pmatrix} b & 2k \\ 1 & 0 \end{pmatrix}, \tag{1407}$$

which has eigenvalues

$$\lambda = \frac{b \pm \sqrt{b^2 + 8k}}{2}, \tag{1408}$$

and so $\left(\pm\sqrt{-k/a}, 0\right)$ form saddle points, as before. $\square$

S11.3 Let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary Γ, and assume that $a(x)$ is a continuous function on $\overline{D}$. Show that solutions to $u_t = \Delta u - a(x)u$ vanishing on Γ with $u(x,0) \geq 0$ will be nonnegative for all $t > 0$.

Solution:

Let u be a solution to the given PDE so that

$$\begin{cases} u_t - \Delta u + au = 0 & \text{in } D \times (0,\infty), \\ u = 0 & \text{on } \partial D \times (0,\infty), \\ u \geq 0 & \text{on } D \times \{t = 0\}. \end{cases} \tag{1409}$$

We must show $u \geq 0$ in $D \times (0,\infty)$. Since $a \in C(\overline{D})$ and $\overline{D}$ is closed and bounded, $\overline{D}$ is compact and $a(\overline{D})$ is compact. Thus a is bounded in $\overline{D}$ by some $B > 0$. Choose $\lambda > B$ and let $w(x,t) := u(x,t)e^{-\lambda t}$. Then note the facts $e^{-\lambda t} > 0$ and u satisfies (1409) together imply $w = 0$ on $\partial D \times (0,\infty)$ and $w \geq 0$ on $D \times \{t = 0\}$. Furthermore,

$$w_t - \Delta w = (u_t - \lambda u - \Delta u)e^{-\lambda t} = (-a - \lambda)w \;\text{ in } D \times (0,\infty). \tag{1410}$$

We proceed using a "first time" argument. Let $\varepsilon > 0$ and, by way of contradiction, suppose there is a point in $D \times (0,\infty)$ at which $w = -\varepsilon$. Let $(x_0, t_0) \in D \times (0,\infty)$ be such a point with $t_0 > 0$ the smallest time at which this condition $w = -\varepsilon$ occurs. This implies $w_t(x_0,t_0) \leq 0$. The function $w(\cdot, t_0)$ has a local minimum at x_0 and so $\Delta w(x_0,t_0) \geq 0$. Consequently,

$$0 \geq w_t(x_0,t_0) - \Delta w(x_0,t_0) = \underbrace{(-a(x_0) - \lambda)}_{<0}\underbrace{w(x_0,t_0)}_{=-\varepsilon} > 0, \tag{1411}$$

a contradiction. Note $-a(x_0) - \lambda < 0$ since $\lambda > \sup_{\overline{D}} |a|$. This contradiction shows $w \geq -\varepsilon$ in $D \times (0,\infty)$. Letting $\varepsilon \to 0^+$, we deduce $w \geq 0$ in $D \times (0,\infty)$. Since $w(x,t) = e^{-\lambda t}u(x,t)$ and $e^{-\lambda t} > 0$, we conclude $u \geq 0$ in $D \times (0,\infty)$ also. $\square$

S11.4. Consider the Cauchy problem in the plane

$$\begin{cases} u_x u_x + u_x u_y = 1 & \text{in } \mathbb{R}\times\mathbb{R}, \\ u = f & \text{on } \mathbb{R}\times\{y=0\}, \end{cases} \tag{1412}$$

where $f \in C^2(\mathbb{R})$. When will this be characteristic at $(x_0, 0)$? Assuming that it is not characteristic at $(x_0, 0)$, find a solution defined in a neighborhood of that point. The soultion will be expressed in terms of f and the function $r(x, y)$ defined near $(x_0, 0)$ by $y = (f'(r))^2(x - r - y)$. Show also that $y = (f'(r))^2(x - r - y)$ has a unique local solution with $r(x_0, 0) = x_0$.

Solution:

We proceed using the method of characteristics. Set $F(p, q, z, x, y) = p^2 + pq - 1$. Taking $p = u_x$, $q = u_y$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = 0, \quad p(0) = f'(x_0), \\ \dot{q}(s) = -F_y - F_z q = 0, \quad q(0) = \dfrac{1-p(0)^2}{p(0)} = \frac{1}{f'(x_0)} - f'(x_0), \\ \dot{x}(s) = F_p = 2p + q, \quad x(0) = x_0, \\ \dot{y}(s) = F_q = p, \quad y(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = p(2p+q) + pq = 2, \quad z(0) = f(x_0). \end{cases} \tag{1413}$$

We see this is characteristic[49] when $f'(x_0) = 0$ as $q(0)$ is undefined in this case. Now suppose $f'(x_0) \neq 0$. This implies

$$p(s) = f'(x_0) \quad \text{and} \quad q(s) = \frac{1}{f'(x_0)} - f'(x_0), \tag{1414}$$

and so

$$x(s) - x_0 = \int_0^s 2p(\tau) + q(\tau)\ \mathrm{d}\tau = s\left(\frac{1}{f'(x_0)} + f'(x_0)\right). \tag{1415}$$

Similarly,

$$y(s) - 0 = \int_0^s p(\tau)\ \mathrm{d}\tau = sf'(x_0). \tag{1416}$$

Combining the previous two results reveals

$$x - x_0 - y = x - x_0 - sf'(x_0) = \frac{s}{f'(x_0)} \quad \Longrightarrow \quad f'(x_0)^2(x - x_0 - y) = sf'(x_0) = y. \tag{1417}$$

[49] I don't actually know what this means. But, this is my best guess made by inferring from the context...

We claim there exists a neighborhood $N \subset \mathbb{R}^2$ of $(x_0, 0)$ and a function $r : N \to \mathbb{R}$ such that

$$r(x_0, 0) = x_0 \quad \text{and} \quad y = f'(r)^2(x - r - y), \quad \text{for all } (x, y) \in N. \tag{1418}$$

This shows that, in a neighborhood of $(x_0, 0)$, the function $r(x, y)$ gives the starting point of the characteristic passing through (x, y). From our ODE system and previous results,

$$z(s) - z(0) = \int_0^s 2 \, \mathrm{d}\tau = 2s \quad \Longrightarrow \quad z(s) = z(0) + \frac{2sf'(x_0)}{f'(x_0)} = f(x_0) + \frac{2y}{f'(x_0)}. \tag{1419}$$

Thus,

$$u(x, y) = f(r(x, y)) + \frac{2y}{f'(r(x, y))} \quad \text{in } N. \tag{1420}$$

All that remains is to verify the existence of the claimed function $r(x, y)$. Define $G(x, y, r)$ via

$$G(x, y, r) := y - f'(r)^2(x - r - y), \tag{1421}$$

and so

$$G_r(x, y, r) = 0 - 2f'(r)f''(r)(x - r - y) + f'(r)^2. \tag{1422}$$

Then

$$G(x_0, 0, x_0) = 0 - f'(x_0)(x_0 - x_0 - 0) = 0 \tag{1423}$$

and

$$G_r(x_0, 0, x_0) = 0 - 2f'(x_0)f''(x_0)(x_0 - x_0 - 0) + f'(x_0)^2 = f'(x_0)^2 \neq 0. \tag{1424}$$

Therefore, the implicit function theorem asserts there exists a function $r(x, y)$ defined in a neighborhood N of $(x_0, 0)$ such that $G(x, y, r(x, y)) = 0$ for all $(x, y) \in N$ and $r(x_0, 0) = x_0$. With the definition of G in (1421), we see (1418) holds, and the proof is complete. □

2010 Fall

F10.2.

a) Solve

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix}_x, \tag{1425}$$

with the initial data $(u(x,0), v(x,0)) = (f(x), g(x))$.

b) Find all boundary conditions of the form $au(0,t) + bv(0,t) = 0$ which make the initial value problem in a) well-posed in $x \geq 0$, $t \geq 0$.

Solution:

a) Let $y = (u, v)$ and A be the matrix in the differential equation so that $y_t = Au_x$. Since A is real and symmetric, it is diagonalizable. Observe

$$0 = \det(A - \lambda \text{Id}) = (1-\lambda)^2 - 16 = (\lambda - 5)(\lambda + 3). \tag{1426}$$

Thus the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 5$. Then observe

$$0 = (A - \lambda_1 \text{Id})v_1 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} v_1 \quad \Longrightarrow \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{1427}$$

and

$$0 = (A - \lambda_2 \text{Id})v_2 = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} v_2 \quad \Longrightarrow \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1428}$$

Thus,

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{=:P} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}}_{=:D} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = PDP^{-1}, \tag{1429}$$

and

$$(P|\text{Id}) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right) = (\text{Id}|P^{-1}). \tag{1430}$$

So, our PDe may be rewritten as

$$y_t = Ay_x = PDP^{-1}y_x \quad \Longrightarrow \quad P^{-1}y_t = DP^{-1}y_x. \tag{1431}$$

Taking $w = P^{-1}y$, we see

$$w_t = Dw_x \quad \Longrightarrow \quad \begin{cases} (w_1)_t - 5(w_1)_x = 0, \\ (w_2)_t + 3(w_2)_x = 0. \end{cases} \tag{1432}$$

Define $F(p, q, z, x, t) = q - 5p$. Taking $q = (w_1)_t$, $p = (w_1)_x$, and $z = w$, yields $F = 0$. And the method of characteristics gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = -5, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = -5p + q = 0, \quad z(0) = w_1(x_0, 0). \end{cases} \tag{1433}$$

This implies $t = s$, z is constant along characteristics, and

$$x - x_0 = \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = -5t \quad \Longrightarrow \quad x_0 = x + 5t. \tag{1434}$$

Thus,

$$w_1(x, t) = z(t) = z(0) = w_1(x_0, 0) = w_1(x + 5t, 0). \tag{1435}$$

Likewise,

$$w_2(x, t) = w_2(x - 3t, 0). \tag{1436}$$

Since

$$w(x, 0) = P^{-1}y(x, 0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(f(x) + g(x)) \\ \frac{1}{2}(f(x) - g(x)) \end{pmatrix}, \tag{1437}$$

we deduce

$$w(x, t) = \frac{1}{2}\begin{pmatrix} f(x + 5t) + g(x + 5t) \\ f(x - 3t) - g(x - 3t) \end{pmatrix}. \tag{1438}$$

From this, we conclude

$$y(x,t) = Pw(x,t) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} f(x+5t)+g(x+5t) \\ f(x-3t)-g(x-3t) \end{pmatrix}. \tag{1439}$$

b) By our work in a), the PDE in (1425) is well-posed if and only if the PDE in (1432) is well-posed. And, from (1438) and our boundary condition,

$$\begin{aligned} 0 &= au(0,t) + bv(0,t) \\ &= \frac{1}{2}\left[a\left(f(5t)+g(5t)+f(-3t)-g(-3t)\right) + b\left(f(5t)+g(5t)-f(-3t)+g(-3t)\right)\right] \\ &= \frac{1}{2}\left[(a+b)\left(f(5t)+g(5t)\right) + (a-b)\left(f(-3t)-g(-3t)\right)\right] \\ &= \frac{1}{2}\left[(a+b)w_1(0,t) + (a-b)w_2(0,t)\right]. \end{aligned} \tag{1440}$$

This implies

$$w_2(0,t) = -\frac{a+b}{a-b}w_1(0,t) = -\frac{a+b}{a-b}\cdot\frac{(f(5t)+g(5t))}{2}. \tag{1441}$$

From our work in a), we know the characteristics of w_2 are of the form $x - 3t = C$. Since w_2 is also constant along characteristics, it follows that

$$w_2(x,t) = -\frac{a+b}{2(a-b)}\left(f\left(-\frac{5}{3}(x-3t)\right) + g\left(-\frac{5}{3}(x-3t)\right)\right) \quad \text{for } t > 3x. \tag{1442}$$

Additionally, by (1438),

$$w_2(x,t) = \frac{1}{2}\left(f(x-3t) - g(x-3t)\right) \quad \text{for } t < 3x. \tag{1443}$$

Well-posedness of our PDE therefore comes down to checking that the boundary conditions along the t and x axes align. Namely, the PDE is well-posed provided

$$-\frac{a+b}{2(a-b)}\left(f(0)+g(0)\right) = \lim_{t\to 0^+} w_2(0,t) = \lim_{x\to 0^+} w_2(x,0) = \frac{f(0)-g(0)}{2}. \tag{1444}$$

Equivalently, well-posedness occurs when

$$\frac{a+b}{a-b} = \frac{g(0)-f(0)}{g(0)+f(0)} \quad \Longleftrightarrow \quad 0 = af(0) + bg(0). \tag{1445}$$

We conclude the PDE is well-posed when $0 = af(0) + bg(0)$.

□

F10.3. Consider the competition with limited resources model

$$\dot{x} = (a_1 - b_1 x - c_y y)x, \quad \dot{y} = (a_2 - b_2 x - c_2 y)y, \tag{1446}$$

where a_i, b_i, and c_i are positive constant with $c_1 a_2 > a_1 c_2$ and $b_2 a_1 > b_1 a_2$. Note this implies that $c_1 b_2 > c_2 b_1$.

a) Find the equilibria of this system in the closed quarter plane $x \geq 0$, $y \geq 0$.

b) Show that an equilibrium in the open quarter plane $x > 0$, $y > 0$ must be a saddle.

c) Make a plausible plane diagram for trajectories in the closed quarter plane.

Solution:

a) The equilibria are at $(0,0)$, $(0, a_2/c_2)$, $(a_1/b_1, 0)$, and $(\overline{x}, \overline{y})$, where

$$\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{1447}$$

Thus

$$\begin{aligned} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} &= \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \frac{1}{b_1 c_2 - b_2 c_1} \begin{pmatrix} c_2 & -c_1 \\ -b_2 & b_1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \frac{1}{b_1 c_2 - b_2 c_1} \begin{pmatrix} a_1 c_2 - a_2 c_1 \\ a_2 b_1 - a_1 b_2 \end{pmatrix}. \end{aligned} \tag{1448}$$

b) The point $(\overline{x}, \overline{y})$ is in the open quarter plane, due to our hypotheses regarding the constants a_i, b_i, and c_i. The Jacobian for the system is given by

$$J(x,y) = \begin{pmatrix} \partial \dot{x}/\partial x & \partial \dot{x}/\partial y \\ \partial \dot{y}/\partial x & \partial \dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -b_2 y & a_2 - b_2 x - 2c_2 y \end{pmatrix}. \tag{1449}$$

Using the relation in (1447), we see

$$J(\overline{x},\overline{y}) = \begin{pmatrix} (a_1 - b_1\overline{x} - c_1\overline{y}) - b_1\overline{x} & -c_1\overline{x} \\ -b_2\overline{y} & (a_2 - b_2\overline{x} - c_2\overline{y}) - c_2\overline{y} \end{pmatrix} = \begin{pmatrix} -b_1\overline{x} & -c_1\overline{x} \\ -b_2\overline{y} & -c_2\overline{y} \end{pmatrix}. \tag{1450}$$

To verify $(\overline{x},\overline{y})$ is a saddle point, it suffices to verify $J(\overline{x},\overline{y})$ has a positive real and a negative real eigenvalue. Each eigenvalue λ satisfies

$$0 = \det(\lambda \mathrm{Id} - J(\overline{x},\overline{y})) = (\lambda + b_1\overline{x})(\lambda + c_2\overline{y}) - c_1 b_2 \overline{xy} = \lambda^2 + (b_1\overline{x} + c_2\overline{y})\lambda + \overline{xy}\,(b_1c_2 - b_2c_1)\,, \tag{1451}$$

and so

$$\lambda = \frac{-(b_1\overline{x} + c_2\overline{y}) \pm \sqrt{(b_1\overline{x} + c_2\overline{y})^2 - 4\overline{xy}(b_1c_2 - b_2c_1)}}{2}. \tag{1452}$$

Since $\overline{x} > 0$, $\overline{y} > 0$, and $b_1c_2 < b_2c_1$,

$$-4\overline{xy}(b_1c_2 - b_2c_1) > 0, \tag{1453}$$

from which we deduce

$$\sqrt{(b_1\overline{x} + c_2\overline{y})^2 - 4\overline{xy}(b_1c_2 - b_2c_1)} > |b_1\overline{x} + c_2\overline{y}|\,. \tag{1454}$$

Therefore, (1452) and (1454) together imply $J(\overline{x},\overline{y})$ has a positive real and a negative real eigenvalue. Whence $(\overline{x},\overline{y})$ is a saddle point.

c) Omitted.

□

F10.4. Use the method of characteristics to find a solution to

$$\begin{cases} u_t + uu_x = -x & \text{in } \mathbb{R} \times [0,\infty), \\ u = f & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1455}$$

You will not be able to find $u(x,t)$ explicitly. However, if $f'(x) \geq 0$, show that the solution will exist for $t \in [0, \pi/2)$.

Solution:

We proceed using the method of characteristics. Define $F(p,q,z,x,t) = q + zp + x$. Then taking $q = u_t$, $p = u_x$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = -x, \quad z(0) = f(x_0). \end{cases} \tag{1456}$$

This implies $t = s$ and

$$x(s) + \ddot{x}(s) = x(s) + \dot{z}(s) = x(s) - x(s) = 0, \tag{1457}$$

Likewise, $z(s) + \ddot{z}(s) = 0$. Thus,

$$z = c_1 \sin(t) + c_2 \cos(t) \quad \text{and} \quad x = c_3 \sin(t) + c_4 \cos(t), \tag{1458}$$

for some scalars $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Note

$$f(x_0) = z(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 \quad \text{and} \quad -x_0 = \dot{z}(0) = c_1 \cos(0) + c_2 \sin(0) = c_1. \tag{1459}$$

Similarly,

$$x_0 = x(0) = c_3 \sin(0) + c_4 \cos(0) = c_4 \quad \text{and} \quad f(x_0) = z(0) = \dot{x}(0) = c_3 \cos(0) + c_4 \sin(0) = c_3. \tag{1460}$$

Therefore,

$$u(x,t) = f(x_0) \cos(t) - x_0 \sin(t), \tag{1461}$$

where x_0 satisfies

$$x = f(x_0)\sin(t) + x_0\cos(t). \tag{1462}$$

All that remains is to verify the solution exists for $t \in [0, \pi/2)$. It suffices to show the characteristics cannot cross for $t \in [0, \pi/2)$. We are given that the solution exists at $t = 0$. By way of contradiction, suppose there exists distinct $y_1, y_2 \in \mathbb{R}$ such that the characteristics originating from these points cross at some time $t \in (0, \pi/2)$. Then

$$f(y_1)\sin(t) + y_1\cos(t) = f(y_2)\sin(t) + y_2\cos(t) \quad \Longrightarrow \quad f(y_2) - f(y_1) = \left(-\frac{\cos(t)}{\sin(t)}\right)(y_2 - y_1). \tag{1463}$$

By the mean value theorem, there exists $y^\star$ between y_1 and y_2 such that

$$f(y_2) - f(y_1) = f'(y^\star)(y_2 - y_1). \tag{1464}$$

The previous two results imply there exists $y^\star \in \mathbb{R}$ such that

$$f'(y^\star) = \frac{f(y_2) - f(y_1)}{y_2 - y_1} = -\frac{\cos(t)}{\sin(t)} < 0, \tag{1465}$$

where the final inequality holds since $t \in (0, \pi/2)$. This contradicts the fact $f' \geq 0$ everywhere. Whence the initial assumption was false and we conclude the characteristics do not cross for $t \in [0, \pi/2)$. $\square$

F10.7. Consider the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } D \times (0,\infty), \\ u = 0 & \text{on } D \times \{t = 0\}, \\ u = f & \text{on } \partial D \times (0,\infty). \end{cases} \tag{1466}$$

Find an expansion for the solution to the problem in terms of eigenfunctions of Δ and the solution of the Dirichlet problem $\Delta w = 0$ in D and $w = f$ on ∂D. What is the leading term in the asymptotic expansion of $u(x,t) - w(x)$ as $t \to \infty$?

Solution:

Define $v := u - w$ so that

$$\begin{cases} v_t - \Delta v = 0 & \text{in } D \times (0,\infty), \\ v = -w & \text{on } D \times \{t = 0\}, \\ v = 0 & \text{on } \partial D \times (0,\infty). \end{cases} \tag{1467}$$

We proceed using separation of variables. To this end, assume $v(x,t) = F(x)G(t)$ for some functions F and G. Plugging this into our PDE yields

$$F(x)G'(t) - \Delta F(x)G(t) = 0 \quad \Longrightarrow \quad \frac{G'(t)}{G(t)} = \frac{\Delta F(x)}{F(x)}. \tag{1468}$$

Since the left and right hand sides of the final equality are independent of each other, each side equals $-\mu$, where $\mu \in \mathbb{R}$ is a constant. Consequently,

$$\begin{cases} -\Delta F = \mu F & \text{in } D, \\ F = 0 & \text{on } \partial D. \end{cases} \tag{1469}$$

Since the Laplacian operator is symmetric elliptic, there exists an orthogonal basis of eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ with associated eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. Moreover, for each $n \in \mathbb{N}$

$$0 \leq \int_D |\nabla \varphi_n|^2 \ \mathrm{d}x = -\int_D \varphi_n \Delta \varphi_n \ \mathrm{d}x + \int_{\partial D} \varphi_n \frac{\partial \varphi_n}{\partial \nu} \ \mathrm{d}\sigma = \lambda_n \int_D \varphi_n^2 \ \mathrm{d}x. \tag{1470}$$

If $\lambda_n = 0$, then (1470) implies $\nabla \varphi_n = 0$ in D, i.e., φ_n is constant, which is not possible since $\varphi_n = 0$ on ∂D and the zero function is *not* an eigenfunction. Whence $\lambda_n > 0$ for each $n \in \mathbb{N}$ as the integral on the right hand side of (1470) is positive and $\lambda_n \neq 0$.

The above result regarding the Laplacian operator implies there exists a collection of functions $\{g_n\}_{n\in\mathbb{N}}$ such that

$$g_n'(t) = -\lambda_n g(t), \quad \text{for all } n \in \mathbb{N}, \tag{1471}$$

where we are supposing, by the superposition principle, our solution is of the form

$$v(x,t) = \sum_{n\in\mathbb{N}} g_n(t)\varphi_n(x). \tag{1472}$$

Expanding $-w$ in terms of the eigenfunctions yields

$$-w(x) = \sum_{n\in\mathbb{N}} \alpha_n \varphi_n(x), \quad \text{where} \quad \alpha_n = \frac{\langle -w, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} \tag{1473}$$

and $\langle \cdot, \cdot \rangle$ is the L^2 scalar product on D. Thus, for each $n \in \mathbb{N}$,

$$\begin{cases} g_n' = -\lambda_n g_n & \text{in } (0,\infty), \\ g_n = \alpha_n & \text{on } \{t=0\} \end{cases} \quad \Longrightarrow \quad g_n(t) = \alpha_n e^{-\lambda_n t}. \tag{1474}$$

Compiling our results, we write

$$v(x,t) = \sum_{n\in\mathbb{N}} \alpha_n \varphi_n(x) e^{-\lambda_n t}, \tag{1475}$$

from which we deduce

$$u(x,t) = w(x) + \sum_{n\in\mathbb{N}} \alpha_n \varphi_n(x) e^{-\lambda_n t}. \tag{1476}$$

Since $\lambda_n > 0$ for each $n \in \mathbb{N}$, each term of $v(x,t)$ vanishes as $t \to \infty$. So, there is no leading term in the asymptotic expansion of $v(x,t)$ as $t \to \infty$. □

F10.8. Let $u(x,t)$ be the solution to

$$\begin{cases} u_{tt} + a^2(x,t)u_t - \Delta u = 0 & \text{in } D \times (0,\infty), \\ u = 0 & \text{on } \partial D \times [0,\infty)) \\ u = f & \text{on } D \times \{t=0\}, \\ u_t = g & \text{on } D \times \{t=0\}. \end{cases} \tag{1477}$$

Prove that $\int_D u^2 \ \mathrm{d}x$ is bounded for $t \in [0,\infty)$. You may assume that D is a bounded domain with smooth boundary, f and g are smooth functions vanishing on ∂D, and that a is a smooth function on $D \times [0,\infty)$.

Solution:

Define the energy $E : [0,\infty) \to \mathbb{R}$ via

$$E(t) := \int_D \|\nabla u(x,t)\|^2 + u_t^2(x,t) \ \mathrm{d}x. \tag{1478}$$

Since f and g are smooth and D is bounded,

$$E(0) = \int_D \|\nabla f\|^2 + g^2 \ \mathrm{d}x \leq \int_D \|\nabla\|^2_{L^2\infty(D)} + \|g\|^2_{L^\infty(D)} \ \mathrm{d}x = |D| \left[\|\nabla\|^2_{L^2\infty(D)} + \|g\|^2_{L^\infty(D)}\right] < \infty, \tag{1479}$$

i.e., $E(0)$ is bounded. Differentiating in time reveals

$$\begin{aligned} \dot{E} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_D \|\nabla u\|^2 + u_t^2 \ \mathrm{d}x\right] \\ &= 2\int_D \nabla u \cdot \nabla u_t + u_t u_{tt} \ \mathrm{d}x \\ &= 2\int_D u_t\left(-\Delta u + u_{tt}\right) \ \mathrm{d}x + \int_{\partial D} u_t \frac{\partial u}{\partial \nu} \ \mathrm{d}\sigma \\ &= 2\int_D u_t\left(-a^2 u_t\right) \ \mathrm{d}x \\ &= -2\int_D (au_t)^2 \ \mathrm{d}x \\ &\leq 0, \end{aligned} \tag{1480}$$

where the boundary term vanishes since $u = 0$ on $\partial D \times [0,\infty)$, whereby $u_t = 0$ on $\partial D \times [0,\infty)$. Since the integrand in the definition of E is nonnegative, we therefore know $0 \leq E(t) \leq E(0)$ for all $t \in [0, \infty)$. Furthermore, by Poincaré's inequality, there exists $C > 0$, dependent only on D, such that

$$\|u\|_{L^2(D)} \leq C\|\nabla u\|_{L^2(D)}. \tag{1481}$$

Therefore,

$$\|u\|^2_{L^2(D)} \leq C^2\|\nabla u\|^2_{L^2(D)} \leq C^2 \int_D \|\nabla u\|^2 + u_t^2 \ \mathrm{d}x = C^2 E(t) \leq C^2 E(0) < \infty, \tag{1482}$$

i.e., $\|u\|^2_{L^2(D)}$ is bounded for all $t \in [0, \infty)$. □

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S10.1. Consider the generalize eigenvalue problem

$$\begin{cases} y'' - y = -\lambda x^2 y' & \text{in } (0,1), \\ y = 0 & \text{on } \partial(0,1). \end{cases} \tag{1483}$$

Show that all eigenvalues $-\lambda$ must be bigger than 1.

Solution:

Let u be an eigenfunction of the given ODE with eigenvalue $-\lambda$. This implies

$$\int_0^1 u^2 \ \mathrm{d}x > 0. \tag{1484}$$

Furthermore,

$$\int_0^1 -\lambda x^2 y' y \ \mathrm{d}x = \int_0^1 y'' y - y^2 \ \mathrm{d}x = -\int_0^1 (y')^2 + y^2 \ \mathrm{d}x + \left[y' y \right]_{x=0}^1 = -\int_0^1 (y')^2 + y^2 \ \mathrm{d}x, \tag{1485}$$

and integrating by parts reveals

$$-\int_0^1 x^2 y' y \ \mathrm{d}x = \int_0^1 y \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 y \right] \ \mathrm{d}x - \left[y(x^2 y) \right]_{x=0}^1 = \int_0^1 2xy^2 + x^2 y' y \ \mathrm{d}x, \tag{1486}$$

which implies

$$-\int_0^1 x^2 y' y \ \mathrm{d}x = \int_0^1 xy^2 \ \mathrm{d}x. \tag{1487}$$

Combining (1485) and (1487) reveals

$$-\lambda = \frac{\int_0^1 (y')^2 + y^2 \ \mathrm{d}x}{-\int_0^1 x^2 y' y \ \mathrm{d}x} = \frac{\int_0^1 (y')^2 + y^2 \ \mathrm{d}x}{\int_0^1 xy^2 \ \mathrm{d}x} \geq \frac{\int_0^1 (y')^2 + y^2 \ \mathrm{d}x}{\int_0^1 y^2 \ \mathrm{d}x} = 1 + \frac{\int_0^1 (y')^2 \ \mathrm{d}x}{\int_0^1 y^2 \ \mathrm{d}x} > 1. \tag{1488}$$

The division is well-defined due to (1484). The inequality holds since y' is not identically zero, which follow from (1484) and the fact $y = 0$ on the boundary. This completes the proof. □

S10.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let u be a C^2 solution of the following problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega \times (0,\infty), \\ u = g & \text{on } \Omega \times \{t = 0\}, \\ u = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{1489}$$

Suppose g is bounded and compactly supported in Ω. Using an appropriate energy, show there exists $C > 0$ such that $|u(x,t)| \leq C\exp(-t)$ as $t \to \infty$.

Solution:

Let u be a C^2 solution to the given PDE. We obtain the result by using the "L^p trick." For each $p > 2$, let $\|\cdot\|_p$ be the associated $L^p(\Omega)$ norm. Since Ω is bounded with smooth boundary, it has finite measure. This implies for each $t \in [0,\infty)$

$$\lim_{p\to\infty} \|u(\cdot,t)\|_p = \|u(\cdot,t)\|_\infty. \tag{1490}$$

Now fix any $p > 2$ and define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(v) := |v|^p$. Then note φ is convex since

$$\varphi''(u) = \left(n|u|^{p-2}u\right)' = n(n-1)|u|^{p-2} \geq 0. \tag{1491}$$

Next define the energy e via

$$e(t) := \int_\Omega \varphi(u)\ \mathrm{d}x = \|u(\cdot,t)\|_p^p. \tag{1492}$$

Then differentiating in time yields

$$\begin{aligned} \dot{e}(t) &= \int_\Omega \dot{\varphi}(u)u_t\ \mathrm{d}x \\ &= \int_\Omega \dot{\varphi}(u)\left(\Delta u - u\right)\ \mathrm{d}x \\ &= \int_\Omega -p\varphi(u) + \dot{\varphi}(u)\Delta u\ \mathrm{d}x \\ &= \int_\Omega -p\varphi(u) - \ddot{\varphi}(u)|Du|^2\ \mathrm{d}x - \int_{\partial\Omega} \dot{\varphi}(u)\frac{\partial u}{\partial \nu}\ \mathrm{d}\sigma \\ &\leq \int_\Omega -p\varphi(u)\ \mathrm{d}x \\ &= -pe(t). \end{aligned} \tag{1493}$$

The second equality holds by (1489), the third since $u\dot{\varphi}(u) = u \cdot pu|u|^{p-2} = p|u|^p = p\varphi(u)$. The fourth line follows using integration by parts and the fifth inequality holds since $\ddot{\varphi} \geq 0$ due to convexity and $\dot{\varphi}(u) = 0$ on ∂D. Using Gronwall's inequality, we deduce

$$\|u(\cdot,t)\|_p^p = e(t) \leq e(0)\exp\left(\int_0^t -p \ \mathrm{d}\tilde{t}\right) = e(0)\exp(-pt) = \|g\|_p^p \exp(-pt). \tag{1494}$$

Combining (1490) and (1494), we see for $(x,t) \in \Omega \times (0,\infty)$

$$|u(x,t)| \leq \|u(\cdot,t)\|_\infty = \lim_{p\to\infty} \|u(\cdot,p)\|_p = \lim_{p\to\infty} \left(\|g\|_p^p \exp(-pt)\right)^{1/p} = \lim_{p\to\infty} \|g\|_p \exp(-t) = \|g\|_\infty \exp(-t), \tag{1495}$$

where the final equality holds and is finite since g is bounded. This completes the proof. □

S10.3. Let Ω be a bounded region in $\mathbb{R}^n$ with smooth boundary. Prove C^2 solutions to the following problem are unique:

$$\begin{cases} -\Delta u + a(x)u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = f(x) & \text{on } \partial\Omega, \end{cases} \tag{1496}$$

where $a(x) > 0$, $f(x) \in C^2(\overline{\Omega})$, and ν is the outward normal vector along $\partial\Omega$.

Solution:

Suppose u and v are solutions to the PDE and set $w = u - v$. It suffices to show w is zero everywhere in Ω. Observe

$$\begin{cases} -\Delta w + aw = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1497}$$

Using this, we see

$$0 \leq \int_\Omega |Dw|^2 \ \mathrm{d}x = -\int_\Omega w\Delta w \ \mathrm{d}x + \int_{\partial\Omega} w\frac{\partial w}{\partial \nu} \ \mathrm{d}\sigma = -\int_\Omega w\Delta w \ \mathrm{d}x = -\int_\Omega aw^2 \ \mathrm{d}x \leq 0, \tag{1498}$$

where the final inequality holds since $a > 0$ in Ω. This implies

$$0 = \int_\Omega |Dw|^2 \ \mathrm{d}x, \tag{1499}$$

and so $Dw = 0$ in Ω, i.e., w equals a constant β in Ω. However, (1498) also implies

$$0 = \int_\Omega aw^2 \ \mathrm{d}x = \beta^2 \int_\Omega a \ \mathrm{d}x. \tag{1500}$$

Since $a > 0$ in Ω, the integral on the right hand side of (1500) is positive, which implies $\beta = 0$. Thus w is identically zero in Ω, and we are done. $\square$

S10.4. Let u solve the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \\ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1501}$$

where g and h are compactly supported.

a) Find an energy associated with u.

b) Show that $u(\cdot, t)$ is compactly supported at each $t > 0$.

Solution:

a) Consider the energy $e : [0, \infty) \to \mathbb{R}$ defined via

$$e(t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t) + u_t^2(x,t) + u^2(x,t) \ \mathrm{d}x. \tag{1502}$$

As we will show in b) below, $u(\cdot, t)$ is compactly supported, and so this energy does not blow up.

b) We first verify that if $u = 0$ on $B(x, r) \times \{t = 0\}$ for $r > 0$, then $u = 0$ in the cone

$$K(x, r) := \{(x, t) : t \in [0, r), \ x \in B(x, r - t)\} \cup \{(x, \tau)\}. \tag{1503}$$

For notational compactness, set $S(t) := B(x, r - t)$. Define the energy $E : [0, r) \to \mathbb{R}$ via

$$E(t) := \frac{1}{2} \int_{S(t)} u_x^2 + u_t^2 + u^2 \ \mathrm{d}x. \tag{1504}$$

Our hypothesis implies $E(0) = 0$. Then differentiating in time reveals

$$\begin{aligned} \dot{E}(t) &= \int_{S(t)} u_x u_{xt} + u_t u_{tt} + u u_t \ \mathrm{d}x + \frac{1}{2} \int_{\partial S(t)} (u_t u_x^2 + u_t^2 + u^2) v \cdot \nu \ \mathrm{d}\sigma, \\ &= \int_{S(t)} u_x u_{xt} + u_t u_{tt} + u u_t \ \mathrm{d}x - \frac{1}{2} \int_{\partial S(t)} u_x^2 + u_t^2 + u^2 \ \mathrm{d}\sigma \\ &= \int_{S(t)} u_t \underbrace{(-u_{xx} + u_{tt} + u)}_{=0} \ \mathrm{d}x + \int_{\partial S(t)} u_t u_x - \frac{1}{2} \left(u_x^2 + u_t^2 + u^2 \right) \ \mathrm{d}\sigma \\ &= \int_{\partial S(t)} u_t u_x - \frac{1}{2} \left(u_x^2 + u_t^2 + u^2 \right) \ \mathrm{d}\sigma, \end{aligned} \tag{1505}$$

where $v = -\nu$ is the Eulerian velocity of the boundary of $S(t)$. Since for all $a, b \in \mathbb{R}$ we have

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2 \quad \Longrightarrow \quad ab \leq \frac{1}{2}(a^2+b^2), \tag{1506}$$

we may write

$$\dot{E}(t) \leq \int_{\partial S(t)} \frac{1}{2}(u_t^2 + u_x^2) - \frac{1}{2}\left(u_x^2 + u_t^2 + u^2\right)\ \mathrm{d}\sigma = -\frac{1}{2}\int_{\partial S(t)} u^2\ \mathrm{d}\sigma \leq 0. \tag{1507}$$

Thus, $E(t)$ is nonincreasing. Since $E(0) = 0$ and the integrand in the definition of E is always nonnegative, it follows that $E(t) = 0$ for all $t \in [0, r)$. This implies $u(\cdot, t) = 0$ in $S(t)$ for each $t \in [0, r)$, from which we deduce, by the continuity of u,

$$u(x, r) = \lim_{t \to (r)^-} u(x,t) = \lim_{t \to (r)^-} 0 = 0. \tag{1508}$$

These two facts show $u = 0$ in $K(x, r)$.

We now show u is compactly supported for all time. Let $t^\star \in [0, \infty)$. Since $u(\cdot, 0)$ is compactly supported, there exists $r > 0$ such that $u(x, 0) = 0$ for all x satisfying $|x| > r$. Now pick any z such that $|z| > r + 2t^\star$. Then $u(\cdot, 0) = 0$ in $B(z, r+t^\star)$. Our above result implies $u = 0$ in $K(z, r+t^\star)$, and in particular $u(z, t^\star) = 0$. Since z was an arbitrary point in $\mathbb{R} - \overline{B(0, r+2t^\star)}$, it follows that $u(z, t^\star) = 0$ for all $z \in \mathbb{R} - \overline{B(0, r+2t^\star)}$. Whence $\operatorname{spt}(u(\cdot, t^\star)) \subseteq \overline{B(0, r+2t^\star)}$, i.e., $u(\cdot, t^\star)$ is compactly supported. Since $t^\star$ was arbitrarily chosen, we conclude u is compactly supported for all time.

$\square$

S10.6. Solve the PDE

$$\begin{cases} u_x^2 + yu_y - u = 0 & \text{in } \mathbb{R}\times(1,\infty),\\ u = \dfrac{x^2}{4}+1 & \text{on } \mathbb{R}\times\{y=1\}. \end{cases} \tag{1509}$$

Solution:

We proceed using the method of characteristics. Define $F(p,q,z,x,y) = p^2 + yq - z$. Then taking $p = u_x$, $q = u_y$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = p, & p(0) = \frac{x_0}{2},\\ \dot{q}(s) = -F_y - F_z q = -q + q = 0 & q(0) = z(0) - p(0)^2 = 1,\\ \dot{x}(s) = F_p = 2p, & x(0) = x_0,\\ \dot{y}(s) = F_q = y, & y(0) = 1,\\ \dot{z}(s) = F_p p + F_q q = 2p^2 + yq, & z(0) = \dfrac{x_0^2}{4}+1. \end{cases} \tag{1510}$$

This implies $y = e^s$, $p = \frac{x_0}{2}e^s$, and $q = 1$. Thus

$$x = x_0 + \int_0^s \dot{x}(t)\ \mathrm{d}t = x_0 + \int_0^s x_0 e^t\ \mathrm{d}t = x_0 e^s, \tag{1511}$$

and

$$z = z(0) + \int_0^s 2p(t)^2 + y(t)q(t)\ \mathrm{d}t = \left(\frac{x_0^2}{4}+1\right) + \int_0^s \frac{x_0^2}{2}e^{2t} + e^t\ \mathrm{d}t = \frac{x_0^2}{4} + 1 + \left[\frac{x_0^2}{4}e^{2t} + e^t\right]_{t=0}^s, \tag{1512}$$

whereupon simplifying reveals

$$z = \frac{x_0^2}{4}e^{2s} + e^s = \frac{(xe^{-s})^2}{4}\cdot e^{2s} + e^s = \frac{x^2}{4} + y. \tag{1513}$$

Hence,

$$\boxed{u(x,y) = \frac{x^2}{4} + y.} \tag{1514}$$

$\square$

S10.7. Let $u : [0,1] \to \mathbb{R}$ be piecewise H^1 with a discontinuity at x_Γ. That is, if $u^- : [0, x_\Gamma) \to \mathbb{R}$ with $u^-(x) = u(x)$ for $x \in [0, x_\Gamma)$ and $u^+ : (x_\Gamma, 1] \to \mathbb{R}$ for $x \in (x_\Gamma, 1]$, then $u^- \in H^1(0, x_\Gamma)$ and $u^+ \in H^1(x_\Gamma, 1)$. Furthermore, define the jump in u at x_Γ as

$$[u] := \lim_{x \to x_\Gamma^+} u(x) - \lim_{x \to x_\Gamma^-} u(x), \tag{1515}$$

and $\overline{u}$ as

$$\overline{u} := \frac{1}{2}\left(\lim_{x \to x_\Gamma^+} u(x) + \lim_{x \to x_\Gamma^-} u(x) \right). \tag{1516}$$

Show that if

$$\begin{cases} \partial_x(\beta u_x) = 0 & \text{in } (0, x_\Gamma) \cup (x_\Gamma, 1), \\ [\beta u_x] = b, & \\ u = 0 & \text{on } \partial(0,1), \\ [u] = a, & \end{cases} \tag{1517}$$

where β is piecewise C^∞, but with a discontinuity at x_Γ and $\beta 9x) \geq \varepsilon > 0$, then $e(u) \leq e(v)$ for all piecewise H^1 functions v that also satisfy

$$v(0) = v(1) = 0 \quad \text{and} \quad [v] = a. \tag{1518}$$

Here we set

$$e(u) = \frac{1}{2}\left[\int_0^{x_\Gamma} u_x^2 \beta \ \mathrm{d}x + \int_{x_\Gamma}^1 u_x^2 \beta \ \mathrm{d}x \right] + \overline{u} b. \tag{1519}$$

Solution:

Let u be a solution to the given PDE. Note the set of all piecewise H^1 functions v satisfying (1518) equals $u + \mathcal{A}$, where $\mathcal{A} := \{w : w(0) = w(1) = 0,\ [w] = 0,\ w \text{ is piecewise } H^1\}$. So, it suffices to show that $u + 0$ is a minimizer of e over $u + \mathcal{A}$. To do this, we first we show e is convex. Then we show u is an extremizer of e.

Define $\varphi : \mathbb{R} \to \mathbb{R}$ via $\varphi(v) := v^2$. Then φ is convex since $\varphi'' = 2 > 0$. Consequently, for all $t \in (0,1)$ and $v, w \in (u + \mathcal{A})$,

$$\begin{aligned} & e((1-t)w+tv) \\ &= \frac{1}{2}\left[\int_0^{x_\Gamma} \varphi((1-t)w_x+tv_x)\beta \text{ d}x + \int_0^{x_\Gamma} \varphi((1-t)w_x+tv_x)\beta \text{ d}x\right] + \overline{(1-t)w+tv}b \\ &= \frac{1}{2}\left[\int_0^{x_\Gamma} \varphi((1-t)w_x+tv_x)\beta \text{ d}x + \int_0^{x_\Gamma} \varphi((1-t)w_x+tv_x)\beta \text{ d}x\right] + (1-t)\overline{w}b + t\overline{v}b \\ &\leq \frac{1}{2}\left[\int_0^{x_\Gamma} \left((1-t)\varphi(w_x)+t\varphi(v_x)\right)\beta \text{ d}x + \int_0^{x_\Gamma} \left((1-t)\varphi(w_x)+t\varphi(v_x)\right)\beta \text{ d}x\right] + (1-t)\overline{w}b + t\overline{v}b \\ &= (1-t)e(w) + te(v), \end{aligned} \tag{1520}$$

and so e is convex.

Let $q \in \mathcal{A}$. Then define $\tau : \mathbb{R} \to \mathbb{R}$ by $\tau(\varepsilon) := e(u + \varepsilon q)$. We claim $\tau'(0) = 0$. Since e is convex, it follows that $\varepsilon = 0$ is a minimizer of τ, i.e.,

$$e(u) \leq e(u + \varepsilon q), \quad \text{for all } \varepsilon \in \mathbb{R}. \tag{1521}$$

In particular, $e(u) \leq e(u + q)$. Since q was arbitrarily chosen, this result holds for all $q \in \mathcal{A}$. Whence

$$e(u) \leq e(u + q), \quad \text{for all } q \in \mathcal{A}, \tag{1522}$$

which is precisely what we set out to show.

All that remains is to verify $\tau'(0) = 0$. Since $\partial_x(u_x\beta) = 0$, we know $u_x\beta$ is constant on $(0, x_\Gamma)$ and $(x_\Gamma, 1)$. Thus, there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$u_x\beta = \begin{cases} \alpha_1 & \text{in } (0, x_\Gamma), \\ \alpha_2 & \text{in } (x_\Gamma, 1), \end{cases} \tag{1523}$$

and the fact $[\beta u_x] = b$ implies $\alpha_2 - \alpha_1 = b$. Since $[q] = 0$, we further know there exists $\alpha^\star$ such that

$$\alpha^\star = \lim_{x \to x_\Gamma^-} q(x) = \lim_{x \to x_\Gamma^+} q(x). \tag{1524}$$

This implies

$$\begin{aligned} e(u+\varepsilon q) &= \frac{1}{2}\left[\int_0^{x_\Gamma}(u_x^2+2\varepsilon u_x q_x+\varepsilon^2 q_x^2)\beta\ \mathrm{d}x+\int_{x_\Gamma}^1(u_x^2+2\varepsilon u_x q_x+\varepsilon^2 q_x^2)\beta\ \mathrm{d}x\right]+\overline{u+\varepsilon q}b \\ &= e(u)+\varepsilon\left[\int_0^{x_\Gamma}u_x q_x\beta\ \mathrm{d}x+\int_{x_\Gamma}^1 u_x q_x\beta\ \mathrm{d}x+\overline{q}b\right]+\varepsilon^2\left[\int_0^{x_\Gamma}q_x^2\beta\ \mathrm{d}x+\int_{x_\Gamma}^1 q_x^2\beta\ \mathrm{d}x\right] \\ &= e(u)+\varepsilon\left[\alpha_1\int_0^{x_\Gamma}q_x\ \mathrm{d}x+\alpha_2\int_{x_\Gamma}^1 q_x\ \mathrm{d}x+\overline{q}b\right]+\varepsilon^2\left[\int_0^{x_\Gamma}q_x^2\beta\ \mathrm{d}x+\int_{x_\Gamma}^1 q_x^2\beta\ \mathrm{d}x\right] \\ &= e(u)+\varepsilon\left[\alpha_1(\alpha^\star-q(0))+\alpha_2\left(q(1)-\alpha^\star\right)\right)+\alpha^\star(\alpha_2-\alpha_1)]++\varepsilon^2\left[\int_0^{x_\Gamma}q_x^2\beta\ \mathrm{d}x+\int_{x_\Gamma}^1 q_x^2\beta\ \mathrm{d}x\right] \\ &= e(u)++\varepsilon^2\left[\int_0^{x_\Gamma}q_x^2\beta\ \mathrm{d}x+\int_{x_\Gamma}^1 q_x^2\beta\ \mathrm{d}x\right], \end{aligned} \tag{1525}$$

where we note $q(0)=q(1)=0$. From this, we conclude

$$\tau'(0)=\lim_{\varepsilon\to 0}\frac{e(u+\varepsilon q)-e(u)}{\varepsilon}=\lim_{\varepsilon\to 0^+}\varepsilon\left[\int_0^{x_\Gamma}q_x^2\beta\ \mathrm{d}x+\int_{x_\Gamma}^1 q_x^2\beta\ \mathrm{d}x\right]=0. \tag{1526}$$

□

S10.8. Find a solution of the inhomogeneous initial value problem

$$\begin{cases} u_t + au_x = f(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1527}$$

Solution:

Since the PDE is linear, we may apply the superposition principle to say a solution u may be expressed via $u = v + w$, where

$$\begin{cases} v_t + av_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ v = \phi & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1528}$$

and

$$\begin{cases} w_t + aw_x = f & \text{in } \mathbb{R}\times(0,\infty), \\ w = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1529}$$

Set $F(p,q,z,x,t) = q + ap$. Then taking $p = v_x$, $q = v_t$, and $z = v$, we deduce $F = 0$ and the method of characteristics gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = a, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = ap + q = 0, \quad z(0) = \phi(x_0). \end{cases} \tag{1530}$$

This implies $t = s$, z is constant along characteristics, and

$$x - x_0 = \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = \int_0^t a\ \mathrm{d}\tau = at \quad \Longrightarrow \quad x_0 = x - at. \tag{1531}$$

Therefore,

$$v(x,t) = z(t) = z(0) = \phi(x_0) = \phi(x - at). \tag{1532}$$

To solve for w, we use Duhammel's principle to write

$$w(x,t) = \int_0^t \tilde{w}(x,t;s)\ \mathrm{d}s, \tag{1533}$$

where

$$\begin{cases} \tilde{w}_t(x,t;s) + a\tilde{w}_x(x,t;s) = 0 & \text{in } R \times (s,\infty), \\ \tilde{w}(x,t;s) = f(x,s) & \text{on } \mathbb{R} \times \{t = s\}. \end{cases} \tag{1534}$$

In likewise fashion to when we solved for v, we see

$$\tilde{w}(x,t;s) = f(x - a(t-s), t) \quad \text{in } \mathbb{R} \times [s,\infty). \tag{1535}$$

Thus,

$$w(x,t) = \int_0^t f(x - a(t-s), s) \ \mathrm{d}s, \tag{1536}$$

from which we conclude a solution u to our PDE is given by

$$u(x,t) = \phi(x - at) + \int_0^t f(x - a(t-s), s) \ \mathrm{d}s. \tag{1537}$$

□

2009 Fall

F09.1. Let $u(x)$ be harmonic in the open ball $\{x \in \mathbb{R}^n : |x| < R\}$. Assume that $u(x) \geq 0$. Show that the following *Harnack inequality* holds,

$$\frac{R^2 - |x|^2}{(R+|x|)^n} u(0) \leq R^{2-n} u(x) \leq \frac{R^2 - |x|^2}{(R-|x|)^n} u(0), \quad \text{for all } |x| < R. \tag{1538}$$

Solution:

We assume $u \in C\left(\overline{D}\right)$, where $D := \{x \in \mathbb{R}^n : |x| < R\}$ so that we may define $g : \partial D \to \mathbb{R}$ by $g(x) := u(x)$ for all $x \in \partial D$. Then Poisson's formula for the ball states

$$u(x) = \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial D} \frac{g(y)}{|x-y|^n} \ \mathrm{d}\sigma(y), \quad \text{for all } x \in D. \tag{1539}$$

Though direct computation, we find

$$u(0) = \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial D} g(y) \ \mathrm{d}\sigma(y). \tag{1540}$$

Together with the fact $|y - x| \leq |y| + |x| = R + |x|$ for $y \in \partial D$, this implies

$$\begin{aligned} u(x) &\geq \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial D} \frac{g(y)}{(R+|x|)^n} \ \mathrm{d}\sigma(y) \\ &= \frac{R^2 - |x|^2}{(R+|x|)^n} \cdot R^{n-2} \cdot \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial D} g(y) \ \mathrm{d}\sigma(y) \\ &= \frac{R^2 - |x|^2}{(R+|x|)^n} \cdot R^{n-2} \cdot u(0). \end{aligned} \tag{1541}$$

Upon multiplying by R^{2-n}, this verifies the left hand inequality in (1538). Likewise, $|y - x| \geq ||y| - |x|| = R - |x|$ for all $y \in \partial D$ and $x \in D$, and so

$$\begin{aligned} u(x) &\leq \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial D} \frac{g(y)}{(R-|x|)^n} \ \mathrm{d}\sigma(y) \\ &= \frac{R^2 - |x|^2}{(R-|x|)^n} \cdot R^{n-2} \cdot \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial D} g(y) \ \mathrm{d}\sigma(y) \\ &= \frac{R^2 - |x|^2}{(R-|x|)^n} \cdot R^{n-2} \cdot u(0). \end{aligned} \tag{1542}$$

Together (1541) and (1542) yield (1538), and the proof is complete. □

F09.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $V \in C(\overline{\Omega})$. Show that for $\varepsilon > 0$ small enough, the Dirichlet problem

$$\begin{cases} (-\Delta + \varepsilon V)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1543}$$

has a unique solution in the space $H_0^1(\Omega)$, for each $f \in L^2(\Omega)$.

Solution:

As verified from integrating a smooth solution to the PDE by parts, the weak formulation of our PDE is

$$\int_\Omega Du \cdot Dv + \varepsilon V uv \text{ d}x = \int_\Omega fv \text{ d}x, \quad \text{for all } v \in H, \tag{1544}$$

where $H := H_0^1(\Omega)$. To this end, define the bilinear form $B : H \times H \to \mathbb{R}$ and the linear form $\ell(f) : H \to \mathbb{R}$, respectively, via

$$B[u,v] := \int_\Omega Du \cdot Dv + \varepsilon V uv \text{ d}x \quad \text{and} \quad \ell(v) := \int_\Omega fv \text{ d}x. \tag{1545}$$

We claim that if $\varepsilon > 0$ is sufficiently small, then B is coercive and bounded and ℓ is bounded. Therefore, the Lax-Milgram theorem asserts there exists a unique $\overline{u} \in H$ such that

$$B[\overline{u}, v] = \ell(v), \quad \text{for all } v \in H, \tag{1546}$$

i.e., by (1544) and (1545), $\overline{u}$ is the unique weak solution to the given PDE. All that remains is to verify the three assumptions for $\varepsilon > 0$ sufficiently small. Observe B is bounded since

$$\begin{aligned} |B[u,v]| &\leq \| |Du||Dv| \|_{L^1(\Omega)} + \varepsilon C_1 \|uv\|_{L^1(\Omega)} \\ &\leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \varepsilon C_1 \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq (1 + \varepsilon C_1) \|u\|_H \|v\|_H, \end{aligned} \tag{1547}$$

where $C_1 := \max_{\overline{\Omega}} |V|$, which exists since V is continuous and $\overline{\Omega}$ is compact. The first inequality holds from application of the Cauchy-Schwarz and triangle inequalities. The second inequality is an application of the Cauchy-Schwarz inequality (or Hölder's inequality). Likewise, ℓ is bounded since

$$|\ell(v)| \leq \|vf\|_{L^1(\Omega)} \leq \|v\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_H, \tag{1548}$$

where we note $f \in L^2(\Omega)$. Furthermore, Poincaré's theorem asserts there exists a $C_2 > 0$, depending only on Ω, such that

$$\|w\|_{L^2(\Omega)} \leq C_2 \|Dw\|_{L^2(\Omega)}, \quad \text{for all } w \in H. \tag{1549}$$

Assuming $\varepsilon > 0$ satisfies

$$1 - \varepsilon C_1 C_2 > 0, \tag{1550}$$

we deduce

$$\begin{aligned}
B[u,u] &= \int_\Omega |Du|^2 + \varepsilon V u^2 \ \mathrm{d}x \\
&\geq \int_\Omega |Du|^2 - \varepsilon C_1 u^2 \ \mathrm{d}x \\
&= \|Du\|_{L^2(\Omega)} - \varepsilon C_1 \|u\|_{L^2(\Omega)} \\
&\geq [1 - \varepsilon C_1 C_2] \, \|Du\|_{L^2(\Omega)} \\
&\geq [1 - \varepsilon C_1 C_2] \cdot \frac{1}{2} \left[\|Du\|^2_{L^2(\Omega)} + \frac{1}{C_2} \|u\|^2_{L^2(\Omega)} \right] \\
&\geq \frac{1 - \varepsilon C_1 C_2}{2} \cdot \min\{1, 1/C_2\} \|u\|^2_H,
\end{aligned} \tag{1551}$$

and so B is coercive. Thus, if $\varepsilon > 0$ is sufficiently small that (1550) holds, then each of the needed assumptions holds. This completes the proof. □

F09.4. Let $Lu = -u_{xx} + V(x)u$, where $V(x)$ is real-valued, $Au = 4u_{xxx} - 3((Vu)_x + Vu_x)$. A page of exciting computations shows that the commutator $LA - AL$ is given by

$$(LA - AL)u = (6VV_x - V_{xxx})u. \tag{1552}$$

Do not do that computation. Instead suppose that V depends on the parameter t as well as x, and is a solution of the evolution equation $V_t = 6VV_x - V_{xxx}$ (the Korteweg-De Vries) equation. Suppose that $u(x,t)$ satisfies

$$L(t)u = -u_{xx} + V(t)u = \lambda(t)u \quad \text{and} \quad \int_{\mathbb{R}} u^2 \ \mathrm{d}x = 1, \tag{1553}$$

i.e., u is a normalized eigenfunction for the operator $L(t)$. Show that $\lambda(t)$ must be independent of t.

Solution:

It suffices to show λ is constant in time. Differentiating in time reveals

$$(Lu)_t = Lu_t + V_t u = Lu_t + (LA - AL)u = L(\partial_t + A)u - A\lambda u. \tag{1554}$$

Thus, expanding the derivative on the left hand side,

$$\dot{\lambda}u + \lambda u_t = L(\partial_t + A)u - A\lambda u \quad \Longrightarrow \quad \dot{\lambda}u + \lambda(\partial_t + A)u = L(\partial_t + A)u. \tag{1555}$$

Next observe that for all such normalized eigenfunctions v and w, we have

$$\int_{\mathbb{R}} vLw \ \mathrm{d}x = \int_{\mathbb{R}} v(-w_{xx} + Vw) \ \mathrm{d}x = \int_{\mathbb{R}} -wv_{xx} + Vvw \ \mathrm{d}x = \int_{\mathbb{R}} wLv \ \mathrm{d}x, \tag{1556}$$

where the second equality follows from integrating by parts twice and noting the boundary terms vanish since $|v|, |w| \longrightarrow 0$ as $|x| \longrightarrow 0$. Consequently,

$$\int_{\mathbb{R}} \dot{\lambda}u^2 + \lambda(\partial_t + A)u^2 \ \mathrm{d}x = \int_{\mathbb{R}} [L(\partial_t + A)u]\, u \ \mathrm{d}x = \int_{\mathbb{R}} (\partial_t + A)u(Lu) \ \mathrm{d}x = \int_{\mathbb{R}} \lambda(\partial_t + A)u^2 \ \mathrm{d}x. \tag{1557}$$

Subtracting the common terms from both sides, we see

$$0 = \int_{\mathbb{R}} \dot{\lambda}u^2 \ \mathrm{d}x = \dot{\lambda} \int_{\mathbb{R}} u^2 \ \mathrm{d}x = \dot{\lambda}, \tag{1558}$$

which implies λ is constant in time. This completes the proof. ☐

F09.5. Solve the Hamilton-Jacobi equation

$$\begin{cases} u_t + \frac{1}{2}(u_x)^2 - x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = \alpha x & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1559}$$

for some scalar $\alpha \in \mathbb{R}$. The solution is linear in x, but we want you to use the method of characteristics to solve this problem. The linearity in x is a check on your answer.

Solution:

We proceed using the method of characteristics. Define $F(p,q,z,x,t) := q + \frac{1}{2}p^2 - x$. Taking $q = u_t$ and $p = u_x$ and $z = u$, we see $F = 0$ and the method of characteristics gives rise to the ODE system

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = 1, & p(0) = \alpha, \\ \dot{q}(s) = -F_t - F_z q = 0, & q(0) = x^0 - \frac{\alpha^2}{2}, \\ \dot{x}(s) = F_p = p, & x(0) = x^0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = p^2 + q = \dfrac{p^2}{2} + x, & z(0) = \alpha x^0. \end{cases} \tag{1560}$$

This implies $t = s$, $p = t + \alpha$, and so

$$x - x^0 = \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = \int_0^t p(\tau)\ \mathrm{d}\tau = \frac{p^2-\alpha^2}{2} \quad\Longrightarrow\quad x^0 = x - \frac{p^2-\alpha^2}{2}. \tag{1561}$$

Therefore,

$$z = z(0) + \int_0^t \frac{p^2(\tau)}{2} + x\ \mathrm{d}\tau = \alpha x^0 + \int_0^t p^2(\tau) + x^0 - \frac{\alpha^2}{2}\ \mathrm{d}\tau = p x^0 + \frac{p^3-\alpha^3}{3} - \frac{t\alpha^2}{2}. \tag{1562}$$

Substituting for x^0 reveals

$$z = p\left(x - \frac{p^2-\alpha^2}{2}\right) + \frac{2p^3 - 2\alpha^3 - 3\alpha^2 t}{6} = px + \frac{1}{6}\left[3p\alpha^2 - p^3 - 2\alpha^3 - 3\alpha^2 t\right], \tag{1563}$$

and so

$$\boxed{u(x,t) = (t+\alpha)x + \frac{1}{6}\left[3(t+\alpha)\alpha^2 - (t+\alpha)^3 - 2\alpha^3 - 3\alpha^2 t\right].} \tag{1564}$$

□

F09.6. Let $x(t)$ be a nonnegative differentiable function such that

$$\dot{x}(t) \geq \frac{1}{1+tx(t)} + t - 1, \quad \text{for all } t \geq 0. \tag{1565}$$

Show that $x(t) \geq 1 - \exp(-t^2/2)$ for $t \geq 0$.

Hint: Derive a differential equation for the function $t \mapsto 1 - \exp(-t^2/2)$.

Solution:

Following the hint, set $f(t) := 1 - \exp(-t^2/2)$. Then

$$\dot{f} = t\exp(-t^2/2) = t\left[1 - (1 - \exp(-t^2/2))\right] = t\left[1 - f\right]. \tag{1566}$$

We claim $\dot{x} \geq t\left[1 - x\right]$. Indeed,

$$\dot{x} - t(1-x) \geq \left[\frac{1}{1+tx} + t - 1\right] - t(1-x) = \frac{1}{1+tx}\left[1 + (1+tx)(xt-1)\right] = \frac{(xt)^2}{1+tx} \geq 0, \tag{1567}$$

where the final inequality holds since the numerator is nonnegative and $1 + tx \geq 1 + 0$ as x is nonnegative. Since x is nonnegative, we also know $x(0) \geq 0 = 1 - \exp(0) = f(0)$. By way of contradiction, suppose there exists $\tau > 0$ such that $f(\tau) > x(\tau)$. By the intermediate value theorem, since $x(0) \geq f(0)$ and x and f are continuous, it follows that there exists $\tau_1 \in [0, \tau)$ such that $x(\tau_1) = f(\tau_1)$. Choose $\delta > 0$ to be the smallest positive number such that $f(\tau - \delta) = x(\tau - \delta)$. Then $f(t) > x(t)$ for all $x \in (\tau - \delta, \tau)$. Whence

$$x(\tau) = x(\tau-\delta) + \int_{\tau-\delta}^{\tau} \dot{x}(s)\ \mathrm{d}s \geq x(\tau-\delta) + \int_{\tau-\delta}^{\tau} s[1-x(s)]\ \mathrm{d}s \geq f(\tau-\delta) + \int_{\tau-\delta}^{\tau} s[1-f(s)]\ \mathrm{d}s = f(\tau), \tag{1568}$$

which contradicts our assumption that $f(\tau) > x(\tau)$. Therefore, we conclude $x(t) \geq f(t)$ for all $t \geq 0$. □

F09.7. Let $u(x,t)$ solve the wave equation

$$\begin{cases} (\partial_{tt} - \Delta)u = 0 & \text{in } \mathbb{R}\times\mathbb{R}, \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1569}$$

Show that the function

$$\tilde{u}(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-s^2/4t}u(x,s)\ \mathrm{d}s, \tag{1570}$$

defined for $(x,t)\in\mathbb{R}\times(0,\infty)$, satisfies the initial value problem

$$\begin{cases} (\partial_t - \Delta)\tilde{u} = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \tilde{u} = \phi & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1571}$$

Solution:

Using the change of variables $z = s/\sqrt{4t}$, we may write

$$\tilde{u}(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}u(x,\sqrt{4t}z)\ \mathrm{d}z, \tag{1572}$$

and so

$$\begin{aligned} \lim_{t\to0^+}\tilde{u}(x,t) &= \lim_{t\to0^+}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\cdot u(x,z\sqrt{4t})\ \mathrm{d}z \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\cdot u(x,0)\ \mathrm{d}z \\ &= u(x,0)\cdot\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\ \mathrm{d}z \\ &= u(x,0) \\ &= \phi(x). \end{aligned} \tag{1573}$$

We now carefully justify the second equality. Since u solves the wave equation and has compact initial data, it follows that $u(x,\cdot)$ has compact support for all $x\in\mathbb{R}$. Also with the fact u is continuous, we deduce $u(x,\cdot) \in L^\infty(\mathbb{R})$. This implies

$$\int_{-\infty}^{\infty} \left| e^{-z^2} u(x, z\sqrt{4t}) \right| \ \mathrm{d}z \leq \|u(x,\cdot)\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} e^{-z^2} \ \mathrm{d}z = \sqrt{\pi}\|u(x,\cdot)\|_{L^\infty(\mathbb{R})} < \infty, \tag{1574}$$

which shows the integrand in the first line of (1573) is dominated by an integrable function. Thus, the dominated convergence theorem asserts the limit can be brought inside the integral since the pointwise limit of $u(x, z\sqrt{4t})$ exists as $t \to 0^+$. Thus (1573) holds.

Next observe

$$\begin{aligned}
\tilde{u}_t(x,t) &= \partial_t \left[\int_{-\infty}^{\infty} \frac{e^{-s^2/4t}}{\sqrt{4\pi t}} u(x,s) \ \mathrm{d}s \right] \\
&= \int_{-\infty}^{\infty} \partial_{ss} \left[\frac{e^{-s^2/4t}}{\sqrt{4\pi t}} \right] u(x,s) \ \mathrm{d}s \\
&= \int_{-\infty}^{\infty} e^{-s^2/4t} \partial_{ss} u(x,s) \ \mathrm{d}s + \left[\partial_s \left(\frac{e^{-s^2/4t}}{\sqrt{4\pi t}} \right) u(x,s) - \left(\frac{e^{-s^2/4t}}{\sqrt{4\pi t}} \right) u_s(x,s) \right]_{s=-\infty}^{\infty} \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} \partial_{ss} u(x,s) \ \mathrm{d}s \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} \partial_{xx} u(x,s) \ \mathrm{d}s \\
&= \partial_{xx} \left[\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} u(x,s) \ \mathrm{d}s \right] \\
&= \partial_{xx} \tilde{u}(x,t).
\end{aligned} \tag{1575}$$

The third equality holds since the terms in braces is the fundamental solution of the heat equation. The fourth equality follows from integrating by parts twice. The fifth equality holds since u has compact support. The sixth equality since u solves the wave equation. Thus, (1573) and (1575) hold, which verify (1571), and the proof is complete. □

2009 Spring

S09.1. Let $h(t)$ and $a(t)$ be continuous and bounded functions on $[0,\infty)$, with $a(t) \geq 0$. Let $x(t)$ be a continuous function such that

$$x(t) \leq h(t)\int_0^t a(s)x(s)\ \mathrm{d}s + \frac{1}{1+t^2}, \quad \text{for all } t \in [0,\infty). \tag{1576}$$

Assume that

$$\int_0^\infty |h(t)|\ \mathrm{d}t < \infty. \tag{1577}$$

Prove $x(t)$ is bounded above on $[0,\infty)$.

Solution:

If

$$\sup_{t\in[0,\infty)} x(t) \leq 5, \tag{1578}$$

then $x(t)$ is certainly bounded above. Now consider the case when (1578) does not hold. For each $t \in [0,\infty)$ observe

$$x(t) \leq C|h(t)|\int_0^t S_t\ \mathrm{d}s + 1 = Ct|h(t)|S_t + 1. \tag{1579}$$

where $C := \max\{1, \|a\|_{L^\infty(\mathbb{R})}\}$, $S_t := \sup_{\tau\in[0,t]} x(\tau)$, and $x \in L^\infty([0,t])$ since x is continuous and $[0,t]$ is bounded. We claim

$$\lim_{t\to\infty} t|h(t)| = 0. \tag{1580}$$

This implies there exists $T_1 \in (0,\infty)$ such that

$$t|h(t)| < \frac{1}{4C}, \quad \text{for all } t \geq T_1. \tag{1581}$$

Choose $T_2 \in (0,\infty)$ large enough such that $S_{T_2} = \sup_{t\in[0,T_2)} x(t) \geq 4$. Taking $T := \max\{T_1, T_2\}$ yields

$$x(t) \leq Ct|h(t)|S_t + 1 \leq \frac{1}{4}S_t + 1 \leq \frac{1}{4}S_t + \frac{1}{4}S_t \leq \frac{S_t}{2}, \quad \text{for all } t \geq T. \tag{1582}$$

We claim

$$x(t) \leq S_T, \quad \text{for all } t \in [0,\infty). \tag{1583}$$

This immediately follows for $t \in [0, T]$. By way of contradiction, now suppose there is $t > T$ such that $x(t) > S_T$. By the continuity of x, this would imply there exists $\tau \in (T, t)$ such that the supremum S_τ is attained at τ, i.e.,

$$S_\tau = x(\tau) \leq \frac{S_\tau}{2} \quad \Longrightarrow \quad 1 \leq \frac{1}{2}, \tag{1584}$$

a contradiction (n.b. the division by S_τ is justified since $S_\tau \geq 4$.) Therefore (1583) holds, as claimed, and we see $x(t)$ is bounded above.

All that remains is to verify (1580). By way of contradiction, suppose this equality does not hold. This implies there exists $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} t|h(t)| \geq \varepsilon. \tag{1585}$$

Consequently, there exists T_2 such that

$$|h(t)| \geq \frac{\varepsilon}{t}, \quad \text{for all } t \geq T_2. \tag{1586}$$

Thus,

$$\begin{aligned} \int_0^\infty |h(t)| \ \mathrm{d}t &= \int_0^{T_2} |h(t)| \ \mathrm{d}t + \int_{T_2}^\infty |h(t)| \ \mathrm{d}t \\ &\geq \int_0^{T_2} |h(t)| \ \mathrm{d}t + \varepsilon \int_{T_2}^\infty \frac{1}{t} \ \mathrm{d}t \\ &= \int_0^{T_2} |h(t)| \ \mathrm{d}t + \lim_{T\to\infty} \ln(T) - \ln(T_2) \\ &= +\infty. \end{aligned} \tag{1587}$$

This contradicts (1577), and so our assumption that (1580) does not hold was false. □

S09.2. Let $p \in C^1([0,1])$ and $q \in C([0,1])$ be real-valued with $p > 0$. Show that the eigenvalue problem

$$-(pu')' + qu = \lambda u, \quad u(0) = u(1) = 0 \tag{1588}$$

has the following properties:

a) All the eigenvalues are simple;

b) There are at most finitely many negative eigenvalues.

Solution:

a) Let u and v be two eigenfunctions with a common eigenvalue λ. Letting $\mathcal{L}$ be the differential operator, it follows that

$$\begin{aligned} 0 = \lambda uv - \lambda uv &= (\mathcal{L}u)v - u(\mathcal{L}v) \\ &= \left[-(pu')' + qu\right] v - u\left[-(pv')' + qv\right] \\ &= (pv')'u - (pu')'v \\ &= \left[(pv'u)' - pv'u'\right] - \left[(pu'v)' - pu'v'\right] \\ &= \left[p(v'u - vu')\right]'. \end{aligned} \tag{1589}$$

Integrating once and dividing by p reveals there exists $C \in \mathbb{R}$ such that

$$(v'u - vu') = \frac{C}{p} \quad \text{in } [0,1]. \tag{1590}$$

However, the boundary conditions imply $v'u - vu' = 0$ on $\partial[0,1]$. Thus $C = 0$ and we deduce

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu' = 0 \quad \text{in } [0,1], \tag{1591}$$

where $W(u,v)$ is the Wronskian. This establishes that u and v are linearly dependent. Therefore, all the eigenvalues are simple.

b) Since this is a regular Sturm Liouville problem, the eigenvalues $\{\lambda_n\}$ are countable and are such that $\lambda_n \to \infty$ as $n \to \infty$. Consequently, it suffices to show the eigenvalues are bounded from below. By Sturm Liouville theory, the small eigenvalue satisfies

$$\lambda = \min_{u \in \mathcal{A}} \frac{\langle u, \mathcal{L}u \rangle}{\langle u, u \rangle}, \tag{1592}$$

where $\mathcal{A} := \{u \in C^2(0,1) \cap C^1[0,1] : u(0) = u(1) = 0\}$ and $\langle \cdot, \cdot \rangle$ is the L^2 scalar product on $[0,1]$. For all $u \in \mathcal{A}$,

$$\begin{aligned} \langle u, \mathcal{L}u \rangle &= \int_0^1 u(-(pu')' + qu) \text{ d}x \\ &= \int_0^1 qu^2 + p(u')^2 \text{ d}x - \cancelto{0}{\left[u(pu')\right]_0^1} \\ &= \int_0^1 qu^2 + p(u')^2 \text{ d}x \\ &\geq \alpha_1 \langle u, u \rangle + \alpha_2 \langle u', u' \rangle, \end{aligned} \tag{1593}$$

where $\alpha_1 := \min_{[0,1]} q$ and $\alpha_2 := \min_{[0,1]} p$. By Poincaré's theorem, there exists $C > 0$ dependent only on $[0,1]$ such that

$$\langle u, u \rangle = \int_0^1 u^2 \text{ d}x \leq C \int_0^1 (u')^2 \text{ d}x = C \langle u', u' \rangle, \quad \text{for all } u \in \mathcal{A}. \tag{1594}$$

Therefore,

$$\frac{\langle u, \mathcal{L}u \rangle}{\langle u, u \rangle} \geq \frac{\alpha_1 \langle u, u \rangle + \alpha_2 \langle u', u' \rangle}{\langle u, u \rangle} = \alpha_1 + \alpha_2 \cdot \frac{\langle u', u' \rangle}{\langle u, u \rangle} \geq \alpha_1 + \frac{\alpha_2}{C}, \quad \text{for all } u \in \mathcal{A}, \tag{1595}$$

from which it follows

$$\lambda = \min_{u \in \mathcal{A}} \frac{\langle u, \mathcal{L}u \rangle}{\langle u, u \rangle} \geq \alpha_1 + \frac{\alpha_2}{C} > -\infty. \tag{1596}$$

This shows the smallest eigenvalue is bounded below by a constant, and the proof is complete.

□

S09.3. Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^\infty(\Omega)$ be harmonic in Ω so that $\Delta u = 0$. Show there exists a constant $C = C(n)$, depending only on the dimension n, such that

$$|\nabla u(x)| \leq \frac{C}{d(x)} \sup_\Omega |u|, \quad \text{for all } x \in \Omega, \tag{1597}$$

where

$$d(x) := \inf_{y \in \partial\Omega} |x - y| \tag{1598}$$

is the Euclidean distance from x to the boundary of Ω. Generalize (1597) to obtain similar bounds for higher order derivatives of u.

Hint: Use the Poisson formula for the function u in a ball.

Solution:

Observe $\partial^\beta u$ is harmonic for each multi-index β since

$$\Delta \partial^\beta u = \sum_{i=1}^n \partial_{x_i x_i} \partial^\beta u = \partial^\beta \left(\sum_{i=1}^n u_{x_i x_i} \right) = \partial^\beta (\Delta u) = \partial^\beta 0 = 0. \tag{1599}$$

Fix any $x \in \Omega$. Since Ω is open, $d(x) > 0$. Let $\mu_x := d(x)/2$. Then Poisson's formula for u in a ball yields, for all $i \in \{1, 2, \dots, n\}$,

$$|u_{x_i}(x)| = \left| \fint_{B(x,\mu_x)} u(z) \ \mathrm{d}z \right| = \frac{1}{\alpha(n)\mu_x^n} \left| \int_{B(x,\mu_x)} u(z) \ \mathrm{d}z \right| = \frac{1}{\alpha(n)\mu_x^n} \left| \int_{\partial B(x,\mu_x)} u(z)\nu^i \ \mathrm{d}\sigma \right|, \tag{1600}$$

where the final equality holds via integration by parts and $\alpha(n)$ is the measure of the unit ball in $\mathbb{R}^n$. This implies

$$\begin{aligned} |u_{x_i}(x)| &\leq \frac{1}{\alpha(n)\mu_x^n} \int_{\partial B(x,\mu_x)} |u(z)|\nu^i \ \mathrm{d}\sigma \\ &\leq \frac{1}{\alpha(n)\mu_x^n} \int_{\partial B(x,\mu_x)} \sup_\Omega |u| \ \mathrm{d}\sigma \\ &= \frac{1}{\alpha(n)\mu_x^n} \cdot n\alpha(n)\mu_x^{n-1} \cdot \sup_\Omega |u| \\ &= \frac{n}{\mu_x} \sup_\Omega |u| \\ &= \frac{2n}{d(x)} \sup_\Omega |u|, \end{aligned} \tag{1601}$$

where the third line follows from integration by parts and $\alpha(n)$ denotes the measure of the unit ball in $\mathbb{R}^n$. This implies

$$|\nabla u(x)| = \left(\sum_{i=1}^n u_{x_i}^2(x)\right)^{1/2} \leq \left(\sum_{i=1}^n \frac{2n}{d(x)} \sup_\Omega |u|\right) = \frac{2n^{3/2}}{d(x)} \sup_\Omega |u|. \tag{1602}$$

Taking $C = 2n^{3/2}$, we see (1597) holds, as desired.

In similar fashion to above, we see, for $i, j \in \{1, 2, \ldots, n\}$,

$$\left|\partial_{x_i x_j} u(x)\right| \leq \frac{1}{\alpha(n)\mu_x^n} \int_{\partial B(x,\mu_x)} |u_{x_i}(z)| \nu^j \ \mathrm{d}\sigma \leq \frac{1}{\alpha(n)\mu_x^n} \int_{\partial B(x,\mu_x)} \sup_{\partial B(x,\mu_x)} |u_{x_i}| \ \mathrm{d}\sigma = \frac{n}{\mu_x} \sup_{\partial B(x,\mu_x)} |u_{x_i}|. \tag{1603}$$

However, due to our choice of μ_x,

$$\sup_{\partial B(x,\mu_x)} |u_{x_i}| \leq \sup_{z \in \partial B(x,\mu_x)} \frac{2n}{d(z)} \sup_\Omega |u| \leq \frac{2n}{\mu_x} \sup_\Omega |u|, \tag{1604}$$

and so

$$\left|\partial_{x_i x_j} u(x)\right| \leq \frac{2n^2}{\mu_x^2} \sup_\Omega |u| = \left(\frac{2n}{d(x)}\right)^2 \cdot 2 \sup_\Omega |u|. \tag{1605}$$

This inspires us to prove that for each multi-index β with $|\beta| = k$ we have

$$|\partial^\beta u(x)| \leq \left(\frac{2n}{d(x)}\right)^k \cdot 2^f(k) \sup_\Omega |u|, \tag{1606}$$

where for each nonnegative integer k

$$f(k) := \sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}. \tag{1607}$$

We proceed by induction. The base case is given in (1601). Inductively, suppose (1606) holds for any multi-index β with $|\beta| = k$. Now let γ be any multi-index with $|\gamma| = k+1$. This implies there exists $i \in \{1, 2, \ldots, n\}$ and a multi-index β with $|\beta| = k$ such that $\partial^\gamma = \partial^\beta \partial_{x_i}$. Consequently, in likewise manner to above,

$$\begin{aligned}
|\partial^\gamma u(x)| &= \left|\partial^\beta u_{x_i}(x)\right| \\
&= \frac{1}{|B(x,\mu_x)|}\left|\int_{\partial B(x,\mu_x)} \partial^\beta u(z)\nu^i \, \mathrm{d}\sigma\right| \\
&\leq \frac{|\partial B(x,\mu_x)|}{|B(x,\mu_x)|}\sup_{z\in\partial B(x,\mu_x)} |\partial^\beta u(z)| \\
&\leq \frac{n}{\mu_x}\sup_{z\in\partial B(x,\mu_x)}\left(\frac{2n}{d(z)}\right)^k \cdot 2^{f(k)}\sup_\Omega |u| \qquad (1608)\\
&= \frac{2n}{d(x)}\cdot\left(\frac{2n}{d(x)/2}\right)^k \cdot 2^{f(k)}\sup_\Omega |u| \\
&= \left(\frac{2n}{d(x)}\right)^{k+1} \cdot 2^{k+f(k)}\sup_\Omega |u| \\
&= \left(\frac{2n}{d(x)}\right)^{k+1} 2^{f(k+1)}\sup_\Omega |u|,
\end{aligned}$$

and we have closed the induction. By the principle of mathematical induction, the claim follows. □

S09.4. Let Ω be a bounded open set and let $V \in C(\overline{\Omega})$ satisfy $V(x) \geq 0$. Show that for each $f \in L^2(\Omega)$, the Dirichlet problem

$$\begin{cases} (-\Delta + V)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1609}$$

has a unique solution in the space $H_0^1(\Omega)$.

Solution:

Let us momentarily assume u is smooth. Then for all $v \in C_c^\infty(\Omega)$, we may multiply the PDE and by v and integrate by parts to obtain

$$0 = \int_\Omega (-\Delta u + Vu - f)v \text{ d}x = \int_\Omega Du \cdot Dv + Vuv - fv \text{ d}x + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \text{ d}x = \int_\Omega Du \cdot Dv + Vuv \text{ d}x - \int_\Omega fv \text{ d}x, \tag{1610}$$

where the boundary term vanishes due to the compact support of v. Note the right hand side of the above makes sense even if u is not C^2, but rather only in $u \in H_0^1(\Omega)$. So, letting $H := H_0^1(\Omega)$ and defining $\ell : H \to \mathbb{R}$ and $B : H \times H \to \mathbb{R}$ by

$$B[u,v] := \int_\Omega Du \cdot Dv + Vuv \text{ d}x \quad \text{and} \quad \ell(v) := \int_\Omega vf \text{ d}x, \tag{1611}$$

using the weak formulation of the PDE, it suffices to show there exists a unique $\overline{u} \in H$ such that

$$B[\overline{u}, v] = \ell(v), \quad \text{for all } v \in H. \tag{1612}$$

We claim the bilinear form B is coercive and bounded and the linear form ℓ is bounded. Thus, the assumptions of the Lax-Milgram theorem hold, from which we deduce there exists a unique solution $\overline{u} \in H$ such that (1612) holds.

All that remains is to verify our claims. Observe ℓ is bounded since the fact $f \in L^2(\Omega)$ implies

$$|\ell(v)| \leq \|vf\|_{L^1(\Omega)} \leq \|v\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \leq \|v\|_H \|f\|_{L^2(\Omega)}, \quad \text{for all } v \in H. \tag{1613}$$

Similarly, B is bounded since, for all $u, v \in H$,

$$\begin{aligned} |B[u,v]| &= \left| \int_\Omega Du \cdot Dv + Vuv \ \mathrm{d}x \right| \\ &\leq \|Du \cdot Dv\|_{L^1(\Omega)} + \|V\|_{L^\infty(\Omega)} \|uv\|_{L^1(\Omega)} \\ &\leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \|V\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2\Omega} \\ &\leq \|u\|_H \|v\|_H + \|V\|_{L^\infty(\Omega)} \|u\|_H \|v\|_H \\ &= \left(1 + \|V\|_{L^\infty(\Omega)}\right) \|u\|_H \|v\|_H. \end{aligned} \tag{1614}$$

Next note Poincaré's inequality asserts there exists $\alpha > 0$, dependent only on Ω, such that

$$\|u\|_{L^2(\Omega)} \leq \alpha \|Du\|_{L^2(\Omega)}, \quad \text{for all } u \in H. \tag{1615}$$

Therefore, together with the negativity of V, we deduce

$$\begin{aligned} B[u,u] &= \int_\Omega |Du|^2 + Vu^2 \ \mathrm{d}x \\ &\geq \|Du\|^2_{L^2(\Omega)} \\ &= \frac{1}{2} \left[\|Du\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)} \right] \\ &\geq \frac{1}{2} \left[\|Du\|^2_{L^2(\Omega)} + \frac{1}{\alpha^2} \|u\|^2_{L^2(\Omega)} \right] \\ &\geq \frac{\min\{1, 1/\alpha^2\}}{2} \left[\|Du\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right] \\ &= \frac{\min\{1, 1/\alpha^2\}}{2} \|u\|^2_H, \end{aligned} \tag{1616}$$

i.e., B is coercive. Since all our claims have been verified, the proof is now complete. □

S09.5. Consider the complementary error function

$$F(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\ \mathrm{d}t. \tag{1617}$$

Show

$$F(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right) \quad \text{as } x \to \infty. \tag{1618}$$

Show also that this estimate for large x can be refined to a complete asymptotic expansion

$$F(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{k=0}^\infty \frac{a_k}{x^{2k}}, \tag{1619}$$

for some coefficients a_k. (You do not have to determine each a_k.)

Solution:

First observe differentiating yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t^2}\right] = -2te^{-t^2} \quad \Longrightarrow \quad e^{-t^2} = -\frac{1}{2t}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t^2}\right]. \tag{1620}$$

Consequently,

$$\int_x^\infty e^{-t^2}\ \mathrm{d}t = \int_x^\infty -\frac{1}{2t}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t^2}\right]\ \mathrm{d}t = \left[-\frac{1}{2t}\cdot e^{-t^2}\right]_x^\infty - \int_x^\infty \frac{1}{2t^2}e^{-t^2}\ \mathrm{d}t = \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{1}{2t^2}e^{-t^2}\ \mathrm{d}t. \tag{1621}$$

Now for each nonnegative integer k define $f_k : \mathbb{R} \to \mathbb{R}$ by $f_k(t) := t^{-k}e^{-t^2}$. Then

$$\int_x^\infty f_k(t)\ \mathrm{d}t = \int_x^\infty -\frac{1}{2t^{k+1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t^2}\right]\ \mathrm{d}t = \frac{e^{-x^2}}{2x^{k+1}} - \frac{k+1}{2}\int_x^\infty f_{k+2}(t)\ \mathrm{d}t. \tag{1622}$$

Additionally, for $x > 0$ and $k > 1$ we see

$$\int_x^\infty f_k(t)\ \mathrm{d}t = \int_x^\infty \frac{e^{-t^2}}{t^k}\ \mathrm{d}t \leq \int_x^\infty t^{-k}\ \mathrm{d}t = \left[\frac{t^{1-k}}{1-k}\right]_x^\infty = \frac{x^{1-k}}{k-1} = \mathcal{O}(x^{1-k}), \quad \text{as } x \to \infty. \tag{1623}$$

Thus, for $k > 1$,

$$\int_x^\infty f_k(t) = \mathcal{O}(x^{1-k}), \quad \text{as } x \to \infty. \tag{1624}$$

Compiling our results, we see, as $x \to \infty$,

$$\begin{aligned}
F(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty f_0(t)\ \mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^\infty f_2(t)\ \mathrm{d}t \right] \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2 x^3} + \frac{1 \cdot 3}{2^2} \int_x^\infty f_4(t)\ \mathrm{d}t \right] \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2 x^3} + \frac{3e^{-x^2}}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^3} \int_x^\infty f_6(t)\ \mathrm{d}t \right] \\
&= \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 - \frac{1}{2x^2} + \frac{3}{2^2 x^4} - \frac{1 \cdot 3 \cdot 5}{2^2} \int_x^\infty f_6(t)\ \mathrm{d}t \right] \\
&= \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 - \frac{1}{2x^2} + \frac{3}{2^2 x^4} + \mathcal{O}(x^{-5}) \right] \\
&= \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 + \mathcal{O}(x^{-2}) \right].
\end{aligned} \tag{1625}$$

This verifies (1618). Furthermore, from the form of the fourth line in (1625), we see

$$F(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 + \sum_{k=1}^{N} (-1)^k \cdot \frac{1 \cdot 3 \cdots (2k-1)}{2^k x^{2k}} \right) - \frac{1 \cdot 3 \cdots (2N-1)}{2^N} \int_x^\infty f_{2N}(t)\ \mathrm{d}t \tag{1626}$$

Since our earlier work implies

$$\frac{1 \cdot 3 \cdots (2N-1)}{2^N} \int_x^\infty f_{2N}(t)\ \mathrm{d}t = \mathcal{O}(x^{1-2N}), \quad \text{as } x \to \infty, \tag{1627}$$

we deduce for large x

$$F(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 + \sum_{k=1}^{\infty} (-1)^k \cdot \frac{1 \cdot 3 \cdots (2k-1)}{2^k x^{2k}} \right), \tag{1628}$$

as desired. □

S09.6. Consider an initial value proble for the focusing cubic non-linear Schrödinger equation

$$iu_t = -\frac{1}{2}u_{xx} - |u|^2 u, \quad u(x,0) = \varphi(x). \tag{1629}$$

Show that the following quantities are conserved (assuming u vanishes as $|x| \to \infty$ together with all of its derivatives)

a) Mass

$$\int_{\mathbb{R}} |u(x,t)|^2 \, \mathrm{d}x \tag{1630}$$

b) Energy

$$\int_{\mathbb{R}} \frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4 \, \mathrm{d}x. \tag{1631}$$

Solution:

a) First note

$$u_t = \frac{1}{i}\left[-\frac{u_{xx}}{2} - |u|^2 u\right] \quad \Longrightarrow \quad \overline{u}_t = -\frac{1}{i}\left[-\frac{\overline{u}_{xx}}{2} - |u|^2\overline{u}\right] = \frac{1}{i}\left[\frac{\overline{u}_{xx}}{2} + |u|^2\overline{u}\right]. \tag{1632}$$

Then differentiating the mass in time yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u(x,t)|^2 \, \mathrm{d}x &= \int_{\mathbb{R}} \partial_t \left[u\overline{u}\right] \, \mathrm{d}x \\ &= \int_{\mathbb{R}} u_t\overline{u} + u\overline{u}_t \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \left[-\frac{u_{xx}\overline{u}}{2i} - \frac{|u|^4}{i}\right] + \left[\frac{\overline{u}_{xx}u}{2i} + \frac{|u|^4}{i}\right] \, \mathrm{d}x \\ &= \frac{1}{2i}\int_{\mathbb{R}} \overline{u}_{xx}u - u_{xx}\overline{u} \, \mathrm{d}x \\ &= \frac{1}{2i}\int_{\mathbb{R}} -\overline{u}_x u_x + u_x\overline{u}_x \, \mathrm{d}x \\ &= 0, \end{aligned} \tag{1633}$$

where the fifth line holds via integration by parts, noting we assume u and all its derivatives vanish as $|x| \to \infty$. This shows the mass is constant in time, as desired.

b) Likewise,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4 \ \mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}} \partial_t \left[u_x\overline{u}_x - (u\overline{u})^2\right] \ \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}} u_x\overline{u}_{xt} + u_{xt}\overline{u}_x - 2|u|^2(u_t\overline{u} + u\overline{u}_t) \ \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}} u_x\overline{u}_{xt} + u_{xt}\overline{u}_x \ \mathrm{d}x \\
&= -\frac{1}{2}\int_{\mathbb{R}} u_{xx}\overline{u}_t + u_t\overline{u}_{xx} \ \mathrm{d}x \\
&= -\frac{1}{2}\int_{\mathbb{R}} \overline{u}_t \cdot 2\left[-iu_t - |u|^2 u\right] + u_t \cdot 2\left[i\overline{u}_t - |u|^2\overline{u}\right] \ \mathrm{d}x \\
&= \int_{\mathbb{R}} |u|^2 \left[u\overline{u}_t + \overline{u}u_t\right] \ \mathrm{d}x \\
&= 0,
\end{aligned} \tag{1634}$$

where we recall in (1633) we showed $u_t\overline{u} + u\overline{u}_t = 0$.

□

S09.7. Solve the PDE

$$\begin{cases} u_t + u_x^2 = 0, \\ u(x,0) = -x^2. \end{cases} \tag{1635}$$

Find the time T for which $|u| \to \infty$ as $t \to T$.

Solution:

Set[50] $g(x) = -x^2$ and $H(p) := p^2$. Then we see the PDE can be written as the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1636}$$

The associated Lagrangian L is given by the Fenchel transform

$$L(v) := \sup_{p \in \mathbb{R}} pv - H(p) = pv - p^2. \tag{1637}$$

Differentiating and using the fact this expression is a quadratic and concave down, we discover

$$0 = \frac{\mathrm{d}}{\mathrm{d}p}\left[pv - p^2\right] = v - 2p \quad \Longrightarrow \quad p = \frac{v}{2}. \tag{1638}$$

Consequently,

$$L(v) = v\left(\frac{v}{2}\right) - \left(\frac{v}{2}\right)^2 = \frac{v^2}{4}. \tag{1639}$$

Then by the Hopf-Lax formula

$$u(x,t) = \min_{y \in \mathbb{R}} \left(t \cdot L\left(\frac{x-y}{t}\right) + g(y) \right) = \min_{y \in \mathbb{R}} \left(\frac{(x-y)^2}{4t} - y^2 \right). \tag{1640}$$

In similar fashion to above, differentiating with respect to y yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{(x-y)^2}{4t} - y^2\right] = \frac{y-x}{2t} - 2y \quad \Longrightarrow \quad y = \frac{x}{1-4t}. \tag{1641}$$

Therefore for $(x,t) \in \mathbb{R} \times [0, 1/4)$

$$u(x,t) = \frac{1}{4t}\left(x - \frac{x}{1-4t}\right)^2 - \left(\frac{x}{1-4t}\right)^2 = -\frac{x^2}{1-4t}, \tag{1642}$$

and we see $\lim_{t \to 1/4} |u(x,t)| = +\infty$ for $x \neq 0$. □

[50] As in Zane and Peter's notes, we note that this problem can be solved using the method of characteristics.

S09.8. Consider the hyperbolic equation

$$u_{tt} + 3u_{xt} + 2u_{xx} = 0 \tag{1643}$$

in the quarter-plane $Q = \{(x,t) : x, t > 0\}$. Assign boundary conditions along $t = 0$ and $x = 0$ such that the boundary value problem in Q will have a unique solution.

Solution:

We shall prescribe conditions for which the PDE uniquely admits the zero solution. The PDE may be "factored" and rewritten as

$$0 = (\partial_t + \partial_x)(\partial_t + 2\partial_x)u. \tag{1644}$$

Set $v := u_t + 2u_x$. Then set $F(p,q,z,x,t) := q + p$. Taking $p = v_x$ and $q = v_t$ and $z = v$, we see $F = 0$ and the method of characteristics gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = 1, \quad x(0) = x^0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = p + q = 0, \quad z(0) = z^0. \end{cases} \tag{1645}$$

From this, we see v is constant along characteristics, which are of the form $t = x + C$ for scalars $C \in \mathbb{R}$. Thus, v is uniquely defined by its values along the x and t axes, i.e., by

$$v(0,t) = u_t(0,t) + 2u_x(0,t), \quad \text{and} \quad v(x,0) = u_t(x,0) + 2u_x(x,0). \tag{1646}$$

In similar fashion, define $\tilde{F}(p,q,z,x,t) = q + 2p$. Taking $q = v_t$ and $p = u_x$ and $z = u$, we see $\tilde{F} = 0$ and the method of characteristics gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = \tilde{F}_p = 2, \quad x(0) = x^0, \\ \dot{t}(s) = \tilde{F}_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = \tilde{F}_p p + F_q q = 2p + q = 0, \quad z(0) = z^0. \end{cases} \tag{1647}$$

Thus u is constant along characteristics, which are of the form $t = \frac{1}{2}x + C$ for scalars $C \in \mathbb{R}$. Whence u is RETURN AND FINISH...... Something is missing with argument... □

S09.9. Consider the boundary value problem in a smooth bounded domain D in $\mathbb{R}^n$

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ \dfrac{\partial u}{\partial n} + a(x)u = f & \text{on } \partial D, \end{cases} \tag{1648}$$

where n is the outer normal to ∂D.

a) Find a functional whose Euler-Lagrange equation leads to the boundary value problem above.

b) Assume that $a(x) > 0$. Prove that this boundary value has a unique smooth solution.[51]

Solution:

a) Define the functional $E : H^2(D) \to \mathbb{R}$ via

$$E[u] := \frac{1}{2}\int_D |Du|^2 \text{ d}x + \int_{\partial D} \frac{au^2}{2} - uf \text{ d}\sigma. \tag{1649}$$

Then for all $v \in H^2(D)$ and $\varepsilon \in \mathbb{R}$,

$$\begin{aligned} E[u+\varepsilon v] &= \int_D \frac{1}{2}|Du|^2 + \varepsilon Du\cdot Dv + \frac{\varepsilon^2}{2}|Dv|^2 \text{ d}x + \int_{\partial D} \frac{au^2}{2} + \varepsilon auv + \frac{\varepsilon^2 av^2}{2} - uf - \varepsilon vf \text{ d}\sigma \\ &= E[u] + \varepsilon\left[\int_D Du\cdot Dv \text{ d}x + \int_{\partial D}(au-f)v \text{ d}\sigma\right] + \varepsilon^2\left[\frac{1}{2}\int_D |Dv|^2 \text{ d}x + \int_{\partial D}\frac{av^2}{2} \text{ d}\sigma\right]. \end{aligned} \tag{1650}$$

Therefore,

$$\begin{aligned} \delta E(u,v) &= \lim_{\varepsilon\to 0^+} \frac{E[u+\varepsilon v] - E[u]}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+}\left[\int_D Du\cdot Dv \text{ d}x + \int_{\partial D}(au-f)v \text{ d}\sigma\right] + \varepsilon\left[\frac{1}{2}\int_D |Dv|^2 \text{ d}x + \int_{\partial D}\frac{av^2}{2} \text{ d}\sigma\right] \\ &= \int_D Du\cdot Dv \text{ d}x + \int_{\partial D}(au-f)v \text{ d}\sigma. \end{aligned} \tag{1651}$$

The PDE satisfied by each extremizer of E yields the Euler-Lagrange equation. Supposing

$$\delta E(u,v) = 0, \quad \text{for all } v \in H^2(D), \tag{1652}$$

[51] We interpret this to mean "Prove this PDE admits at most one solution" rather than also proving existence.

we obtain, from integrating by parts the expression in (1651),

$$0 = -\int_D \Delta uv \, \mathrm{d}x + \int_{\partial D} \left(\frac{\partial u}{\partial n} + au - f \right) v \, \mathrm{d}\sigma, \tag{1653}$$

from which it follows, by the arbitrariness of v, that u satisfies (1648).

b) Suppose u and q are two solutions of the given PDE, and set $w := u - q$. It suffices to show w is identically zero. Observe that

$$\begin{cases} \Delta w = 0 & \text{in } D, \\ \dfrac{\partial w}{\partial n} + aw = 0 & \text{on } \partial D, \end{cases} \tag{1654}$$

and so

$$0 = \int_D w \Delta w \, \mathrm{d}x = -\int_D |Dw|^2 \, \mathrm{d}x + \int_{\partial D} w \frac{\partial w}{\partial n} \, \mathrm{d}\sigma = -\underbrace{\left[\int_D |Dw|^2 \, \mathrm{d}x + \int_{\partial D} aw^2 \, \mathrm{d}\sigma \right]}_{\geq 0} \leq 0, \tag{1655}$$

where we have used the fact $aw^2 \geq 0$. This implies

$$\int_D |Dw|^2 \, \mathrm{d}x = 0, \tag{1656}$$

and so $|Du| = 0$ in D. Whence w is constant in D. Furthermore,

$$0 = \int_{\partial D} aw^2 \, \mathrm{d}\sigma \quad \Longrightarrow \quad aw^2 = 0 \quad \text{on } \partial D. \tag{1657}$$

However, since $a > 0$, we deduce $w = 0$ on ∂D. Together with the fact w is constant in D, we conclude w is identically zero, as desired.

□

2008 Spring

S08.1 Consider the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0 \quad \text{in } (0,\ell), \\ y'(\ell) + y(\ell) = 0 \\ y(0) = 0. \end{cases} \tag{1658}$$

a) Show that if f and g satisfy the boundary conditions, then $[f'g - g'f]_{x=0}^{\ell} = 0$.

b) Prove all eigenfunctions u_1 and u_2 are orthogonal in the L^2 sense.

c) Find an equation satisfied by the eigenvalues, and find the corresponding eigenfunfctions. Show graphically that there are an infinite number of positive eigenvalues with $\lim_{n\to\infty} \lambda_n = +\infty$.

Solution:

a) Suppose f and g satisfy the given boundary conditions. Then

$$\begin{aligned} [f'g - g'f]_{x=0}^{\ell} &= f'(\ell)g(\ell) - g'(\ell)f(\ell) + f'(0)g(0) - g'(0)f(0) \\ &= f'(\ell)\left[g(\ell) + g'(\ell)\right] - g'(\ell)\left[f(\ell) + f'(\ell)\right] \\ &= f'(\ell)0 - g'(\ell)0 \\ &= 0. \end{aligned} \tag{1659}$$

b) Let $\langle \cdot, \cdot \rangle$ denote the $L^2(0,\ell)$ scalar product, i.e.,

$$\langle f, g\rangle := \int_0^{\ell} f(x)g(x)\ \mathrm{d}x. \tag{1660}$$

Let u_1 and u_2 be eigenfunctions with distinct eigenvalues λ_1 and λ_2, respectively. Then observe

$$\begin{aligned} \lambda_1 \langle u_1, u_2 \rangle &= \langle \lambda u_1, u_2 \rangle \\ &= \langle -u_1'', u_2 \rangle \\ &= \langle u_1', u_2' \rangle + \left[-u_1' u_2 \right]_{x=0}^{\ell} \\ &= \langle u_1, -u_2'' \rangle + \left[u_1 u_2' - u_1' u_2 \right]_{x=0}^{\ell} \\ &= \langle u_1, \lambda_2 u_2 \rangle \\ &= \lambda_2 \langle u_1, u_2 \rangle . \end{aligned} \tag{1661}$$

This implies $0 = (\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle$. Since $\lambda_1 \neq \lambda_2$, we conclude $\langle u_1, u_2 \rangle = 0$, as desired.

c) We claim all the eigenvalues are positive. By way of contradiction, suppose the ODE has an eigenfunction v with eigenvalue $\lambda = 0$. Then integrating twice yields $v = c_1 x + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$. The first boundary condition implies $0 = v(0) = c_1 0 + c_2 = c_2$. Thus $c_2 = 0$. The second boundary condition implies $0 = v'(\ell) + v(\ell) = c_1 + c_1 \ell = c_1(1+\ell)$, which implies $c_1 = 0$ since $1+\ell > 0$. Thus v is identically zero, a contradiction to the fact eigenfunctions are nonzero.

Now suppose there is an eigenfunction v with eigenvalue $\lambda < 0$. Let $\mu = \sqrt{-\lambda}$ and note $\mu > 0$. Then v is of the form

$$v(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}. \tag{1662}$$

The first boundary condition implies

$$0 = v(0) = c_1 + c_2 \quad \Longrightarrow \quad v(x) = c_1 \left(e^{-\mu x} - e^{\mu x} \right). \tag{1663}$$

The second boundary condition implies

$$0 = v'(\ell) + v(\ell) = c_1 \left[-\mu e^{-\mu \ell} - \mu e^{\mu \ell} + e^{-\mu \ell} - e^{\mu \ell} \right] \quad \Longrightarrow \quad 0 = \underbrace{(1-\mu) e^{-\mu \ell} - (1+\mu) e^{\mu \ell}}_{=: f(\mu)}, \tag{1664}$$

where we define $f : \mathbb{R} \to \mathbb{R}$ to be the underbraced quantity. Note the second equality holds since

$c_1 \neq 0$ as the eigenfunction v is nontrivial. Now note $f(0) = 0$ and

$$f'(\mu) = \left[-e^{-\mu\ell} - e^{\mu\ell}\right] - \mu(1-\mu)e^{-\mu\ell} - \mu(1+\mu)e^{\mu\ell} = -\mu^2\left[e^{\mu\ell} - e^{-\mu\ell}\right] - (1+\mu)\left[e^{\mu\ell} + e^{-\mu\ell}\right] < 0, \tag{1665}$$

where, since $\mu > 0$, $e^{\mu\ell} - e^{-\mu\ell} > 0$. This implies $f(\mu) < f(0) = 0$ for all $\mu > 0$. This contradicts (1664), from which we deduce no eigenfunctions exists for the given ODE with negative eigenvalues.

Finally, suppose $\lambda > 0$. Then the general solution v of the ODE is given by

$$v = c_1 \sin(\alpha x) + c_2 \cos(\alpha x), \tag{1666}$$

where $\alpha = \sqrt{\lambda}$. The condition $v(0) = 0$ implies $c_2 = 0$. The second condition implies

$$0 = c_1\left[\sin(\alpha\ell) + \alpha\cos(\alpha\ell)\right] \quad \Longrightarrow \quad \alpha = -\tan(\alpha\ell), \tag{1667}$$

where $c_1 \neq 0$ since v is an eigenfunction. Then set $g(\alpha) = \alpha + \tan(\alpha\ell)$. Since $\lim_{\alpha\to(k\pi/2)^-} = +\infty$ and $\lim_{\alpha\to((k-1)\pi/2)^+} = -\infty$ for each $k \in \mathbb{N}$ and g is continuous, it follows from the intermediate value theorem that g has a root α_k in each interval $[(k-1)\pi/2, k\pi/2]$. Consequently, there are infinitely many positive eigenvalues.

□

S08.2 Use the method of characteristics to solve the Eikonal equation $(u_x)^2 + (u_y)^2 = 1$ with initial values $u_\Gamma = 1$ on the unit circle $\Gamma = \{(x, y) : x^2 + y^2 = 1\}$.

Solution:

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then $\Gamma = \partial\Omega$. For notational convenience, for each $(x, y) \in \Omega$ we write $w = (x, y)$. Now define $F(p, z, w) = |p|^2 - 1$. Then, using the method of characteristics and taking $p = Dw$ and $z = u$, we obtain the ODE system

$$\begin{cases} \dot{p}(s) = -F_w - F_z p = 0, \quad p(0) = p^0 \\ \dot{w}(s) = F_p = 2p, \quad w(0) = w^0 \\ \dot{z}(s) = F_p p = 2|p|^2 = 2, \quad z(0) = 1. \end{cases} \tag{1668}$$

The above system implies $z(s) = 2s + 1$, $w(s) = 2sp^0 + w^0$, and $p(s) = p^0$. Employing the use of polar coordinates (r, θ) with the fact $u = 0$ on Γ, we see

$$0 = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = u_x(-r\sin\theta) + u_y(r\cos\theta) = -u_x y + u_y x = p \cdot q \ \text{ on } \Gamma, \tag{1669}$$

where $q := (-y, x)$. Since $w \cdot q = -xy + xy = 0$ and $0 = p \cdot q$ on Γ, we have $w \cdot p = \pm|p||w|$. Then observe for all $w \in \Omega$

$$|w|^2 = |2sp^0 + w^0|^2 = 4s^2|p^0|^2 + 4sp^0 \cdot w^0 + |w^0|^2 = 4s^2|p^0|^2 \pm 4s|p^0||w^0| + |w^0|^2 = 4s^2 \pm 4s + 1, \tag{1670}$$

where the third equality holds by our above argument and $|p^0| = 1$ since $F = 0$ and $|w^0| = 1$ since initial points for the characteristics occur on the unit circle Γ. We now have two cases:

Case 1: $|w|^2 = 4s^2 + 4s + 1$**.** In this case, we see $z^2 = (2s + 1)^2 = 4s^2 + 4s + 1 = |w|^2$, which implies $u(x, y) = \sqrt{x^2 + y^2}$ and we note the positive square root must be taken in order to satisfy the boundary condition $u = 1$ on Γ.

Case 2: $|w|^2 = 4s^2 - 4s + 1$**.** In similar fashion, in this case, we see $|w|^2 = 4s^2 - 4s + 1 = (2s - 1)^2 = (z - 2)^2$, which implies $u(x, y) = 2 - \sqrt{x^2 + y^2}$.

The above two cases show two solutions to the PDE are $u(x, y) = \sqrt{x^2 + y^2}$ and $u(x, y) = 2 - \sqrt{x^2 + y^2}$.

□

S08.4. Solve the following IVP

$$\begin{cases} u_{tt} + u_{xt} + -20u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = \phi & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t = \psi & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{1671}$$

Solution:

The given PDE may be "factored" as

$$(\partial_t + 5\partial_x)(\partial_t - 4\partial_x)\, u = 0 \ \text{ in } \mathbb{R} \times (0,\infty). \tag{1672}$$

Set $v(x,t) := (u_t - 4u_x)(x,t)$. Then

$$v_t + 5u_x = 0 \ \text{ in } \mathbb{R} \times (0,\infty). \tag{1673}$$

Now let $F(p,q,z,x,t) = q + 5p$. Using the method of characteristics, taking $q = u_t$, $p = u_x$, and $z = u$, we then have $F = 0$ and obtain the ODE system

$$\begin{cases} \dot{x}(s) = F_p = 5, \quad x(0) = x^0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = 5p + q = 0, \quad z(0) = v(x^0, 0). \end{cases} \tag{1674}$$

This implies $s = t$, z is constant along characteristics, and $x = x^0 + 5t$. Thus

$$v(x,t) = z(s) = z(x^0) = v(x^0, 0) = v(x - 5t, 0) \ \text{ in } \mathbb{R} \times (0,\infty). \tag{1675}$$

Setting $f(x,t) := v(x - 5t, 0)$ and using our initial conditions, we see

$$f(x,t) = u_t(x - 5t, 0) - 4u_x(x - 5t, 0) = \psi(x - 5t) - 4\phi(x - 5t). \tag{1676}$$

Consequently, we find the solution u satisfies

$$\begin{cases} u_t - 4u_x = f(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1677}$$

By linearity of the PDE, we may separate u and write $u = w_1 + w_2$ where

$$\begin{cases} (w_1)_t - 4(w_1)_x = 0 & \text{in } \mathbb{R}\times\{t=0\}, \\ w_1 = \phi & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1678}$$

and

$$\begin{cases} (w_2)_t - 4(w_2)_x = f & \text{in } \mathbb{R}\times\{t=0\}, \\ w_2 = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1679}$$

Following the argument in the method of characteristics used for v, we deduce $w_1(x,t) = \phi(x+4t)$. Now, using Duhamel's principle and this knowledge of the transport equation, we know

$$\begin{aligned} w_2(x,t) &= \int_0^t f(x+4(t-s),s)\ \mathrm{d}s \\ &= \int_0^t \psi(x+4(t-s)-5s) - 4\phi'(x+4(t-s)-5s)\ \mathrm{d}s \\ &= -\frac{1}{9}\int_{x+4t}^{x-5t} \psi(\xi) - 4\phi'(\xi)\ \mathrm{d}\xi \\ &= \frac{4}{9}\left[\phi(x-5t)-\phi(x+4t)\right] + \frac{1}{9}\int_{x-5t}^{x+4t}\psi(\xi)\ \mathrm{d}\xi. \end{aligned} \tag{1680}$$

Compiling our results, we conclude

$$\begin{aligned} u(x,t) &= w_1(x,t) + w_2(x,t) \\ &= \phi(x+4t) + \frac{4}{9}\left[\phi(x-5t)-\phi(x+4t)\right] + \frac{1}{9}\int_{x-5t}^{x+4t}\psi(\xi)\ \mathrm{d}\xi \\ &= \boxed{\frac{4}{9}\phi(x-5t) + \frac{5}{9}\phi(x+4t) + \frac{1}{9}\int_{x-5t}^{x+4t}\psi(\xi)\ \mathrm{d}\xi.} \end{aligned} \tag{1681}$$

□

S08.6. Consider the differential equation

$$u_t = -\varepsilon\Delta u + \Delta^3 u, \tag{1682}$$

on the interval $[0, 2\pi]$ with periodic boundary conditions. Find the largest value of ε_0 so that the solution of the PDE stays bounded as $t \longrightarrow \infty$, if $\varepsilon < \varepsilon_0$. Justify your answer.

Solution:

Since u satisfies the PDE on $[0, 2\pi]$ and is periodic, it may be expressed via

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)e^{ikx}, \tag{1683}$$

where each term $\hat{u}(k,t)$ is a Fourier coefficient. Transforming our PDE reveals, for each $k \in \mathbb{Z}$,

$$\hat{u}_t(k,t) = -\varepsilon(ik)^2\hat{u}(k,t) + (ik)^6\hat{u}(k,t) = \left[\varepsilon k^2 - k^6\right]\hat{u}(k,t), \tag{1684}$$

where we have used basic properties of the Fourier transform to replace the Laplacian terms. Integrating with respect to t yields

$$\hat{u}(k,t) = \hat{u}(k,0)\exp\left(k^2\left[\varepsilon - k^4\right]t\right), \quad \text{for all } k \in \mathbb{Z}. \tag{1685}$$

Therefore,

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,0)\exp\left(k^2[\varepsilon - k^4]t\right)\cdot e^{ikx}. \tag{1686}$$

The only term in the series that is not constant with respect to time is the middle term. Thus, in order for u to remain finite as $t \longrightarrow \infty$, we must have

$$k^2\left[\varepsilon - k^4\right] \le 0, \quad \text{for all } k \in \mathbb{Z}, \tag{1687}$$

which is true precisely when

$$\varepsilon < k^4, \quad \text{for all } k \in \mathbb{Z} - \{0\}. \tag{1688}$$

The infimum of the right hand side of the inequality is obtained at $k = 1$, which gives $k^4 = 1$. Thus, we see the solution remains bounded as $t \longrightarrow \infty$, if $\varepsilon < \varepsilon_0$, where $\boxed{\varepsilon_0 := 1.}$ □

2008 Fall

F08.2 Let $g : [0,\infty) \to \mathbb{R}$ be a continuous function with $g(0) = 0$. Derive an integral formula for the solution of the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}^+ \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ u = g & \text{on } \{x = 0\} \times (0,\infty), \end{cases} \tag{1689}$$

in terms of g.

Solution:

We first define an extension of u with domain $\mathbb{R} \times (0,\infty)$. Then we use Duhamel's principle to obtain the integral formula for this extension, which reduces to u in $[0,\infty) \times [0,\infty)$. Let us momentarily assume $g \in C^1$. We will use integration by parts to obtain a final expression that only requires continuity of g. Set $v(x,t) := u(x,t) - g(t)$ and

$$\tilde{v}(x,t) := \begin{cases} v(x,t) & \text{in } [0,\infty) \times [0,\infty), \\ -v(-x,t) & \text{in } (-\infty,0) \times [0,\infty). \end{cases} \tag{1690}$$

Since $v = 0$ on $\{x = 0\} \times (0,\infty)$, this odd extension ensures $\tilde{v} = 0$ in $\{x = 0\} \times (0,\infty)$. Then observe

$$\tilde{v}_{xx}(x,t) = \begin{cases} v_{xx}(x,t) = u_{xx}(x,t) & \text{in } [0,\infty) \times [0,\infty), \\ -v_{xx}(-x,t) = -u_{xx}(-x,t) & \text{in } (-\infty,0) \times [0,\infty), \end{cases} \tag{1691}$$

and

$$\tilde{v}_t(x,t) = \begin{cases} v_t(x,t) = u_t(x,t) - g'(t) & \text{in } [0,\infty) \times [0,\infty), \\ -v_t(-x,t) = -u_t(-x,t) + g'(t) & \text{in } (-\infty,0) \times [0,\infty), \end{cases} \tag{1692}$$

which implies $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = f(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ \tilde{v} = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \\ \tilde{v} = 0 & \text{on } \{x = 0\} \times (0,\infty), \end{cases} \tag{1693}$$

where

$$f(x,t) := \begin{cases} g'(t) & \text{in } [0,\infty)\times[0,\infty), \\ -g'(t) & \text{in } (-\infty,0)\times[0,\infty). \end{cases} \tag{1694}$$

Duhamel's principle asserts we can build a solution by writing

$$u(x,t) = \int_0^t \int_{\mathbb{R}} \Phi(x-\xi, t-s) f(\xi, s) \ \mathrm{d}\xi \mathrm{d}s, \tag{1695}$$

where Φ is the fundamental solution of the heat equation given by

$$\Phi(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right). \tag{1696}$$

Using the definition of Φ and f, we write

$$\begin{aligned} \tilde{v}(x,t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left[\int_0^\infty g'(s)\exp\left(-\frac{|x-\xi|^2}{4(t-s)}\right) \ \mathrm{d}\xi - \int_{-\infty}^0 g'(s)\exp\left(-\frac{|x-\xi|^2}{4(t-s)}\right) \ \mathrm{d}\xi\right] \mathrm{d}s \\ &= \int_0^t \frac{g'(s)}{\sqrt{4\pi(t-s)}} \int_0^\infty \exp\left(-\frac{|x-\xi|^2}{4(t-s)}\right) - \exp\left(-\frac{|x+\xi|^2}{4(t-s)}\right) \ \mathrm{d}\xi \mathrm{d}s. \end{aligned} \tag{1697}$$

From above, we know $\tilde{v} = u - g$ in $[0,\infty)\times[0,\infty)$. Hence

$$u(x,t) = g(t) + \int_0^t \frac{g'(s)}{\sqrt{4\pi(t-s)}} \int_0^\infty \exp\left(-\frac{|x-\xi|^2}{4(t-s)}\right) - \exp\left(-\frac{|x+\xi|^2}{4(t-s)}\right) \ \mathrm{d}\xi \mathrm{d}s. \tag{1698}$$

For $s \in [0,t]$, letting

$$q(s) := \begin{cases} \dfrac{1}{\sqrt{4\pi(t-s)}} \displaystyle\int_0^\infty \exp\left(-\frac{|x-\xi|^2}{4(t-s)}\right) - \exp\left(-\frac{|x+\xi|^2}{4(t-s)}\right) \ \mathrm{d}\xi & \text{if } s\in[0,t), \\ 0 & \text{if } s = t, \end{cases} \tag{1699}$$

reveals

$$\begin{aligned} u(x,t) &= g(t) + \int_0^t g'(s)q(s) \ \mathrm{d}s \\ &= g(t) - \int_0^t g(s)q'(s) \ \mathrm{d}s + [gq]_{s=0}^t \\ &= g(t) - \int_0^t g(s)q'(s) \ \mathrm{d}s + g(t)q(t) - g(0)q(0) \\ &= g(t) - \int_0^t g(s)q'(s) \ \mathrm{d}s, \end{aligned} \tag{1700}$$

where the final line holds since $q(t) = 0$ and $g(0) = 0$. Note also the final line gives an expression solely in terms of g, not including any of its derivatives. $\square$

F08.3. Consider the initial value problem for Burgers' equation

$$\begin{cases} u_t + u_x u = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = g & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1701}$$

Find the entropy solution of this problem with the initial data

$$g(x) = \begin{cases} 0 & \text{if } x > 1, \\ 1-x & \text{if } x\in(0,1), \\ 1 & \text{if } x < 0. \end{cases} \tag{1702}$$

Also find the maximum time interval $[0, t^*0$ on which the solution is continuous.

Solution:

We proceed by using the method of characteristics. Let $F(p,q,z,x,t) := q + pz$. Then taking $q = u_t$, $p = u_x$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z & x(0) = x^0, \\ \dot{t}(s) = F_q = 1 & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0 & z(0) = g(x^0). \end{cases} \tag{1703}$$

This implies $t = s$, z is constant along characteristics, and

$$x(t) = tz(0) + x^0 = tg(x^0) + x^0 = \begin{cases} x^0 & \text{if } x^0 > 1, \\ t(1-x^0) + x^0 & \text{if } x^0 \in (0,1), \\ 1 + x^0 & \text{if } x^0 < 0. \end{cases} \tag{1704}$$

Written more concisely, we have

$$x = \begin{cases} x^0 & \text{if } x^0 > 1, \\ (1-t)x^0 + t & \text{if } x^0 \in (0,1), \\ 1 + x^0 & \text{if } x^0 < 0. \end{cases} \tag{1705}$$

Thus the initial characteristics are vertical in the (x,t) plane for $x^0 > 1$ and have slope 1 for $x^0 \leq 0$. For time $t \in (0,1)$ and $x^0 \in (0,1)$, we see each characteristic is a line segment originating at $(x^0, 0)$ and intersecting $(1,1)$. Thus, the characteristics crash at time $t^* = 1$, and so the maximum time interval $[0, t^*)$ on which the solution is continuous occurs when $\boxed{t^* = 1.}$

We now identify u for $t \in [0,1)$. If $x > 1$, then the fact the characteristics are vertical there and $g = 0$ for $x > 1$ implies $u = 0$. Similarly, if $x - t < 0$, then the origin of the characteristic was at a point $x^0 = x - t < 0$, for which $g(x^0) = 1$, and so $u = 1$ in such a case. In the final case, where $t \leq x \leq 1$, we see

$$x = (1-t)x^0 + t \quad \Longrightarrow \quad x^0 = \frac{x-t}{1-t} \quad \Longrightarrow \quad g(x^0) = 1 - x^0 = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}. \tag{1706}$$

Compiling these results reveals, for $(x,t) \in \mathbb{R} \times [0,1)$,

$$u(x,t) = \begin{cases} 0 & \text{if } x > 1, \\ 1 & \text{if } x < t, \\ \dfrac{1-x}{1-t} & \text{if } t \leq x < 1. \end{cases} \tag{1707}$$

To obtain the entropy solution for $t \geq 1$, we apply the Rankine-Hugenoit condition to get that the shock curve parameterized as $(s(t), t)$ satisfies $(s(1), 1) = (1,1)$ and

$$\dot{s}(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2} \cdot 1^2 - \frac{1}{2} \cdot 0^2}{1 - 0} = \frac{1}{2}, \tag{1708}$$

where $f(u) := u^2/2$ and $u_\ell = 1$ and $u_r = 0$ are the limiting values approaching the shock from the left and the right, respectively. Thus

$$s(t) = \frac{1+t}{2}, \tag{1709}$$

and, for $(x,t) \in \mathbb{R} \times (1, \infty)$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < (1+t)/2, \\ 0 & \text{if } x > (1+t)/2. \end{cases} \tag{1710}$$

Together (1707) and (1710) identify $u(x,t)$, as desired. $\square$

F08.6. Consider the first order system of equations

$$u_t + \sum_{j=1}^{n} A_j u_{x_j} = 0, \tag{1711}$$

where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$, and the A_j's are symmetric $m \times m$ matrices with constant real entries. Use an energy argument to show that the domain of dependence of (x_0, t_0), $t_0 > 0$, for a solution of the system above is contained in the cone

$$\{|x - x_0| \leq \Lambda(t_0 - t)\}, \tag{1712}$$

where

$$\Lambda := \max_{\|\xi\|=1, 1\leq \ell \leq m} |\lambda_\ell(\xi)\|, \tag{1713}$$

and, for $\ell = 1, \ldots, m$, $\lambda_\ell(\xi)$ is the ℓ-th eigenvalue of the matrix $A(\xi) := \sum_{j=1}^{n} \xi_j A_j$.

Solution:

Fix $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and define $S(t)$ via

$$S(t) := B(x_0, \Lambda(t_0 - t)), \quad \text{for all } t \in [0, t_0), \tag{1714}$$

and $S(t_0) := \{x_0\}$. Then observe the cone in (1712) may be expressed as

$$K(x_0, t_0) := \{(x,t) : x \in S(t),\ t \in [0, t_0]\} = \{|x - x_0| \leq \Lambda(t_0 - t)\}. \tag{1715}$$

In order to verify the domain of dependence assertion, it suffices to show that if $u(\cdot, 0) = 0$ in $S(0) \subseteq K(x_0, t_0)$, then $u = 0$ in $K(x_0, t_0)$. Suppose $u = 0$ in $S(0)$ and define the energy $E : [0, t_0) \to \mathbb{R}$ by

$$E(t) := \frac{1}{2} \int_{S(t)} |u(x,t)|^2 \ \mathrm{d}x, \tag{1716}$$

and note $E(0) = 0$ by our assumption. Differentiating in time reveals

$$\dot{E}(t) = \int_{S(t)} u \cdot u_t \ \mathrm{d}x + \frac{1}{2} \int_{\partial S(t)} |u|^2 v \cdot \nu \ \mathrm{d}\sigma, \tag{1717}$$

where $v = -\Lambda \nu$ is the Eulerian velocity of the boundary and ν is the outward normal along $\partial S(t)$. This implies

$$\dot{E}(t) = -\int_{S(t)} u \cdot \sum_{j=1}^{n} A_j u_{x_j} \ \mathrm{d}x - \frac{\Lambda}{2}\int_{\partial S(t)} |u|^2 \ \mathrm{d}\sigma. \tag{1718}$$

Integrating by parts for a fixed $j \in [n]$ yields

$$\begin{aligned} -\int_{S(t)} u \cdot A_j u_{x_j} \ \mathrm{d}x &= \int_{S(t)} u_{x_j} \cdot A_j u \ \mathrm{d}x + \int_{\partial S(t)} (u \cdot Au)\nu_j \ \mathrm{d}\sigma \\ &= \int_{S(t)} A_j u_{x_j} \cdot u \ \mathrm{d}x + \int_{\partial S(t)} (u \cdot Au)\nu_j \ \mathrm{d}\sigma, \end{aligned} \tag{1719}$$

where the final line holds since A_j is symmetric. Using the symmetry of the dot product reveals

$$-\int_{S(t)} u \cdot A_j u_{x_j} \ \mathrm{d}x = \frac{1}{2}\int_{\partial S(t)} (u \cdot Au)\nu_j \ \mathrm{d}\sigma. \tag{1720}$$

Combining (1718) and (1720) yields

$$\dot{E}(t) = \frac{1}{2}\int_{\partial S(t)} u \cdot \sum_{j=1}^{n} \nu^j A_j u - \Lambda |u|^2 \ \mathrm{d}\sigma = \frac{1}{2}\int_{\partial S(t)} u \cdot A(\nu)u - \Lambda|u|^2 \ \mathrm{d}\sigma \leq \frac{1}{2}\int_{\partial S(t)} \Lambda|u|^2 - \Lambda|u|^2 \ \mathrm{d}\sigma = 0, \tag{1721}$$

where the inequality holds by our choice of Λ. Since the integrand in the definition of $E(t)$ is nonnegative, it follows that $E(t) \geq 0$ for all $t \in [0, t_0)$. Combined with the facts that E is nonincreasing and $E(0) = 0$, we deduce $E(0) = 0$ for all $t \in [0, t_0)$.

The above result shows $u(\cdot, t) = 0$ in $S(t)$ for all $t \in [0, t_0)$ (since otherwise E would be positive at some $t \in [0, t_0)$). In particular, $u(x_0, t) = 0$ for all $t \in [0, t_0)$. Together with the continuity of u, this implies

$$u(x_0, t_0) = \lim_{t \to t_0^-} u(x_0, t) = \lim_{t \to t_0^-} 0 = 0. \tag{1722}$$

Whence $u = 0$ in $K(x_0, t_0)$, as desired. This completes the proof. □

F08.7. Suppose u is a smooth solution of the following problem

$$\begin{cases} u_{xxt} + u_{xx} - u^3 = 0 & \text{in } [0,1]\times(0,\infty), \\ u = 0 & \text{on } \partial[0,1]\times(0,\infty), \\ u = x(x-1) & \text{on } [0,1]\times\{t=0\}. \end{cases} \tag{1723}$$

Derive a differential inequality for $w(t) := \int_0^1 (u_x)^2(x,t)\ \mathrm{d}x$ and show u uniformly tends to zero as $t \longrightarrow \infty$.

Solution:

First observe, using the given PDE and integration by parts,

$$\dot{w}(t) = 2\int_0^1 u_x u_{xt}\ \mathrm{d}x = -2\int_0^1 u u_{xxt}\ \mathrm{d}x = -2\int_0^1 u(u^3 - u_{xx})\ \mathrm{d}x = -2\int_0^1 u^4 - u u_{xx}\ \mathrm{d}x, \tag{1724}$$

where the boundary terms cancel since $u = 0$ on $\partial[0,1]\times(0,\infty)$. Since $-u^4 \leq 0$, it follows that

$$\dot{w}(t) \leq -2\int_0^1 u_x^2\ \mathrm{d}x = -2w(t). \tag{1725}$$

Then, using Grownwall's inequality, we deduce

$$w(t) \leq w(0)\exp\left(\int_0^t -2\ \mathrm{d}\tilde{t}\right) = w(0)\exp(-2t). \tag{1726}$$

Given our initial data, we see

$$w(0) = \int_0^1 [\partial_x(x(x-1))]^2\ \mathrm{d}x = \int_0^1 [2x-1]^2\ \mathrm{d}x = \int_0^1 4x^2 - 4x + 1\ \mathrm{d}x = \frac{4}{3} - 2 + 1 = \frac{1}{3}. \tag{1727}$$

This implies we have the differential inequality

$$w(t) \leq \frac{1}{3}\cdot\exp(-2t). \tag{1728}$$

With this inequality, we see for $(x,t) \in (0,1) \times (0,\infty)$

$$|u(x,t)| = \left| \int_0^x u_x(\xi,t)\ \mathrm{d}\xi \right| \leq \int_0^1 |u_x(\xi,t)|\ \mathrm{d}\xi \leq \left(\int_0^1 1^2\ \mathrm{d}\xi \right)^{1/2} w(t)^{1/2} = \frac{1}{\sqrt{3}} \exp(-t). \tag{1729}$$

The first inequality holds by the triangle inequality and the fact $x \leq 1$. The second inequality is a special case of Hölder's inequality. Because we obtained the inequality in (1729) and $u = 0$ on $\partial[0,1] \times (0,\infty)$, it follows that

$$\|u(\cdot,t)\|_{L^\infty([0,1])} \leq \frac{1}{\sqrt{3}} \exp(-t). \tag{1730}$$

Whence

$$0 \leq \lim_{t\to\infty} \|u(\cdot,t) - 0\|_{L^\infty([0,1])} \leq \lim_{t\to\infty} \frac{1}{\sqrt{3}} \exp(-t) = 0, \tag{1731}$$

and so by the squeeze lemma we conclude u tends to zero uniformly as $t \longrightarrow \infty$, as desired. □

F08.8. Suppose that $q(x)$ is a real-valued continuous function such that $\int_0^1 q(x)\ \mathrm{d}x = 0$, but $q(x)$ is not identically zero. Show that $Lu = -u'' + qu$ with the boundary conditions $u'(0) = u'(1) = 0$ must have a strictly negative eigenvalue by showing that $\int_0^1 uLu\ \mathrm{d}x$ can be negative.

Solution:

Taking $p = 1$, we see the eigenvalue may be expressed via

$$\begin{cases} Lu = -(pu')' + qu = \lambda u \quad \text{in } (0,1), \\ 0 \cdot u(0) + 1 \cdot u'(0) = 0, \\ 0 \cdot u(1) + 1 \cdot u'(1) = 0, \end{cases} \tag{1732}$$

which is in regular Sturm-Liouville form. Whence, by Sturm-Liouville theory, the smallest eigenvalue λ for this problem satisfies

$$\lambda = \min_{u \in \mathcal{A}} \frac{\langle u, Lu \rangle}{\langle u, u \rangle}, \tag{1733}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 scalar product on $(0,1)$ and $\mathcal{A} := \{u \in C^2(0,1) \cap C^1[0,1] : u'(0) = u'(1) = 0\}$. Let $v \in \mathcal{A}$ be any function satisfying $\int_0^1 vq\ \mathrm{d}x \neq 0$, which is possible through our hypothesis concerning q. Then choose $\alpha \in \mathbb{R}$ such that

$$\langle v, Lv \rangle + 2\alpha \int_0^1 qv\ \mathrm{d}x < 0, \tag{1734}$$

which can be done since α is multiplied by a nonzero quantity. Observe

$$\langle v + \alpha, L(v + \alpha) \rangle = \langle v, Lv \rangle + \langle v, L\alpha \rangle + \langle \alpha, Lv \rangle + \langle \alpha, L\alpha \rangle, \tag{1735}$$

and

$$\begin{aligned} \langle \alpha, L\alpha \rangle &= \int_0^1 \alpha(0 + q\alpha)\ \mathrm{d}x = \alpha^2 \int_0^1 q\ \mathrm{d}x = 0, \\ \langle v, L\alpha \rangle &= \int_0^1 v(0 + q\alpha)\ \mathrm{d}x = \alpha \int_0^1 qv\ \mathrm{d}x, \\ \langle \alpha, Lv \rangle &= \alpha \int_0^1 -v'' + qv\ \mathrm{d}x = \alpha \int_0^1 qv\ \mathrm{d}x + [1 \cdot (-v')]_0^1 = \alpha \int_0^1 qv\ \mathrm{d}x, \end{aligned} \tag{1736}$$

where the final line follows from integration by parts with the boundary conditions. This implies

$$\langle v + \alpha, L(v + \alpha) \rangle = \langle v, Lv \rangle + 2\alpha \int_0^1 qv\ \mathrm{d}x < 0, \tag{1737}$$

where the final inequality holds by (1734). By our choice of v, we know v is not constant, which implies $v+\alpha$ is not identically zero, and so

$$\langle v+\alpha, v+\alpha\rangle > 0 \quad \Longrightarrow \quad \lambda = \min_{u\in\mathcal{A}} \frac{\langle u, Lu\rangle}{\langle u,u\rangle} \leq \frac{\langle v+\alpha, L(v+\alpha)\rangle}{\langle v+\alpha, v+\alpha\rangle} < 0. \tag{1738}$$

Whence the eigenvalue problem admits an eigenfunction solution with negative eigenvalue. □

REMARK: The solution above does not provide any insight into why we might know to take this approach. This is explained as follows. First, (1733) is straightforward to obtain. Since λ equals the minimum value of the shown fraction over all choices of u, it suffices to find a particular choice of u for which the given fraction is negative. The simplest function that satisfies the boundary conditions is a constant function. This is why we need to choose α. Then, in order to utilize the fact q is not identically zero, we also must introduce a second term $v \in \mathcal{A}$ to get that $\int_0^1 qv \ \mathrm{d}x \neq 0$. We then want $\langle v+\alpha, L(v+\alpha)\rangle < 0$. Expanding this out reveals the condition we seek for α. △

2007 Fall

F07.1 Let ϕ be continuous and bounded in $\mathbb{R}^n$. Assume $\lim_{|x|\to\infty} \phi(x) = \phi_0$. Consider the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times [0,\infty), \\ u = \phi & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1739}$$

Prove that $\lim_{t\to\infty} u(x,t) = \phi_0$.

Solution:

Since u is a solution to the heat equation, its integral representation is given by the convolution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \phi(\xi) \exp\left(-\frac{|x-\xi|^2}{4t}\right)\ \mathrm{d}\xi = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \phi\left(z\sqrt{4t} + x\right) \exp\left(-|z|^2\right)\ \mathrm{d}z, \tag{1740}$$

where the final equality holds by using the change of variables to $z = (\xi - x)/\sqrt{4t}$ so that $\mathrm{d}z = \mathrm{d}\xi/(4t)^{n/2}$. Since ϕ is bounded by some $M > 0$,

$$\int_{\mathbb{R}^n} \phi(x + \sqrt{4t}z) \exp(-|z|^2)\ \mathrm{d}z \leq M \int_{\mathbb{R}^n} \exp(-|z|^2)\ \mathrm{d}z = M\pi^{n/2}. \tag{1741}$$

This shows the integrand is dominated by an integrable function, and so the dominated convergence theorem may be applied. Indeed, for each fixed $x \in \mathbb{R}^n$, it follows that

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty} \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \phi\left(x + \sqrt{4t}z\right) \exp\left(-|z|^2\right)\ \mathrm{d}z \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \lim_{t\to\infty} \phi\left(x + \sqrt{4t}z\right) \exp\left(-|z|^2\right)\ \mathrm{d}z \\ &= \frac{\phi_0}{\pi^{n/2}} \int_{\mathbb{R}^n} \exp\left(-|z|^2\right)\ \mathrm{d}z \\ &= \frac{\phi_0}{\pi^{n/2}} \cdot \pi^{n/2} \\ &= \phi_0, \end{aligned} \tag{1742}$$

as desired. The dominated convergence theorem may be applied to obtain the second line, bringing the limit inside the integrand. The third line holds since as $t \to \infty$ the norm of the argument of ϕ in the second line also goes to infinity. This completes the proof. $\square$

F07.2. Let $A_i(x)$ for $i = 1, 2$ be smooth functions in a bounded domain $\Omega \subset \mathbb{R}^n$ such that $A_1 = A_2$ on $\partial\Omega$. Assume that

$$\Delta A_1 + \sum_{j=1}^{n} \left(\frac{\partial A_1}{\partial x_j}\right)^2 = \Delta A_2 + \sum_{j=1}^{n} \left(\frac{\partial A_2}{\partial x_j}\right)^2 \tag{1743}$$

Solution:

Set $w := A_1 - A_2$ so that

$$\begin{cases} \Delta w = \|DA_2\|^2 - \|DA_1\|^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1744}$$

It suffices to show $w = 0$ in Ω. Since Ω is bounded and $(A_1 + A_2)_{x_1}$ is smooth on $\overline{\Omega}$, it follows that this expression takes on its supremum. Thus, we may set

$$\lambda := 2 \cdot \left| \max_{x \in \overline{\Omega}} (A_1 + A_2)_{x_1} \right|. \tag{1745}$$

Now let $\varepsilon > 0$ and set $v(x) := w(x) + \varepsilon \exp(\lambda x_1)$. Observe that

$$\Delta w = (DA_2 - DA_1) \cdot (DA_2 + DA_1) = -Dw \cdot (DA_2 + DA_1), \tag{1746}$$

and so

$$\Delta v(x) = \Delta w(x) + \lambda^2 \varepsilon \exp(\lambda x_1) = -\left(Dv(x) - \lambda \exp(\lambda x_1)\hat{e}_1\right) \cdot (DA_2(x) + DA_1(x)) + \lambda^2 \varepsilon \exp(\lambda x_1), \tag{1747}$$

where $\hat{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Since v is smooth over $\overline{\Omega}$, it attains its supremum. By way of contradiction, suppose v attains its max at a point $z \in \Omega$ in the interior. This implies, at z, $Dv = 0$ and

$$\begin{aligned} 0 \geq \Delta v &= -\left(0 - \varepsilon\lambda \exp(\lambda z_1)\hat{e}_1\right) \cdot (DA_2 + DA_1) + \lambda^2 \varepsilon \exp(\lambda z_1) \\ &= \varepsilon\lambda \exp\left(\lambda z_1\right) \left[(A_1 + A_2)_{x_1} + \lambda\right] \\ &> 0, \end{aligned} \tag{1748}$$

where we note $\varepsilon\lambda \exp(\lambda z_1) > 0$ and $\lambda > (A_1 + A_2)_{x_1}$, by (1745). This implies $0 > 0$, a contradiction. Therefore, v attains its maximum along the boundary. Because Ω is bounded, there exists $M > 0$ such that $\exp(\lambda x_1) < M$ for all $x \in \overline{\Omega}$. Whence

$$0 = \max_{x\in\partial\Omega} w(x) \leq \max_{x\in\overline{\Omega}} w(x) \leq \max_{x\in\overline{\Omega}} w(x) + \varepsilon\exp(\lambda x_1) = \max_{x\in\overline{\Omega}} v(x) = \max_{x\in\partial\Omega} v(x) \leq \max_{x\in\partial\Omega} \varepsilon M = \varepsilon M. \quad (1749)$$

This implies

$$0 \leq \max_{x\in\overline{\Omega}} w(x) \leq \varepsilon M. \quad (1750)$$

Letting $\varepsilon \to 0^+$ reveals $\max\limits_{\overline{\Omega}} w = 0$, and so $w \leq 0$ in Ω. Note the above argument can be repeated with $\tilde{w} := A_2 - A_1$ to deduce that $-w = \tilde{w} \leq 0$ in Ω. Therefore, $w = 0$ in Ω and the proof is complete. $\square$

F07.5. Consider the initial value problem

$$\begin{cases} u'(t) = cu^{1+\alpha}, \\ \quad u(0) = 0. \end{cases} \tag{1751}$$

where $c > 0$, $\alpha > 0$, and $u_0 \in (0,1)$.

a) Find the solution of this ODE.

b) Find the blowup time t_* at which $u \to +\infty$.

c) Find the value of α that minimizes t_* for fixed values of c and u_0.

Solution:

a) Observe this DOE is separable, and so

$$\int_{u_0}^{u} \frac{\mathrm{d}\tilde{u}}{\tilde{u}^{1+\alpha}} = \int_0^t c \ \mathrm{d}\tilde{t} \quad \Longrightarrow \quad -\alpha^{-1}\left[u^{-\alpha} - u_0^{-\alpha}\right] = ct \quad \Longrightarrow \quad \boxed{u = \left[u_0^{-\alpha} - \alpha ct\right]^{-1/\alpha}.} \tag{1752}$$

b) We claim $t_* = (c\alpha u^{\alpha})^{-1}$. Indeed, for this choice of t_*,

$$\lim_{t\to t_*^-} u(t) = \lim_{t\to t_*^-} \left[u_0^{-\alpha} - \alpha ct\right]^{-1/\alpha} = (\alpha c)^{-1/\alpha} \lim_{t\to t_*^-} \frac{1}{(t_* - t)^{1/\alpha}} = +\infty. \tag{1753}$$

c) We seek $\alpha^\star$ that minimizes the expression $(c\alpha u^{\alpha})^{-1}$. Since the logarithm function is strictly increasing, this is equivalent to finding the $\alpha^\star$ that minimizes $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(\alpha) := \ln\left(c\alpha^{-1}u_0^{-\alpha}\right) = \ln(c) + \ln\left(\alpha^{-1}\right) + \ln\left(u_0^{-\alpha}\right) = \ln(c) - \ln(\alpha) - \alpha\ln(u_0). \tag{1754}$$

Through differentiation, we see $\alpha^\star$ satisfies

$$0 = f'(\alpha^\star) = 0 - \frac{1}{\alpha^\star} - \ln(u_0) \quad \Longrightarrow \quad \boxed{\alpha^\star = -\frac{1}{\ln(u_0)}.} \tag{1755}$$

Since $u_0 \in (0,1)$, we know $\ln(u_0) < 0$, and so $\alpha^\star > 0$, as desired. Lastly, since

$$f''(\alpha^\star) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[-\alpha^{-1} - \ln(u_0)\right]_{\alpha=\alpha^\star} = \frac{1}{(\alpha^\star)^2} > 0, \tag{1756}$$

it follows from the second derivative test that $\alpha^\star$ is, in fact, a local minimizer.

□

2007 Spring

S07.1. Consider a minimizer u of the energy

$$E(u) := \frac{1}{2}\int_\Omega (f-u)^2 + \lambda(\Delta u)^2 \text{ d}x, \tag{1757}$$

where both u and f are periodic on the 2-torus Ω.

a) Show the Euler-Lagrange equation for u is $-(f-u)+\lambda\Delta^2 u = 0$.

b) Compute a solution of this problem in terms of a Fourier series expansion.

c) Discuss how the high frequency modes depend on the value of λ, which imparts smoothing to u.

Solution:

a) For each test function v with $v=0$ on $\partial\Omega$ we see

$$\begin{aligned}
\delta E(u,v) &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[E(u+\varepsilon v) - E(u)\right] \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\frac{1}{2}\int_\Omega (f-u-\varepsilon v)^2 + \lambda(\Delta u + \varepsilon\Delta v)^2 \text{ d}x - \frac{1}{2}\int_\Omega (f-u)^2 + \lambda(\Delta u)^2 \text{ d}x\right] \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\frac{1}{2}\int_\Omega -2\varepsilon v(f-u) + \varepsilon^2 v^2 + 2\lambda\varepsilon\Delta u\Delta v + \lambda\varepsilon^2(\Delta v)^2 \text{ d}x\right] \\
&= \lim_{\varepsilon\to 0^+} \left[\int_\Omega -(f-u)v + \lambda\Delta u\Delta v \text{ d}x + \varepsilon\int_\Omega v^2 + \lambda(\Delta v)^2 \text{ d}x\right] \\
&= \int_\Omega -(f-u)v + \lambda\Delta u\Delta v \text{ d}x \\
&= \int_\Omega \left(-(f-u) + \lambda\Delta^2 u\right) v \text{ d}x,
\end{aligned} \tag{1758}$$

where the final equality holds via integration by parts twice and utilizing the fact $v=0$ on $\partial\Omega$. Since this holds for all the test functions, it follows that

$$0 = -(f-u) + \lambda\Delta^2 u \quad \text{in } \Omega, \tag{1759}$$

as desired.

b) Assume u and f are 2π periodic. Then we take the Fourier transform of the Euler-Lagrange equation to find that for each $(m,n) \in \mathbb{Z}^2$,

$$\begin{aligned} 0 &= -\left(\hat{f}(m,n) - \hat{u}(m,n)\right) + \lambda\widehat{\Delta^2 u} \\ &= -\hat{f}(m,n) + \hat{u}(m,n) + \lambda(m^2+n^2)\widehat{\Delta u} \\ &= -\hat{f}(m,n) + \left(1+\lambda(m^2+n^2)^2\right)\hat{u}(m,n). \end{aligned} \tag{1760}$$

Then solving for each coefficient $\hat{u}(m,n)$ and recalling for each x and y we have

$$u(x,y) = \sum_{(m,n)\in\mathbb{Z}^2} \hat{u}(m,n)e^{i(mx+ny)}, \tag{1761}$$

we deduce

$$u(x,y) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{\hat{f}(m,n)}{1+\lambda(m^2+n^2)^2} e^{i(mx+ny)}. \tag{1762}$$

c) Smoothness of u corresponds to rapid decay of its Fourier coefficients as m and n get large. From (1762), we see, as λ becomes large, the coefficients for larger m and n become smaller. This implies u becomes increasingly smooth as $\lambda \to \infty$.

□

S07.2. Find all solutions to the boundary value problem $\Delta u = x$ in $x^2+y^2<1$, $\partial u/\partial r = y$ on $x^2+y^2=1$.

Solution:

Set $\Omega := \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. Our PDE becomes, using polar coordinates,

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = \Delta u = r\cos\theta & \text{in } \Omega, \\ u_r = \sin\theta & \text{on } \partial\Omega, \end{cases} \tag{1763}$$

By linearity of the PDE, we assume $u = v + w$ where

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v_r = \sin\theta & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta w = r\cos\theta & \text{in } \Omega, \\ w_r = 0 & \text{on } \partial\Omega. \end{cases} \tag{1764}$$

We solve for v using separation of variables, i.e., we assume $v(r,\theta) = f(r)g(\theta)$ so that

$$0 = g(\theta)\left[f''(r) + \frac{1}{r}f'(r)\right] + \frac{1}{r^2}f(r)g''(\theta) \quad\Longrightarrow\quad \frac{r}{f(r)}\left[rf''(r)+f'(r)\right] = -\frac{g''(\theta)}{g(\theta)}, \tag{1765}$$

Since the variables on the right and left hand sides are independent of one another, there exists $\lambda\in\mathbb{R}$ such that

$$g''(\theta) + \lambda g(\theta) = 0 \quad\Longrightarrow\quad g(\theta) = A\sin(\lambda\theta) + B\cos(\lambda\theta). \tag{1766}$$

From the boundary condition in (1764), we deduce $A = 1$, $B = 0$, and $\lambda = 1$. Thus,

$$r^2 f''(r) + rf'(r) - f(r) = 0, \tag{1767}$$

which implies $f(r)$ is of the form x^m and yields

$$0 = m(m-1) + m - 1 = m^2 - 1 \quad\Longrightarrow\quad m = \pm 1. \tag{1768}$$

With the boundary condition in (1764), it follows that $m = 1$. Compiling our results, we see

$$v(r,\theta) = r\sin(\theta). \tag{1769}$$

Similarly, assuming $w = \phi(r)\gamma(\theta)$, we immediately see the ansatz $\gamma(\theta) = \cos\theta$. This implies

$$r^2\phi''(r) + r\phi'(r) - \phi(r) = r^3, \tag{1770}$$

for which we assume a particular solution ϕ is given by $ar^3 + br^2 + cr + d$. Then

$$r^3 = r^2(6ar + 2b) + r(3ar^2 + 2br + c) - (ar^3 + br^2 + cr + d) \quad \Longrightarrow \quad 0 = r^3(8a - 1) + r^2(3b) + +d, \tag{1771}$$

from which equating coefficients reveals $b = d = 0$, and $a = 1/8$. Using the fact $\phi_r(1) = 0$, we know

$$0 = \phi_r(1) = \left[3ar^2 + 2br + c\right]_{r=1} = \left[\frac{3}{8} + 0 + c\right] \quad \Longrightarrow \quad c = -\frac{3}{8}. \tag{1772}$$

Compiling our results, we obtain

$$u(r,\theta) = v(r,\theta) + w(r,\theta) = r\sin(\theta) + \frac{r^3 - 3r}{8} \cdot \cos\theta, \tag{1773}$$

and so

$$u(x,y) = y + \frac{(x^2+y^2)x}{8} - \frac{3x}{8}. \tag{1774}$$

We were asked to find all solutions, and we claim every other solution $\tilde{u}$ to the PDE differ from u only by a constant. Indeed, setting $q := u - \tilde{u}$ yields $\Delta q = 0$ in Ω and $q_r = 0$ on $\partial\Omega$, and so

$$0 = \int_\Omega q\Delta q \, \mathrm{d}x = -\int_\Omega |Dq|^2 \, \mathrm{d}x + \int_\Omega q\frac{\partial q}{\partial n} \, \mathrm{d}\sigma = -\int_\Omega |Dq|^2 \, \mathrm{d}x, \tag{1775}$$

where the final equality holds since the normal vector n is radial along $\partial\Omega$ and $q_r = 0$. This implies Dq is zero in Ω, from which our claim follows. This completes the proof. $\square$

S07.4. Suppose that $\Delta u = 0$ in a bounded domain D and that $u \in C^3(\overline{D})$. Show that $|\nabla u|^2$ takes its maximum value in $\overline{D}$ on the boundary of D.

Solution:

For each $x \in D$, set $v(x) := |\nabla u(x)|^2$ and observe

$$\Delta v = \Delta|\nabla u|^2 = \sum_{i=1}^n \partial_{x_i x_i} \left(\sum_{j=1}^n u_{x_j}^2 \right) = 2\sum_{i=1}^n u_{x_i} u_{x_i x_i x_i} + (u_{x_i x_i})^2 = 2Du \cdot D(\Delta u) + 2\sum_{i=1}^n (u_{x_i x_i})^2 \geq 0, \tag{1776}$$

where the final inequality holds since $\Delta u = 0$ in D and the terms in the summation are nonnegative. Thus v is subharmonic.

Because $\overline{D} \subset \mathbb{R}^n$ is closed and bounded, it is compact. Since $v \in C^2(\overline{D})$, it attains its supremum. Now let $\varepsilon > 0$ and set $v_\varepsilon := v + \varepsilon|x|^2$, and note v_ε is continuous on $\overline{U}$. By way of contradiction, suppose v_ε attains its maximum at an interior point $z \in \text{int}(U)$. This implies

$$0 \geq \Delta v_\varepsilon(z) = \left[\sum_{i=1}^n \partial_{x_i x_i} \left(v(x) + \varepsilon|x|^2 \right) \right]_{x=z} = \Delta v(z) + 2n\varepsilon > 0, \tag{1777}$$

where the final inequality holds since $\varepsilon > 0$ and $\Delta v(z) \geq 0$. This implies $0 > 0$, a contradiction. Consequently, $\max_{\overline{U}} v_\varepsilon = \max_{\partial U} v_\varepsilon$. Then observe

$$\max_{\overline{U}} v \leq \max_{x \in \overline{U}} \left(v(x) + \varepsilon|x|^2 \right) = \max_{x \in \overline{U}} v_\varepsilon(x) = \max_{x \in \partial U} v_\varepsilon(x) = \max_{x \in \partial U} \left(v(x) + \varepsilon|x|^2 \right). \tag{1778}$$

Since $\overline{U}$ is bounded, there is $M > 0$ such that $|x|^2 \leq M$ for all $x \in \overline{U}$. Thus

$$\max_{\overline{U}} v \leq \max_{x \in \partial U} v(x) + \varepsilon|x|^2 \leq \left(\max_{\partial U} v \right) + \varepsilon M. \tag{1779}$$

Letting $\varepsilon \to 0$, we deduce $\max_{\overline{U}} v \leq \max_{\partial U} v$. And, because $\partial U \subset \overline{U}$, $\max_{\overline{U}} v \geq \max_{\partial U} v$. Combining our inequalities, we conclude $v = |\nabla u|^2$ takes its maximum value in $\overline{D}$ on the boundary of D. $\square$

S07.5. Consider the equation

$$\begin{cases} u_t + (u^2)_x = au^2 & \text{in } \mathbb{R}\times(0,\infty), \\ u = g & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1780}$$

where $a > 0$ and

$$g(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1+x & \text{if } -1 < x < 0, \\ 1-x & \text{if } 0 < x < 1. \end{cases} \tag{1781}$$

a) Solve this problem by the method of characteristics to get functions $w(y,t)$ and $x(y,t)$ such that the solution $u(x,t)$ must satisfy $u(x(y,t),t) = w(y,t)$. To really find $u(x,t)$ you would have to solve $x = x(y,t)$ for $y(x,t)$, but do not attempt to do that.

b) The functions $w(y,t)$ and $x(y,t)$ will not exist for all $t \geq 0$ and $y \in \mathbb{R}$. Find $t^\star$, the largest number such that $w(y,t)$ is finite for $0 \leq t < t^\star$ for all $y \in \mathbb{R}$.

c) Will it be possible to solve for $x = x(y,t)$ for $y(x,t)$ for all t in the interval $[0,t^\star)$? Explain your answer.

Solution:

a) We proceed by using the method of characteristics. Let $F(p,q,z,x,t) = q + 2zp - az^2$. Taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = 2z, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = 2zp + q = az^2, \quad z(0) = g(x_0). \end{cases} \tag{1782}$$

This implies $t = s$ and, using separation of variables,

$$\int_{g(x_0)}^{z} \frac{\mathrm{d}\zeta}{\zeta^2} = \int_0^t a\tau \mathrm{d}\tau \quad \Longrightarrow \quad \frac{1}{g(x_0)} - \frac{1}{z} = at \quad \Longrightarrow \quad z = \frac{1}{\frac{1}{g(x_0)} - at}. \tag{1783}$$

Following the notation of the prompt, set

$$w(y,t) := \frac{1}{\frac{1}{g(y)} - at}. \tag{1784}$$

Then

$$\begin{aligned} x &= x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau \\ &= x_0 + \int_0^t \frac{2}{\frac{1}{g(x_0)} - a\tau}\ \mathrm{d}\tau \\ &= x_0 - \frac{2}{a}\int_0^t \frac{-a\mathrm{d}\tau}{\frac{1}{g(x_0)} - a\tau} \\ &= x_0 - \frac{2}{a}\left(\ln\left(\frac{1}{g(x_0)} - at\right) - \ln\left(\frac{1}{g(x_0)}\right)\right). \end{aligned} \tag{1785}$$

Likewise to above, taking $y = x_0$, we see

$$x = x(y,t) = y - \frac{2}{a}\left(\ln\left(\frac{1}{g(y)} - at\right) - \ln\left(\frac{1}{g(y)}\right)\right) = y - \frac{2}{a}\ln\left(1 - atg(y)\right). \tag{1786}$$

from which we obtain $u(x(y,t),t) = w(y,t)$.

b) We seek to $t^\star$ such that $1/g(y) - at^\star \geq 0$ for all $y \in \mathbb{R}$, with $t^\star$ as large as possible. We assume this equality holds whenever $g(y) = 0$, in which case we assume $1/g(y) = +\infty$ since g is always nonnegative and positive somewhere. Observe

$$\operatorname*{argmin}_y \frac{1}{ag(y)} = \operatorname*{argmax}_y g(y). \tag{1787}$$

Of course, $0 \leq g(y) \leq 1$ for all y, with the final equality strict precisely when $y = 0$. Consequently,

$$t^\star = \inf_y \frac{1}{ag(y)} = \frac{1}{ag(0)} = \frac{1}{a}. \tag{1788}$$

c) No, it will not be possible. We verify this claim by showing two characteristics crash together before time $t^\star = 1/a$. Consider two characteristics that originate at $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 \neq \alpha_2$. ....
(Return and complete.)

□

S07.7.

a) Suppose $a(x)$ is a smooth function (continuous of all order) which vanishes for $|x| > R$. If the derivative of $\phi(x)$ does not vanish for $|x| \leq R$, show that

$$F(k) = \int_{\mathbb{R}} e^{ik\phi(x)} a(x)\ \mathrm{d}x \tag{1789}$$

satisfies $|F(k)| \leq C_N k^{-N}$ for all $N \in \mathbb{N}$ for some sequence of constants C_N.

b) Consider the solution to $\Delta u + k^2 u = 0$ given by

$$u(x, y, k) = \int_{\mathbb{R}} \exp\left(ik\left[x\sin(\alpha) - y\cos(\alpha) - \alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha, \tag{1790}$$

where $a(\alpha)$ is as in part a). Show that $|u(x, y, k)| \leq C_N k^{-N}$ for all N on $x^2 + y^2 < 1$.

c) Suppose that $a(\alpha) = 0$ for $|\alpha| > \pi$. Show that

$$u(1, 0, k) = \frac{a(0)}{k^{1/3}} \int_{\mathbb{R}} \exp\left(-\frac{i\eta^3}{6}\right)\ \mathrm{d}\eta + \mathcal{O}(k^{-2/3}) \quad \text{as} \quad k \to \infty. \tag{1791}$$

Solution:

a) Differentiating yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\exp(ik\phi(x))\right] = ik\phi'(x)\exp(ik\phi(x)) \qquad \Longrightarrow \qquad \exp(ik\phi(x)) = \frac{1}{ik\phi'(x)}\left[\exp(ik\phi(x))\right]'. \tag{1792}$$

Then integrating by parts, we find

$$F(k) = \int_{B(0,R)} \frac{1}{ik\phi'(x)}\left[\exp(ik\phi(x))\right]' a(x)\ \mathrm{d}x = -\frac{1}{ik}\int_{B(0,R)} \exp(ik\phi(x)) \cdot \left[\frac{a(x)}{\phi'(x)}\right]'\ \mathrm{d}x, \tag{1793}$$

where the boundary term vanishes since $a(x) = 0$ for $|x| \geq R$. Note the right hand side is also well-defined since ϕ does not vanish in the domain. However,

$$\left|-\frac{1}{ik}\int_{B(0,2R)} \exp(ik\phi(x)) \cdot \left[\frac{a(x)}{\phi'(x)}\right]'\ \mathrm{d}x\right| \leq \frac{1}{k}\|(a/\phi')'\|_{L^\infty(\overline{B(0,2R)})}, \tag{1794}$$

where we note $(a/\phi')'$ is smooth on the compact set $\overline{B(0,R)}$, and so the supremum is finite. Setting $C_1 := \|(a/\phi')'\|_{L^\infty(\overline{B(0,R)})}$, we see $|F(k)| \leq C_1 k^{-1}$.

Proceeding inductively, we may make repeated used of (1792) and integration by parts. Suppose we apply this N times for some $N \in \mathbb{N}$. Then there the factor in front of the integral in (1793) will be replaced with $(-1/ik)^N$ and the right hand term will become $[a/\phi']^{(N)}$, i.e., it will be differentiated N times. Also, all the boundary terms will vanish, by the same reasoning as before. Furthermore, the denominator of the expanded expression for $[a/\phi']^{(N)}$ will only have terms that are multiples of ϕ', and so the entire expression for the derivative is well-defined and smooth in $B(0,R)$. So, in this case, we set

$$C_N := \|(a/\phi')^{(N)}\|_{L^\infty(\overline{B(0,R)})}. \tag{1795}$$

Choosing the constants in this way for all N implies

$$|F(k)| \leq C_N k^{-N}, \quad \text{for all } N \in \mathbb{N}, \tag{1796}$$

as desired.

b) Set $\phi(\alpha) := x\sin\alpha - y\cos\alpha - \alpha$. Then ϕ is smooth (differentiable of all orders) and, from our result in a), it suffices to show $\phi'(\alpha) \neq 0$ whenever $x^2+y^2<1$. Indeed, in this case,

$$\begin{aligned} \phi'(\alpha) &= x\cos(\alpha) + y\sin(\alpha) - 1 \\ &= \langle (x,y), (\cos(\alpha), \sin(\alpha))\rangle - 1 \\ &\leq \|(x,y)\|\|(\cos(\alpha),\sin(\alpha))\| - 1 \\ &= \sqrt{x^2+y^2}\cdot\sqrt{\cos^2(\alpha)+\sin^2(\alpha)} - 1 \\ &< 1\cdot 1 - 1 \\ &= 0. \end{aligned} \tag{1797}$$

The second line follows from rewriting the first terms as the dot product of two vectors in $\mathbb{R}^2$. Then the third line follows from the Cauchy Schwarz inequality, and the fifth line since $x^2+y^2<1$. Thus, $\phi'(\alpha) < 0$ in $x^2+y^2<1$. So, a) can be applied to assert there exists a sequence $\{C_N\}_{N\in\mathbb{N}}$ such that $|u(x,y,k)| \leq C_N k^{-N}$ for all $N \in \mathbb{N}$ and $x^2+y^2<1$.

c) First observe

$$u(1,0,k) = \int_{-\pi}^{\pi} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha. \tag{1798}$$

In a neighborhood of the origin, we see

$$\exp(ik[\sin(\alpha)-\alpha]) = \exp\left(ik\left[-\frac{\alpha^3}{6}+\mathcal{O}(\alpha^5)\right]\right) = \exp\left(-\frac{ik\alpha^3}{6}\right)\left(1+\mathcal{O}(\alpha^5)\right), \quad \text{as } \alpha \longrightarrow 0, \tag{1799}$$

where we use the expansion of the exponential and the fact $e^{cd} = e^c e^d$ to obtain the final equality. This implies that, as $\alpha \longrightarrow 0$,

$$\begin{aligned} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha) &= \exp\left(-\frac{ik\alpha^3}{6}\right)\left(1+\mathcal{O}(\alpha^5)\right) a(\alpha) \\ &= \exp\left(-\frac{ik\alpha^3}{6}\right)\left(1+\mathcal{O}(\alpha^5)\right)\left(a(0)+a'(0)\alpha+\mathcal{O}(\alpha^2)\right) \\ &= \exp\left(-\frac{ik\alpha^3}{6}\right)\left(a(0)+\mathcal{O}(\alpha)\right). \end{aligned} \tag{1800}$$

Let $\delta^* > 0$ be the radius of convergence of the Taylor series for the left hand side of (1800) and set $\delta := \min\{\delta^*, \pi\}$. Then

$$\begin{aligned} u(1,0,k) &= \int_{-\delta}^{\delta} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha + \int_{-\pi}^{-\delta} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha \\ &\quad + \int_{\delta}^{\pi} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha \\ &\leq \int_{-\delta}^{\delta} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha + \frac{C}{k}, \end{aligned} \tag{1801}$$

for some $C > 0$. The final inequality holds from our result in a), noting that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\sin(\alpha)-\alpha\right] = \cos(\alpha) - 1 < 0, \quad \text{for all } \alpha \in [-\pi,\pi]\backslash(-\delta,\delta). \tag{1802}$$

We seek an integral over all the reals, not just $[-\delta, \delta]$. So observe, following the trick in (1792),

$$\begin{aligned}
\left|\int_\delta^\infty \alpha \exp\left(-\frac{ik\alpha^3}{6}\right)\ \mathrm{d}\alpha\right| &= \left|-\frac{1}{ik}\int_\delta^\infty \exp\left(-\frac{ik\alpha^3}{6}\right)\cdot\left[\frac{\alpha}{\alpha^2}\right]'\ \mathrm{d}\alpha\right| \\
&= \left|\frac{1}{ik}\int_\delta^\infty \exp\left(-\frac{ik\alpha^3}{6}\right)\alpha^{-2}\right| \\
&\leq \frac{1}{k}\int_\delta^\infty \alpha^{-2}\ \mathrm{d}\alpha \\
&= -\frac{1}{k}\left[\alpha^{-1}\right]_\delta^\infty \\
&= \frac{\delta}{k}.
\end{aligned} \tag{1803}$$

This implies

$$\int_\delta^\infty \alpha \exp\left(-\frac{ik\alpha^3}{6}\right)\ \mathrm{d}\alpha = \mathcal{O}(1/k), \quad \text{as } k \longrightarrow \infty. \tag{1804}$$

Likewise,

$$\int_{-\infty}^{-\delta} \alpha \exp\left(-\frac{ik\alpha^3}{6}\right)\ \mathrm{d}\alpha = \mathcal{O}(1/k), \quad \text{as } k \longrightarrow \infty. \tag{1805}$$

Thus,

$$\begin{aligned}
\int_{-\delta}^{\delta} \exp\left(ik\left[\sin(\alpha)-\alpha\right]\right) a(\alpha)\ \mathrm{d}\alpha &= \int_{-\delta}^{\delta} \exp\left(-\frac{ik\alpha^3}{6}\right)\left(a(0)+\mathcal{O}(\alpha)\right)\ \mathrm{d}\alpha \\
&= \int_{-\delta}^{\delta} \exp\left(\right)
\end{aligned} \tag{1806}$$

(THERE IS AN ERROR IN HERE.)

Thus, compiling the results of ??, we see

$$u(1,0,k) = \frac{a(0)}{k^{1/3}}\int_{\mathbb{R}} \exp\left(-\frac{i\eta^3}{6}\right)\ \mathrm{d}\eta + \mathcal{O}(k^{-2/3}+1/k), \quad \text{as } k \longrightarrow \infty. \tag{1807}$$

Since $k^{-2/3}+1/k = \mathcal{O}(k^{-2/3})$ as $k \longrightarrow \infty$, the proof is complete.

□

S07.8. The porous media equation in $\mathbb{R}^n$ is

$$u_t = \Delta u^m, \quad m > 1. \tag{1808}$$

Consider a similarity solution of the form $t^{-\alpha}U(x/t^\beta)$, where U is nonnegative.

a) Compute the values of α and β, depending on the dimension of the space. (*hint:* the PDE conserves $\int_{\mathbb{R}^n} u(x,t)\ \mathrm{d}x$.)

b) Show that $U(\eta)$ satisfies an elliptic PDE of the form

$$C_1 U + C_2 \eta \cdot \nabla U + \Delta(U^m) = 0. \tag{1809}$$

Compute C_1 and C_2 in terms of α and β.

c) Find a family of radially symmetric solutions of the PDE in b). Use the fact that for radially symmetric $f(r)$, $\nabla f = f_r \hat{r}$ and $\Delta f = f_{rr} + \frac{n-1}{r} f_r$, where $\hat{r}$ is the unit vector pointing outward from the origin, and n is the dimension of the space.

d) Find the special solution with unit mass, i.e., $\int u(x,t)\ \mathrm{d}x = 1$.

Solution:

a) Since the mass is conserved we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^n} t^{-\alpha} U(xt^{-\beta})\ \mathrm{d}x\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^n} t^{-\alpha} g(y) t^{n\beta}\ \mathrm{d}y\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[t^{-\alpha+n\beta}\right] \int_{\mathbb{R}^n} g(y)\ \mathrm{d}y \\ &= (\beta n - \alpha) t^{\beta n - \alpha - 1} \int_{\mathbb{R}^n} g(y)\ \mathrm{d}y, \end{aligned} \tag{1810}$$

where $y = xt^{-\beta}$. Assuming u is not the trivial solution, the integral is positive since g is nonnegative. And, because this holds for all $t \in (0,\infty)$, it follows that

$$\alpha = n\beta. \tag{1811}$$

Differentiating in time reveals

$$\begin{aligned} u_t(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[t^{-n\beta}U(\eta)\right] &= -n\beta t^{-n\beta-1}U(\eta) + t^{-n\beta}\nabla U(\eta)\cdot(-\beta t^{-1}\eta) \\ &= t^{-n\beta-1}\left[-n\beta U(\eta) - \beta\nabla U(\eta)\cdot\eta\right], \end{aligned} \tag{1812}$$

where $\eta := xt^{-\beta}$. Next observe

$$\begin{aligned} \Delta(u^m) &= \sum_{i=1}^n \partial_{x_i x_i} u^m \\ &= \sum_{i=1}^n m\partial_{x_i}\left[u^{m-1}u_{x_i}\right] \\ &= mu^{m-2}\sum_{i=1}^n (m-1)u_{x_i}^2 + uu_{x_i x_i} \\ &= mu^{m-2}\left[(m-1)\|\nabla u\|^2 + u\Delta u\right]. \end{aligned} \tag{1813}$$

Since

$$\nabla u(x,t) = t^{-n\beta}\nabla U(xt^{-\beta})\cdot t^{-\beta} = t^{-(n+1)\beta}\nabla U(\eta) \tag{1814}$$

and

$$\Delta u(x,t) = t^{-n\beta}\sum_{i=1}^n \partial_{x_i x_i} U(xt^{-\beta}) = t^{-(n+2)\beta}\Delta U(\eta), \tag{1815}$$

it follows that

$$\begin{aligned} \Delta(u^m) &= m\left[t^{-n\beta}U(\eta)\right]^{m-2}\left[(m-1)t^{-2(n+1)\beta}\|\nabla U(\eta)\|^2 + t^{-n\beta}U(\eta)\cdot t^{-(n+2)\beta}\Delta U(\eta)\right] \\ &= t^{-\beta((m-2)n+2(n+1))}mU(\eta)^{m-2}\left[(m-1)\|\nabla U(\eta)\|^2 + U(\eta)\Delta U(\eta)\right] \\ &= t^{-\beta(mn+2)}mU(\eta)^{m-2}\left[(m-1)\|\nabla U(\eta)\|^2 + U(\eta)\Delta U(\eta)\right]. \end{aligned} \tag{1816}$$

Equating (1812) and (1816) reveals, from the powers of t,

$$-n\beta - 1 = -\beta(mn+2) \quad\Longrightarrow\quad \beta = \frac{1}{(m-1)n+2} \quad\Longrightarrow\quad \alpha = \frac{n}{(m-1)n+2}, \tag{1817}$$

where the final equality holds by (1811).

b) Equating (1812) and (1816) for our choices of α and β yields

$$-n\beta U(\eta) - \beta\nabla U(\eta)\cdot\eta = mU(\eta)^{m-2}\left[(m-1)\|\nabla U(\eta)\|^2 + U(\eta)\Delta U(\eta)\right] = \Delta(U^m(\eta)), \tag{1818}$$

where the final equality holds in analogous fashion to (1813). This implies

$$\alpha U(\eta) + \beta\nabla U(\eta)\cdot\eta + \Delta(U^m(\eta)) = 0, \tag{1819}$$

recalling $\alpha = n\beta$. Thus $C_1 = \alpha$ and $C_2 = \beta$.

c) Let $r := \|\eta\|$ and assume $f(r)$ is a solution to the PDE in (1819). Then the PDE transforms into the ODE

$$0 = n\beta f + \beta f' r + \left((f^m)'' + \frac{n-1}{r}(f^m)'\right), \tag{1820}$$

which implies

$$0 = n\beta r^{n-1} + \beta f' r^n + \left((f^m)'' r^n + (n-1)(f^m)' r^{n-1}\right) = (\beta f r^n)' + (r^{n-1}(f^m)')'. \tag{1821}$$

Integrating with respect to r and assuming the integration constant is zero, we obtain

$$0 = \beta f r^n + r^{n-1}(f^m)' = \beta f r^n + m r^{n-1} f^{m-1} f' \quad\implies\quad 0 = \beta r + m f^{m-2} f'. \tag{1822}$$

Integrating once more yields

$$\int m f^{m-2}\,\mathrm{d}f = -\int \beta r\,\mathrm{d}r \quad\implies\quad \frac{m f^{m-1}}{m-1} = C - \frac{\beta r^2}{2} \quad\implies\quad f = \left[\frac{m-1}{m}\left(C - \frac{\beta r^2}{2}\right)\right]^{1/(m-1)}, \tag{1823}$$

for some scalar $C \in \mathbb{R}$. Since we assume $U(\eta)$ is nonnegative, we then write

$$U(\eta) = \max\left\{\left[\frac{m-1}{m}\left(C - \frac{\beta\|\eta\|^2}{2}\right)\right]^{1/(m-1)},\ 0\right\}, \tag{1824}$$

noting 0 is a solution to the PDE. Then $u(x,t) = t^{-\alpha}U(\eta)$, using α, β, and U as derived above.

d) Setting

$$R(t) := \left(\frac{2Ct^{2\beta}}{\beta}\right)^{1/2}, \tag{1825}$$

it follows (1824) that $u(x,t) = 0$ for all $x \notin B(0, R(t))$. Consequently,

$$1 = \int_{\mathbb{R}^n} u(x,t)\ \mathrm{d}x = \int_{B(0,R(t))} u(x,t)\ \mathrm{d}x = \int_{r=0}^{R(t)} \int_{\partial B(0,r)} t^{-\alpha} U(rt^{-\beta})\ \mathrm{d}\sigma \mathrm{d}r, \tag{1826}$$

where we abusively write $U(x) = U(\|x\|) = U(r)$ since U is radially symmetric. Then

$$1 = n\,|B(0,1)| \int_{r=0}^{R(t)} t^{-\alpha} U(rt^{-\beta})\ r^{n-1}\mathrm{d}r \quad \Longrightarrow \quad \frac{t^\alpha}{n|B(0,1)|} = \int_{r=0}^{R(t)} \left[\frac{m-1}{m}\left(C - \frac{\beta r^2}{2t^{2\beta}}\right)\right]^{1/(m-1)} r^{n-1}\mathrm{d}r. \tag{1827}$$

This implies

$$\frac{t^\alpha}{n|B(0,1)|}\left(\frac{m}{m-1}\cdot\frac{2t^{2\beta}}{\beta}\right)^{1/(m-1)} = \int_{r=0}^{R(t)} \left(R(t)^2 - r^2\right)^{1/(m-1)} r^{n-1}\ \mathrm{d}r. \tag{1828}$$

Then using the trig substitution $r = R\sin\theta$ the integral becomes

$$\int_0^{\pi/2} (R^2 - R^2\sin^2\theta)^{1/(m-1)} (R\sin\theta)^{n-1}\ R\cos\theta \mathrm{d}\theta = R^{n+2/(m-1)} \int_0^{\pi/2} \cos^{(m+1)/(m-1)}\theta \sin^{n-1}\theta\ \mathrm{d}\theta. \tag{1829}$$

Thus

$$R^{n+2/(m-1)} = \frac{t^\alpha}{n|B(0,1)|}\left(\frac{m}{m-1}\cdot\frac{2t^{2\beta}}{\beta}\right)^{1/(m-1)} \left[\int_0^{\pi/2} \cos^{(m+1)/(m-1)}\theta \sin^{n-1}\theta\ \mathrm{d}\theta\right]^{-1}. \tag{1830}$$

With R as in (1830), we then deduce, by (1825),

$$C = R^2 \cdot \frac{\beta}{2t^{2\beta}}. \tag{1831}$$

With this choice of C and U as in (1824), we see $u(x,t) = t^{-\alpha}U(\eta)$ has unit mass, as desired.

□

2006 Fall

F06.3 Consider the PDE

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}\times[0,\infty)\times(0,\infty), \\ u = u_0 & \text{on } \mathbb{R}\times[0,\infty)\times\{t=0\}, \end{cases} \tag{1832}$$

with $u_0 \geq 0$. Compare the following two boundary conditions:

$$u = 0 \quad \text{on } \mathbb{R}\times\{y=0\}\times(0,\infty), \tag{1833}$$

and

$$u_y = 0 \quad \text{on } \mathbb{R}\times\{y=0\}\times(0,\infty). \tag{1834}$$

Denote the solution of (1832) and (1833) by u^D and the solution of (1832) and (1834) by u^N. Show $u^D \leq u^N$ in $\mathbb{R}\times[0,\infty)\times(0,\infty)$.

Solution:

We proceed as follows. First we define a function $\tilde{u}^D : \mathbb{R}^2\times[0,\infty)\to\mathbb{R}$ that extends u^D to all of $\mathbb{R}^2$ by means of a reflection. Then we express $\tilde{u}^D$ in terms of a convolution. We then do similarly with a function $\tilde{u}^N$. Then we obtain our result by directly comparing the integrands in each of the integral representations.

For each function $f : \mathbb{R}\times[0,\infty)\to\mathbb{R}$ define $\tilde{f} : \mathbb{R}^2\to\mathbb{R}$ via

$$\tilde{f}(x,y,t) := \begin{cases} f(x,y,t) & \text{if } y \geq 0, \\ -f(x,-y,t) & \text{if } y < 0. \end{cases} \tag{1835}$$

Then note $\tilde{f} = f$ in $\mathbb{R}\times[0,\infty)$ for each f and $\tilde{u}^D(x,0,t) = u^D(x,0,t) = 0$. The extension $\tilde{u}^D$ satisfies

$$\begin{cases} \tilde{u}^D_t - \Delta\tilde{u}^D = 0 & \text{in } \mathbb{R}^2\times(0,\infty), \\ \tilde{u}^D = \tilde{u}_0 & \text{on } \mathbb{R}^2\times\{t=0\}. \end{cases} \tag{1836}$$

For notational convenience, let $r = (x,y) \in \mathbb{R}^2$. Since $\tilde{u}^D$ solves the heat equation, its solution is given by the convolution

$$\tilde{u}^D(r,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} \tilde{u}_0(\xi)\exp\left(-\frac{|r-\xi|^2}{4t}\right)\,\mathrm{d}\xi. \tag{1837}$$

We may expand this using the definition of $\tilde{u}_0$ by writing

$$\begin{aligned}\tilde{u}(r,t) &= \frac{1}{4\pi t}\left[\int_{-\infty}^{\infty}\int_0^{\infty} u_0(\xi_1,\xi_2)\exp\left(-\frac{|r-\xi|^2}{4t}\right)\,\mathrm{d}\xi_1\mathrm{d}\xi_2 - \int_{-\infty}^{\infty}\int_{-\infty}^{0} u_0(\xi_1,-\xi_2)\exp\left(-\frac{|r-\xi|^2}{4t}\right)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\right]\\ &= \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_0^{\infty} u_0(\xi_1,\xi_2)\left[\exp\left(-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4t}\right) - \exp\left(-\frac{(x-\xi_1)^2+(y+\xi_2)^2}{4t}\right)\right]\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\ &= \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_0^{\infty} u_0(\xi_1,\xi_2)\exp\left(-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4t}\right)\left[1-\exp\left(-\frac{y\xi_2}{t}\right)\right]\,\mathrm{d}\xi_1\mathrm{d}\xi_2\end{aligned} \tag{1838}$$

Now for each function $f:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ define $\overline{f}:\mathbb{R}^2\to\mathbb{R}$ via

$$\overline{f}(x,y,t) := \begin{cases} f(x,y,t) & \text{if } y \geq 0,\\ f(x,-y,t) & \text{if } y < 0.\end{cases} \tag{1839}$$

Then note $\overline{f} = f$ in $\mathbb{R}\times[0,\infty)$ for each f and $\overline{u}^N_y = 0$ since

$$\lim_{y\to 0^-}\overline{u}^N_y(x,y,t) = \lim_{y\to 0^-}\frac{\partial}{\partial y}\left[u^N_y(x,-y,t)\right] = \lim_{y\to 0^-} -u^N_y(x,-y,t) = \lim_{y\to 0^-} -\overline{u}^N_y(x,-y,t) = \lim_{y\to 0^+} -\overline{u}^N_y(x,y,t), \tag{1840}$$

which implies

$$\overline{u}^N_y(x,0,t) = -\overline{u}^N_y(x,0,t). \tag{1841}$$

Thus

$$\begin{cases} \overline{u}^D_t - \Delta\overline{u}^D = 0 & \text{in } \mathbb{R}^2\times(0,\infty),\\ \overline{u}^D = \overline{u}_0 & \text{on } \mathbb{R}^2\times\{t=0\}.\end{cases} \tag{1842}$$

In similar fashion to above, we write

$$\begin{aligned}\overline{u}^N(r,t) &= \frac{1}{4\pi t}\int_{\mathbb{R}^2}\overline{u}_0(\xi)\exp\left(-\frac{|r-\xi|^2}{4t}\right)\ \mathrm{d}\xi\\ &= \frac{1}{4\pi t}\left[\int_{-\infty}^{\infty}\int_0^{\infty}u_0(\xi_1,\xi_2)\exp\left(-\frac{|r-\xi|^2}{4t}\right)\ \mathrm{d}\xi_1\mathrm{d}\xi_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{0}u_0(\xi_1,-\xi_2)\exp\left(-\frac{|r-\xi|^2}{4t}\right)\ \mathrm{d}\xi_1\mathrm{d}\xi_2\right]\\ &= \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_0^{\infty}u_0(\xi_1,\xi_2)\left[\exp\left(-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4t}\right)+\exp\left(-\frac{(x-\xi_1)^2+(y+\xi_2)^2}{4t}\right)\right]\ \mathrm{d}\xi_1\mathrm{d}\xi_2\\ &= \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_0^{\infty}u_0(\xi_1,\xi_2)\exp\left(-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4t}\right)\left[1+\exp\left(-\frac{y\xi_2}{t}\right)\right]\ \mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned} \tag{1843}$$

Since $u_0 \geq 0$ and exponentials are always positive, the integrands in (1838) and (1843) are nonnegative. Consequently, for all $t > 0$ we see the integrand for $\overline{u}^N$ is at least as large as that for $\tilde{u}^D$ (by comparing the final terms), which implies $\overline{u}^N \geq \tilde{u}^D$ in $\mathbb{R}^2 \times (0,\infty)$. Furthermore, since $\tilde{u}^D = u^D$ and $\overline{u}^N = u^N$ in $\mathbb{R}\times[0,\infty)\times[0,\infty)$, we conclude $u^N \geq u^D$, as desired. □

2006 Spring

S06.1. Solve the following initial value problem and verify your solution

$$\begin{cases} u_x + u_y = u^2 & \text{in } \mathbb{R}^2, \\ u = h & \text{on } \mathbb{R} \times \{y = 0\}. \end{cases} \tag{1844}$$

Solution:

We proceed by using the method of characteristics. Define $F(p, q, z, x, y) = p + q - z^2$. Taking $p = u_x$, $q = u_y$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = 1, & x(0) = x_0, \\ \dot{y}(s) = F_q = 1, & y(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = p + q = z^2 & z(0) = h(x_0). \end{cases} \tag{1845}$$

This implies $y = s$ and $x = x_0 + s = x_0 + y$. Using separation of variables, we see

$$\int_{h(x_0)}^{z} \frac{\mathrm{d}\zeta}{\zeta^2} = \int_0^s \mathrm{d}s \quad \Longrightarrow \quad \frac{1}{h(x_0)} - \frac{1}{z} = s \quad \Longrightarrow \quad z = \frac{1}{\frac{1}{h(x_0)} - s} = \frac{1}{\frac{1}{h(x-y)} - y}. \tag{1846}$$

Therefore,

$$u(x, y) = \frac{1}{\frac{1}{h(x-y)} - y}. \tag{1847}$$

Indeed, $u(x, 0) = 1/(1/h(x)) = h(x)$ and

$$u_x + u_y = \left[-u^2 \cdot \left(-\frac{1}{h(x-y)^2}\right) \cdot h'(x-y)\right] + \left[-u^2 \cdot \left(-\frac{1}{h(x-y)^2} \cdot h'(x-y) \cdot (-1) - 1\right)\right] = u^2, \tag{1848}$$

as desired. □

S06.8. Let $u(x,t) \in C^2(\mathbb{R}^n \times \mathbb{R})$ be a solution of the wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in } D, \tag{1849}$$

where

$$\mathcal{D} := \{(x,t) : x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ |x_n| \leq t\}. \tag{1850}$$

Assume that $u = 0$ for $|x'| \geq R$ for some $R > 1$. Suppose also that $u|_{\Gamma_1} = 0$ and $u|_{\Gamma_2} = 0$, where

$$\Gamma_1 := \{(x,t) : x' \in \mathbb{R}^{n-1},\ t - x_n = 0,\ t > 0\} \quad \text{and} \quad \Gamma_2 := \{(x,t) : x' \in \mathbb{R}^{n-1},\ t + x_n = 0,\ t > 0\}. \tag{1851}$$

Prove that $u \equiv 0$.

Solution:

Let $T \in (0, \infty)$, and set $\Omega := \mathcal{D} \cap \{t \leq T\}$. Then note

$$\Omega = \bigcup_{t \in [0,T]} S(t) \times \{t\}, \tag{1852}$$

where $S(t) := \{x : |x_n| \leq t\}$. Thus, multiplying our PDE by u_t and integrating reveals

$$\begin{aligned}
0 &= \int_\Omega u_t(u_{tt} - \Delta u)\ \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{S(t)} u_t(u_{tt} - \Delta u)\ \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \left[\int_{S(t)} u_t u_{tt} + \nabla u \cdot \nabla u_t\ \mathrm{d}x - \int_{\partial S(t)} u_t \frac{\partial u}{\partial n}\ \mathrm{d}\sigma \right] \\
&= \int_0^T \left[\int_{S(t)} \partial_t \left[\frac{1}{2} \left(u_t^2 + |\nabla u|^2 \right) \right]\ \mathrm{d}x - \int_{\partial S(t)} u_t \frac{\partial u}{\partial n}\ \mathrm{d}\sigma \right] \\
&= \int_0^T \left(\partial_t \left[\frac{1}{2} \int_{S(t)} u_t^2 + |\nabla u|^2\ \mathrm{d}x \right] - \int_{\partial S(t)}\ \mathrm{d}\sigma \right) \mathrm{d}t \\
&= \int_0^T \dot{E}(t)\ \mathrm{d}t - \int_0^T \int_{\partial S(t)} ??\ \mathrm{d}\sigma\mathrm{d}t \\
&= E(T) - E(0) - ?????? \\
&\geq E(T) \\
&\geq 0.
\end{aligned} \tag{1853}$$

Therefore, $E(T) = 0$. Since T was arbitrarily chosen, we deduce $E(T) = 0$ for all $T \in (0, \infty)$. Therefore, $u_t = 0$ and $\nabla u = 0$ in $\mathcal{D}$, and so u is constant. Since $u = 0$ at a point in $\mathcal{D}$, it follows that $u \equiv 0$, and we are done. $\square$

2005 Winter

W05.6. Find the Fourier transform of the integrable function $x \mapsto \sin^2(x)/x^2$.

Solution:

Let $f(x) := \sin(x)/x$. Then note

$$\begin{aligned}
\mathcal{F}(\sin(x)) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(x) \exp\left(-2\pi i x \xi\right) \ \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{2i} \left(e^{ix} - e^{-ix}\right) \exp\left(-2\pi i x \xi\right) \ \mathrm{d}x \\
&= \frac{1}{2i\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-2\pi i x (\xi - \frac{1}{2\pi})\right) - \exp\left(-2\pi i x \left(\xi + \frac{1}{2\pi}\right)\right) \ \mathrm{d}x \\
&= \frac{1}{2i\sqrt{2\pi}} \left[\delta\left(\xi - \frac{1}{2\pi}\right) - \delta\left(\xi + \frac{1}{2\pi}\right)\right].
\end{aligned} \tag{1854}$$

□

2005 Fall

F05.4. Consider the heat equation $u_t = u_{yy}$ on the real line with initial data $u_0 = 1$ if $y < 0$ and $u_0 = 0$ if $y > 0$.

a) Show the solution $u(y,t)$ satisfies $\lim_{t\to\infty} u(y,t) = 1/2$.

b) Is the limit uniform in y? Prove your answer.

Solution:

a) Let u be a solution to the given PDE. Then

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = u_0 & \text{on } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1855}$$

where

$$u_0(y) := \begin{cases} 1 & \text{if } y < 0, \\ 0 & \text{if } y > 0. \end{cases} \tag{1856}$$

Then since this is the heat equation we know the solution is given by

$$u(y,t) = (\Phi * u)(y,t) = \int_{\mathbb{R}} \Phi(y-\xi,t)u_0(\xi)\ \mathrm{d}\xi = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \exp\left(-\frac{|y-\xi|^2}{4t}\right) u_0(\xi)\ \mathrm{d}\xi, \tag{1857}$$

where Φ is the fundamental solution of the heat equation given by

$$\Phi(y,t) := \begin{cases} \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{y^2}{4t}\right) & \text{in } \mathbb{R}\times(0,\infty), \\ 0 & \text{if } \mathbb{R}\times(-\infty,0). \end{cases} \tag{1858}$$

We may then use the definition of u_0 and the change of variables $z = (\xi - y)/\sqrt{4t}$ to write

$$u(y,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{0} \exp\left(-\frac{|y-\xi|^2}{4t}\right) u_0(\xi)\ \mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-y/\sqrt{4t}} \exp\left(-z^2\right)\ \mathrm{d}z. \tag{1859}$$

For any fixed $y \in \mathbb{R}$, we deduce

$$\lim_{t\to\infty} u(y,t) = \lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-y/\sqrt{4t}} \exp\left(-z^2\right)\ \mathrm{d}z = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0}\exp\left(-z^2\right)\ \mathrm{d}z = \frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{2}. \tag{1860}$$

b) We claim the limit is *not* uniform in y. By way of contradiction, suppose the limit is uniform in y. Let $\varepsilon \in (0, 1/2)$. Then, by our assumption, there is $T > 0$ such that $t > T$ implies

$$\left\| u(\cdot, t) - \frac{1}{2} \right\|_{L^\infty(\mathbb{R})} < \varepsilon. \tag{1861}$$

However, for each $t \in (0, \infty)$

$$\lim_{y \to -\infty} u(y,t) = \lim_{y \to -\infty} \int_{-\infty}^{-y/\sqrt{4t}} \exp\left(-z^2\right) \, \mathrm{d}z = \int_{-\infty}^{-\infty} \exp\left(-z^2\right) \, \mathrm{d}z = 0. \tag{1862}$$

Now note u is positive in $\mathbb{R} \times (0, \infty)$, which is apparent by (1859) since the integrand on the right hand side is always positive. Let $t^* > T$. Then the positivity of u together with (1862) implies there exists $y^* \in \mathbb{R}$ such that $u(y^*, t^*) \in (0, \frac{1}{2} - \varepsilon)$, and so

$$\frac{1}{2} - u(y^*, t^*) > \varepsilon, \tag{1863}$$

which contradicts (1861). Whence the limit is *not* uniform in y.

□

F05.7. Find the (entropy) solution for all time $t > 0$ of the inviscid Burgers equation $u_t + \frac{1}{2}(u^2)_x = 0$ with the initial condition

$$u(x,0) = \begin{cases} 0 & \text{if } x < -1, \\ x+1 & \text{if } -1 < x < 0, \\ 1 - \frac{1}{2}x & \text{if } 0 < x < 2, \\ 0 & \text{if } x > 2. \end{cases} \tag{1864}$$

Solution:

We proceed by using the method of characteristics. Let $F(p,q,z,x,t) := q + zp$. Then taking $q = u_t$, $p = u_x$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z(s), \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = zp + q = 0, \quad z(0) = g(x_0), \end{cases} \tag{1865}$$

where $g(\alpha) := u(\alpha, 0)$. This implies $t = s$ and z is constant along characteristics. Thus, integrating reveals

$$x = x_0 + \int_0^t \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^t z(\tau)\ \mathrm{d}\tau = x_0 + tg(x_0) = \begin{cases} x_0 & \text{if } x_0 < -1, \\ (1+t)x_0 + t & \text{if } -1 < x_0 < 0, \\ (1 - \frac{t}{2})x_0 + t & \text{if } 0 < x_0 < 2, \\ x_0 & \text{if } x_0 > 2. \end{cases} \tag{1866}$$

Notice the characteristics are linear, and, in particular, are vertical for $x_0 < -1$ and $x_0 > 2$. And, for $x_0 \in (0,2)$ we see

$$\lim_{t\to 2^-} x = \lim_{t \to 2^-} \left(1 - \frac{t}{2}\right) x_0 + t = 2, \tag{1867}$$

i.e., the characteristics crash at time $t = 2$. Note they do not crash before this time. As mentioned, they are vertical in two regions. And, if two characteristics originating at distinct points $\alpha_1, \alpha_2 \in (-1, 0)$ cross at a time $t^\star \in (0,2)$, then we we see

$$(1+t^\star)\alpha_1 + t^\star = (1+t^\star)\alpha_2 + t^\star \quad \Longrightarrow \quad \alpha_1 = \alpha_2, \tag{1868}$$

a contradiction. In similar fashion, we see characteristics both originating in $(0,2)$ don't crash before time $t = 2$. And, the slope of characteristics originating in $(-1,0)$ exceeds that of those originating in $(0,2)$, and so these characteristics cannot cross either. Thus, we see $t = 2$ is, in fact, the first time at which the characteristics crash.

Solving for x_0 in terms of x and t in our above expression reveals that, for all $x \in \mathbb{R} \times (0,2)$,

$$u(x,t) = g(x_0) = \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 2, \\ 1 + x_0 & \text{if } -1 < \frac{x-t}{1+t} < 0, \\ 1 - \frac{1}{2}x_0 & \text{if } 0 < \frac{x-t}{1-\frac{t}{2}} < 2, \end{cases} = \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 2, \\ \dfrac{1+x}{1+t} & \text{if } -(1+t) < (x-t) < 0, \\ \dfrac{2-x}{2-t} & \text{if } 0 < (x-t) < (2-t). \end{cases} \tag{1869}$$

Let $f(u) := \frac{u^2}{2}$. Then, by the Rankine-Hugenoit condition, the shock curve parameterized[52] by $(s(t), t)$ satisfies $(s(2), 2) = (2, 2)$ and

$$\dot{s}(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}u_\ell^2 - \frac{1}{2}u_r^2}{u_\ell - u_r} = \frac{1}{2}u_\ell = \frac{1+s}{2(1+t)}, \tag{1870}$$

where u_ℓ and u_r denote the limiting function values approaching the curve from the left and right, respectively, and we note $u_r = 0$. Using separation of variables, we see

$$\int \frac{\mathrm{d}s}{1+s} = \int \frac{\mathrm{d}t}{2(1+t)} \quad \Longrightarrow \quad \ln(1+s) = \ln((1+t)^{1/2}) + \ln(C) \quad \Longrightarrow \quad s = C(1+t)^{1/2} - 1, \tag{1871}$$

for some scalar $C \in \mathbb{R}$. The initial condition implies $C = \sqrt{3}$. Therefore, for all $(x,t) \in \mathbb{R} \times (0, \infty)$,

$$u(x,t) = \begin{cases} \dfrac{1+x}{1+t} & \text{if } -1 < \frac{x-t}{1+t} < s(t) = \sqrt{3(1+t)} - 1, \\ 0 & \text{if } x < -1 \text{ or } x > s(t) = \sqrt{3(1+t)} - 1. \end{cases} \tag{1872}$$

□

[52] This s is distinct from that used in the method of characteristics earlier.

2004 Fall

F04.2. Let $u(x,t)$ be a bounded solution to the Cauchy problem for the heat equation

$$\begin{cases} \partial_t u = a^2 \partial_x^2 u, t > 0, \ x \in \mathbb{R}, a > 0, \\ u(x,0) = \phi(x). \end{cases} \tag{1873}$$

Here $\phi \in C(\mathbb{R})$ satisfies

$$\lim_{x \to +\infty} \phi(x) = b \quad \text{and} \quad \lim_{x \to -\infty} \phi(x) = c. \tag{1874}$$

Compute the limit of $u(x,t)$ as $t \longrightarrow +\infty$, $x \in \mathbb{R}$. Justify your answer carefully.

Solution:

Define the function $v(x,t) := u(ax,t)$ for each $(x,t) \in \mathbb{R} \times [0,\infty)$. Then v satisfies

$$\begin{cases} v_t - v_{xx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty), \\ \qquad\quad v = g \quad \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1875}$$

where $g(x) := \phi(ax)$. Since v solves the heat equation, it is given by the convolution

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} g(\xi) \exp\left(-\frac{|x-\xi|^2}{4t}\right) \, \mathrm{d}\xi. \tag{1876}$$

Letting $z = (\xi - x)/\sqrt{4t}$, we may write

$$v(x,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} g\left(z\sqrt{4t} + x\right) \exp\left(-z^2\right) \, \mathrm{d}z. \tag{1877}$$

We claim ϕ is bounded and, thus, so also is g. This implies the integrand is dominated by some constant multiplied by $\exp(-z^2)$, which is integrable. Whence we may use the dominated convergence theorem to deduce for $x \in \mathbb{R}$

$$\begin{aligned}
\lim_{t\to\infty} v(x,t) &= \lim_{t\to\infty} \frac{1}{\sqrt{\pi}} \left[\int_0^{+\infty} g\left(z\sqrt{4t}+x\right) \exp\left(-z^2\right)\ \mathrm{d}z + \int_{-\infty}^0 g\left(z\sqrt{4t}+x\right) \exp\left(-z^2\right)\ \mathrm{d}z \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\int_0^{+\infty} \lim_{t\to\infty} g\left(z\sqrt{4t}+x\right) \exp\left(-z^2\right)\ \mathrm{d}z + \int_{-\infty}^0 \lim_{t\to\infty} g\left(z\sqrt{4t}+x\right) \exp\left(-z^2\right)\ \mathrm{d}z \right] \\
&= \frac{1}{\sqrt{\pi}} \left[b\int_0^{+\infty} \exp\left(-z^2\right)\ \mathrm{d}z + c\int_{-\infty}^0 \exp\left(-z^2\right)\ \mathrm{d}z \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\frac{b\sqrt{\pi}}{2} + \frac{c\sqrt{\pi}}{2} \right] \\
&= \frac{b+c}{2},
\end{aligned} \tag{1878}$$

where we use the dominated convergence theorem to obtain the second equality. The first part of the third line holds by using the fact $z > 0$ in the first integrand and $g \longrightarrow b$ as $x \longrightarrow \infty$, by hypothesis. The second part of the third line follows similarly. Therefore for each $x \in \mathbb{R}$ we conclude

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} v(x/a,t) = \frac{b+c}{2}, \tag{1879}$$

as desired. Note the division by a is well-defined since $a > 0$.

All that remains is to verify ϕ is bounded. Since $\phi \longrightarrow b$ as $x \longrightarrow +\infty$, there is $y > 0$ such that $x > y$ implies

$$|\phi(x) - b| < 1 \quad \Longrightarrow \quad |\phi(x)| < |b| + 1. \tag{1880}$$

Consequently, ϕ is bounded on (y_1, ∞). And, since $[0, y_1]$ is closed and bounded, it is compact and therefore $\phi([0, y_1])$ is compact as ϕ is continuous, which implies ϕ is bounded on $[0, y_1]$. This shows ϕ is bounded on $[0, \infty)$. Similar argument allows us to deduce ϕ is bounded on $(-\infty, 0]$ and, thus, ϕ is bounded on $\mathbb{R}$. $\square$

F04.3. Consider the damped wave equation

$$\begin{cases} (\partial_{tt} - \Delta + a\partial_t)u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ u = \phi & \text{on } \mathbb{R}^3 \times \{t = 0\}, \\ u = \psi & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases} \tag{1881}$$

where $a \in C_0^\infty(\mathbb{R}^3)$ is a nonnegative function and $\phi, \psi \in C_0^\infty(\mathbb{R}^3)$. Show the energy $E(t)$ is decreasing in time, where

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 + u_t^2 \ \mathrm{d}x. \tag{1882}$$

Solution:

To proceed, we differentiate E in time to find

$$\dot{E}(t) = \int_{\mathbb{R}^3} Du \cdot Du_t + u_t u_{tt} \ \mathrm{d}x = \int_{\mathbb{R}^3} u_t \left(-\Delta u + u_{tt}\right) \ \mathrm{d}x = \int_{\mathbb{R}^3} u_t \left[-au_t\right] \ \mathrm{d}x = -\int_{\mathbb{R}^3} au_t^2 \ \mathrm{d}x \leq 0. \tag{1883}$$

The equality holds via integration by parts, where the boundary terms vanish since we claim u is compactly supported for all time. The following equality holds by the PDE u satisfies, and the final inequality holds since the integral is nonnegative. Therefore $\dot{E}(t) \leq 0$ for all times $t \in [0, \infty)$, which implies $E(t)$ is nonincreasing in time.

All that remains is to verify our claim that u is compactly supported for all times. To do this, we first show that if there exists a ball $B(z, r) \subset \mathbb{R}^3$ such that $u = u_t = 0$ on $B(z, r) \times \{t = 0\}$, then $u = 0$ in the cone $K(z, r)$ defined by

$$K(z, r) := \{(x, t) \in \mathbb{R}^3 \times [0, r) : x \in B(z, r - t)\}. \tag{1884}$$

For notational convenience, set $S(t) := B(z, r - t)$. Then define the energy $e(t)$ via

$$e(t) := \frac{1}{2}\int_{S(t)} |Du|^2 + u_t^2 \ \mathrm{d}x. \tag{1885}$$

We assume $\phi = \psi = 0$ in $S(0)$, from which it follows that $e(0) = 0$. Then differentiating in time reveals

$$\begin{aligned} \dot{e}(t) &= \int_{S(t)} Du \cdot Du_t + u_t u_{tt} \ \mathrm{d}x + \int_{\partial S(t)} \frac{1}{2}(|Du|^2 + u_t^2) v \cdot n \ \mathrm{d}\sigma \\ &= \int_{S(t)} u_t \left(u_{tt} - \Delta u \right) \ \mathrm{d}x + \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} + \frac{1}{2}(|Du|^2 + u_t^2) v \cdot n \ \mathrm{d}\sigma \\ &= \int_{S(t)} -a u_t^2 \ \mathrm{d}x + \int_{\partial S(t)} u_t \frac{\partial u}{\partial n} - \frac{1}{2}\left(|Du|^2 + u_t^2\right) \ \mathrm{d}\sigma, \end{aligned} \tag{1886}$$

where v is the Eulerian velocity of the boundary $\partial S(t)$ and n is the outward normal along $\partial S(t)$. Together the Cauchy-Schwarz inequality and the fact that

$$0 \leq (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta \quad \Longrightarrow \quad \alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2), \quad \text{for all } \alpha, \beta \in \mathbb{R}, \tag{1887}$$

imply

$$\left| u_t \frac{\partial u}{\partial n} \right| = |u_t| \, |Du \cdot n| \leq |u_t||Du| \leq \frac{1}{2}(u_t^2 + |Du|^2). \tag{1888}$$

Thus, our boundary integral is nonnegative, i.e.,

$$\dot{e}(t) \leq \int_{S(t)} -a u_t^2 \ \mathrm{d}x + \int_{\partial S(t)} \frac{1}{2}\left(|Du|^2 + u_t^2\right) - \frac{1}{2}\left(|Du|^2 + u_t^2\right) \ \mathrm{d}\sigma = -\int_{S(t)} a u_t^2 \ \mathrm{d}x \leq 0, \tag{1889}$$

where the final inequality holds since a is nonnegative. This shows $\dot{e}(t) \leq 0$ for all $t \in [0, r)$, and so $e(t)$ is nonincreasing. Since the integrand in the definition of $e(t)$ is nonnegative, it follows that $0 \leq e(t) \leq e(0) = 0$, whereby we see $e(t) = 0$. Whence $u_t = 0$ and $Du = 0$ in $S(t)$ for each $t \in [0, r)$, i.e., in $K(z, r)$. Consequently, u is constant in $K(z, r)$. Since $u = 0$ in $S(0) \subseteq K(z, r)$, it follows that $u = 0$ everywhere in $K(z, r)$, as desired. In particular, this implies, by the continuity of u,

$$u(z, r) = \lim_{t \to r^-} u(z, t) = \lim_{t \to r^-} 0 = 0. \tag{1890}$$

We now apply our result to show u is compactly supported. Let $T \in (0, \infty)$. Since ϕ and ψ are compactly supported, there exists $R > 0$ such that $\mathrm{spt}(u(\cdot, 0)) \subseteq B(0, R)$. Now choose $z \in \mathbb{R}^3 - B(0, R + T + 1)$. This implies $u = 0$ on $B(z, T) \times \{t = 0\}$, from which our result above with (1890) implies $u(z, T) = 0$. Since z was arbitrarily chosen in $\mathbb{R}^3 - B(0, R + T + 1)$, it follows that $\mathrm{spt}(u(\cdot, T)) \subseteq B(0, R + T + 1)$, i.e., $u(\cdot, T)$ is compactly supported. Since T was chosen arbitrarily, this holds for all times, and the result follows. $\square$

F04.7. Consider the PDE

$$uu_x + u_t + u = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{1891}$$

a) Find the particular solution that satisfies the condition $u(0,t) = e^{-2t}$.

b) Show that at the point $(z,t) = (1/9, \ln(2))$, $u = 1/3$.

Solution:

a) We proceed by using the method of characteristics. Define $F(p,q,\xi,z,t) = \xi p + q + \xi$. Taking $p = u_z$, $q = u_t$, and $\xi = u$ yields $F = 0$ and gives rise, via the method of characteristics, to the ODE system

$$\begin{cases} \dot{z}(s) = F_p = \xi(s), \quad z(0) = 0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = t_0, \\ \dot{\xi}(s) = F_p p + F_q q = \xi p + q = -\xi(s), \quad \xi(0) = e^{-2t_0}. \end{cases} \tag{1892}$$

This implies $t = t_0 + s$ and

$$\xi(s) = \xi(0)e^{-s} = e^{-2t_0 - s}. \tag{1893}$$

Thus,

$$z(s) = z(0) + \int_0^s \dot{z}(\tau)\ \mathrm{d}\tau = 0 + \int_0^s e^{-2t_0-\tau}\ \mathrm{d}\tau = e^{-2t_0}\left[1 - e^{-s}\right] = \xi(s)\left[e^s - 1\right], \tag{1894}$$

and so

$$e^s = \frac{z}{\xi} + 1 = \frac{z+\xi}{\xi}, \tag{1895}$$

where the division is well-defined since $\xi(s) \neq 0$ for all s. Whence we obtain the quadratic equation

$$\xi = e^{-2(t_0+s)+s} = e^{-2t}e^s = e^{-2t}\cdot\frac{z+\xi}{\xi} \quad\Longrightarrow\quad \xi^2 - e^{-2t}\xi - e^{-2t}z = 0. \tag{1896}$$

Using the quadratic formula, noting $\xi > 0$, we deduce

$$\xi = \frac{e^{-2t} + \sqrt{e^{-4t} + 4ze^{-2t}}}{2} = \frac{e^{-2t}}{2}\left[1 + \sqrt{1 + 4ze^{2t}}\right]. \tag{1897}$$

Therefore, we conclude

$$u(x,t) = \frac{e^{-2t}}{2}\left[1 + \sqrt{1 + 4ze^{2t}}\right]. \tag{1898}$$

b) Directly plugging $(1/9, \ln(2))$ into our expression for u reveals

$$u(1/9, \ln(2)) = \frac{e^{-2\ln(2)}}{2}\left[1 + \sqrt{1 + 4\left(\frac{1}{9}\right)e^{2\ln(2)}}\right] = \frac{1}{8}\left[1 + \sqrt{1 + \frac{16}{9}}\right] = \frac{1}{8}\left[1 + \frac{5}{3}\right] = \frac{1}{3}, \quad (1899)$$

as desired.

□

2003 Fall

F03.1. Consider the ODE

$$\dot{u} = v - u^3, \quad \dot{v} = u - v. \tag{1900}$$

a) Find all the stationary points and their type.

b) Draw the phase plane and find all connections between the stationary points.

Solution:

a) The three fixed points of this system are given by $(u, v) = (0, 0)$ and $(u, v) = (1, 1)$ and $(u, v) = (-1, -1)$. The Jacobian matrix for this system is given by

$$J(u, v) = \begin{pmatrix} \partial\dot{u}/\partial u & \partial\dot{u}/\partial v \\ \partial\dot{v}/\partial u & \partial\dot{v}/\partial v \end{pmatrix} = \begin{pmatrix} -3u^2 & 1 \\ 1 & -1 \end{pmatrix}, \tag{1901}$$

which has eigenvalues λ that satisfy

$$0 = (\lambda+3u^2)(\lambda+1)-1 = \lambda^2+(3u^2+1)+(3u^2-1) \implies \lambda = \frac{-(3u^2+1) \pm \sqrt{(3u^2+1)^2 - 4(3u^2-1)}}{2}. \tag{1902}$$

At the origin, we obtain eigenvalues $\lambda = (-1 \pm \sqrt{5})/2$, and so the origin obtain forms a saddle. At $(\pm 1, \pm 1)$ we see

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(3-1)}}{2} = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}, \tag{1903}$$

and so $(\pm 1, \pm 1)$ form stable nodes.

b) The null-cline for $\dot{u} = 0$ is $v = u^3$ and for $\dot{v} = 0$ it is $u = v$. A phase plane is given below.

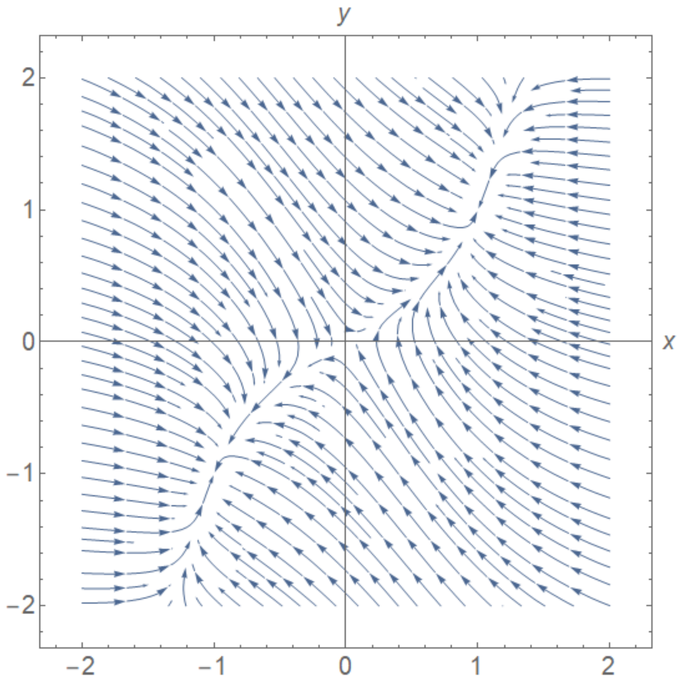

Figure 30: Phase plane for F03.1.

□

F03.3 The function $h(x,t)$ defined by

$$h(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \tag{1904}$$

satisfies the heat equation $h_t + h_{xx} = 0$. Using this result, verify for any smooth function ϕ

$$u(x,t) = \exp\left(\frac{t^3}{3} - xt\right) \int_{-\infty}^{\infty} \phi(\xi) h(x - t^2 - \xi, t) \ \mathrm{d}\xi \tag{1905}$$

satisfies $u_t + xu - u_{xx} = 0$. Given that f is bounded and continuous everywhere in $\mathbb{R}$, establish that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} f(\xi) h(x-\xi, t) \ \mathrm{d}\xi = f(x), \tag{1906}$$

and then show that $u(x,t) \to \phi(x)$ as $t \to 0$.

Solution:

The first part with the derivatives is long and tedious, and so we omit the details here. Let $\varepsilon > 0$ be given and fix $x \in \mathbb{R}$. We must show there is $T > 0$ such that $t \in (0,T)$ implies

$$\left| f(x) - \int_{-\infty}^{\infty} f(\xi) h(x-\xi, t) \ \mathrm{d}\xi \right| < \varepsilon. \tag{1907}$$

□

2002 Winter

W02.5. Consider the boundary value problem

$$\begin{cases} \Delta u + \sum_{k=1}^{n} \alpha_k u_{x_k} - u^3 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1908}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. If the α_k's are constants, and $u(x)$ has continuous derivatives up to second order, prove that u must vanish identically.

Solution:

Set $v := (\alpha_1, \ldots, \alpha_n)$ so that

$$\Delta u + v \cdot Du - u^3 = 0 \ \text{ in } \Omega. \tag{1909}$$

Then

$$\begin{aligned} 0 &= \int_\Omega u\left(\Delta u + v \cdot Du - u^3\right)\ \mathrm{d}x \\ &= \int_\Omega -|Du|^2 - 0 - u^4\ \mathrm{d}x + \int_{\partial\Omega} u\frac{\partial u}{\partial n} + u^2 v \cdot n\ \mathrm{d}\sigma \\ &= \int_\Omega -|Du|^2 - u^4\ \mathrm{d}x, \end{aligned} \tag{1910}$$

where n is the outward normal along $\partial\Omega$ and we note $Dv = 0$. This implies

$$0 \leq \int_\Omega |Du|^2\ \mathrm{d}x = -\int_\Omega u^4\ \mathrm{d}x \leq 0 \quad \Longrightarrow \quad Du = 0 \quad \text{in } \Omega, \tag{1911}$$

and so u is constant in Ω. Combined with the fact $u = 0$ on $\partial\Omega$, it follows that u is identically zero.

□

W02.6. Solve the Cauchy problem

$$\begin{cases} u_t + u^2 u_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = 2 + x & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1912}$$

Solution:

We proceed by using the method of characteristics. Set $F(p,q,z,x,t) = q + z^2 p$. Then taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{x}(s) = F_p = z^2, \quad x(0) = x_0, \\ \dot{t}(s) = F_q = 1, \quad t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = z^2 p + q = 0, \quad z(0) = 2 + x_0. \end{cases} \tag{1913}$$

This implies $t = s$ and z is constant along characteristics. Thus,

$$x(s) = x_0 + \int_0^s \dot{x}(\tau)\ \mathrm{d}\tau = x_0 + \int_0^s z^2(\tau)\ \mathrm{d}\tau = x_0 + sz^2(0) = x_0 + s(2+x_0)^2. \tag{1914}$$

Expanding this quadratic expression reveals

$$x = s(4 + 4x_0 + x_0^2) + x_0 \quad \Longrightarrow \quad sx_0^2 + (4s+1)x_0 + (4s - x) = 0. \tag{1915}$$

Using the quadratic equation, we then obtain

$$x_0 = \frac{-(4s+1) + \sqrt{(4s+1)^2 - 4s(4s-x)}}{2s} = -2 + \frac{-1 + \sqrt{8s + 1 + 4sx}}{2s}. \tag{1916}$$

Therefore,

$$u(x,t) = z(t) = 2 + x_0 = \frac{-1 + \sqrt{8t + 1 + 4tx}}{2t}, \quad \text{for all } (x,t) \in \mathbb{R}\times(0,\infty) \text{ s.t. } 2 + 1/4t + x \geq 0. \tag{1917}$$

□

2001 Fall

F01.2. Consider the differential operator

$$Lu := \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + \alpha(x)u, \tag{1918}$$

where α is a real-valued function. The domain is $x \in [0,1]$, with Neumann boundary conditions $u'(0) = u'(1) = 0$.

a) Find a function ϕ for which L is self-adjoint in the norm

$$\|u\|^2 = \int_0^1 u^2\phi \ \mathrm{d}x. \tag{1919}$$

b) Show L must have a positive eigenvalue if α is not identically zero and

$$\int_0^1 \alpha(x) \ \mathrm{d}x \geq 0. \tag{1920}$$

Solution:

a) In order for L to be self-adjoint in the given norm we need

$$\langle u, Lv\rangle = \langle Lu, v\rangle \tag{1921}$$

to hold for all appropriate functions u and v, where the scalar product is defined by

$$\langle u, v\rangle := \int_0^1 uv\phi \ \mathrm{d}x. \tag{1922}$$

Observe

$$\begin{aligned}\langle u, Lv\rangle &= \int_0^1 u(v'' + 2v' + \alpha v)\phi \text{ d}x \\ &= \int_0^1 -(u\phi)'v' + 2uv'\phi + \alpha uv\phi \text{ d}x + \underbrace{\left[uv'\phi\right]_0^1}_{=0} \\ &= \int_0^1 -u'\phi v' + uv'[2\phi - \phi'] + \alpha uv\phi \text{ d}x \\ &= \int_0^1 (u'\phi)'v + uv'[2\phi - \phi'] + \alpha uv\phi \text{ d}x \\ &= \int_0^1 [u'' + 2u' + \alpha u]v\phi \text{d}x + \int_0^1 [uv' - u'v][2\phi - \phi'] \text{ d}x \\ &= \langle Lu, v\rangle + \int_0^1 [uv' - u'v][2\phi - \phi'] \text{ d}x.\end{aligned} \tag{1923}$$

Taking $\boxed{\phi := \exp(2x)}$ yields $\phi' = 2\phi$ so that the second term on the right hand side above vanishes and we obtain (1922), as desired.

b) The integral inequality (1920) suggests we divide by u, which we presume is well-defined. For a nonzero eigenfunction u with eigenvalue λ we have

$$u'' + 2u' + \alpha u = Lu = \lambda u, \tag{1924}$$

and so

$$\begin{aligned}\lambda = \int_0^1 \lambda \text{ d}x &= \int_0^1 \frac{u''}{u} + 2\frac{u'}{u} + \alpha \text{ d}x \\ &= \int_0^1 -\left(\frac{u'}{u}\right)^2 + 2\frac{u'}{u} + \alpha \text{ d}x + \underbrace{\left[\frac{u'}{u}\right]_0^1}_{=0} \\ &\geq \int_0^1 2\frac{u'}{u} - \left(\frac{u'}{u}\right)^2 \text{ d}x.\end{aligned} \tag{1925}$$

(Return and complete.)

□

2001 Spring

S01.3. Solve the initial value problem

$$\begin{cases} u_t - \frac{1}{2}\left(u_x^2 + x^2\right) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = x & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1926}$$

You will find that the solution blows up in finite time. Explain this in terms of the characteristics for this equation.

Solution:

We proceed by using the method of characteristics. Define $F(p,q,z,x,t) = q - \frac{p^2}{2} - \frac{x^2}{2}$. Then taking $p = u_x$, $q = u_t$, and $z = u$ yields $F = 0$ and gives rise to the ODE system

$$\begin{cases} \dot{p}(s) = -F_x - F_z p = x(s), & p(0) = 1, \\ \dot{q}(s) = -F_t - F_z q = 0, & q(0) = \frac{1}{2}(1 + x_0^2), \\ \dot{x}(s) = F_p = -p(s), & x(0) = x_0, \\ \dot{t}(s) = F_q = 1, & t(0) = 0, \\ \dot{z}(s) = F_p p + F_q q = -p^2(s) + q(s), & z(0) = x_0. \end{cases} \tag{1927}$$

This implies $t = s$ and

$$\ddot{x} = -\dot{p} = -x \quad \implies \quad \ddot{x} + x = 0 \quad \implies \quad x = c_1\cos(t) + c_2\sin(t). \tag{1928}$$

The initial condition $x(0) = x_0$ implies $c_1 = x_0$. The condition $p(0) = 1$ reveals

$$-1 = -p(0) = \dot{x}(0) = -x_0\sin(0) + c_2\cos(0) = c_2 \quad \implies \quad c_2 = -1. \tag{1929}$$

Thus,

$$x = x_0\cos(t) - \sin(t) \quad \text{and} \quad p = -\dot{x} = x_0\sin(t) + \cos(t), \tag{1930}$$

which implies

$$p = \cos(t) + \left(\frac{x + \sin(t)}{\cos(t)}\right)\sin(t) = \cos(t) + \sin^2(t) + x\tan(t). \tag{1931}$$

Thus, for $x \neq 0$,

$$\lim_{t \to (\pi/2)^-} |u_x(x,t)| = \lim_{t \to (\pi/2)^-} \left| \underbrace{\cos(t)}_{\to 0} + \underbrace{\sin^2(t)}_{\to 1} + \underbrace{x \tan(t)}_{\to \pm\infty} \right| = +\infty. \tag{1932}$$

This reveals $|u_x| \to +\infty$ and, thus, $|u| \to +\infty$ by the time $t = \pi/2$.

$\square$

2000 Fall

F00.1. Consider the Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \Delta u + a(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1933}$$

a) Assuming $|a(x)|$ is small enough, prove the uniqueness of the classical solution.

b) Prove the existence of the solution in the Sobolev space $H^1(\Omega)$, assuming $f \in L^2(\Omega)$.

Solution:

a) Let u and v be two classical solutions to the PDE. Setting $w := u - v$, it suffices to show w is identically zero. Note

$$\begin{cases} \Delta w + aw = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1934}$$

Since $w = 0$ on $\partial\Omega$, Poincaré's inequality asserts there exists $C > 0$, dependent only on D, such that

$$\int_\Omega w^2 \ \mathrm{d}x \leq C \int_\Omega |Dw|^2 \ \mathrm{d}x. \tag{1935}$$

However,

$$\int_\Omega aw^2 \ \mathrm{d}x = \int_\Omega -w\Delta w \ \mathrm{d}x = \int_\Omega |Dw|^2 \ \mathrm{d}x - \underbrace{\int_{\partial\Omega} w \frac{\partial w}{\partial n} \ \mathrm{d}\sigma}_{=0} = \int_\Omega |Dw|^2 \ \mathrm{d}x, \tag{1936}$$

and so

$$\|a\|_\infty \int_\Omega w^2 \ \mathrm{d}x \geq \int_\Omega aw^2 \ \mathrm{d}x \geq \frac{1}{C} \int_\Omega w^2 \ \mathrm{d}x \quad \Longrightarrow \quad \left(\|a\|_\infty - \frac{1}{C} \right) \int_\Omega w^2 \ \mathrm{d}x \geq 0. \tag{1937}$$

If $\|a\|_\infty < 1/C$, then it necessarily follows that

$$\int_\Omega w^2 \ \mathrm{d}x = 0, \tag{1938}$$

which implies $w = 0$ in Ω, as desired.

b) We proceed by application of the Lax-Milgram theorem and assume $\|a\|_\infty < 1/C$. Define $H := H^1_0(\Omega)$ and the bilinear form $B : H \times H \to \mathbb{R}$ and the linear form $\ell : H \to \mathbb{R}$ via

$$B[u,v] := \int_\Omega Du \cdot Dv - auv \ \mathrm{d}x \quad \text{and} \quad \ell(v) := \int_\Omega -fv \ \mathrm{d}x. \tag{1939}$$

We claim B is bounded and coercive and ℓ is bounded, from which the Lax-Milgram theorem asserts there exists a unique $\tilde{u} \in H$ such that

$$B[\tilde{u}, v] = \ell(v), \quad \text{for all } v \in H, \tag{1940}$$

i.e., $\tilde{u}$ is the unique weak solution of the PDE.

All that remains is to verify the assumptions of the Lax-Milgram theorem hold. Observe

$$|\ell(v)| \leq \|fv\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_H \tag{1941}$$

and

$$\begin{aligned} |B[u,v]| &\leq \|Du \cdot Dv\|_{L^1(\Omega)} + \|a\|_{L^\infty(\Omega)}\|uv\|_{L^1(\Omega)} \\ &\leq \|Du\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + \|a\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq \left(1 + \|a\|_{L^\infty(\Omega)}\right)\|u\|_H\|v\|_H, \end{aligned} \tag{1942}$$

which shows ℓ and B are bounded. By our hypothesis, there exists $\beta \in (0,1)$ such that $\|a\|_\infty < \beta/C$. This implies B is coercive since

$$\begin{aligned} B[u,u] &= \int_\Omega |Du|^2 + au^2 \ \mathrm{d}x \\ &\geq \|Du\|^2_{L^2(\Omega)} - \|a\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} \\ &\geq (1-\beta)\|Du\|^2_{L^2(\Omega)} + \left(\frac{\beta}{C} - \|a\|_{L^\infty(\Omega)}\right)\|u\|^2_{L^2(\Omega)} \\ &\geq \min\left\{1-\beta, \frac{\beta}{C} - \|a\|_{L^\infty(\Omega)}\right\}\left(\|Du\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right) \\ &= \min\left\{1-\beta, \frac{\beta}{C} - \|a\|_{L^\infty(\Omega)}\right\}\|u\|^2_H, \end{aligned} \tag{1943}$$

and the proof is complete. □

F00.2. Consider the Cauchy problem

$$\begin{cases} u_t - \Delta u + u^2 = f & \text{in } \mathbb{R}^n \times (0,T), \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{1944}$$

Prove the uniqueness of the classical bounded solution, assuming T is small enough.

Solution:

We proceed by applying Banach's fixed point theorem. Fix $M > 0$ and set

$$V := \left\{ v \in C^{2,1}(\mathbb{R}^n \times [0,T)) : \|v\| \leq M \right\}, \tag{1945}$$

where $\|\cdot\|$ denotes the sup norm. Since V is a closed subset of a complete space, V forms a Banach space. Define

$$\phi(v) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \left(f(y,s) - v(y,s)^2 \right) \, \mathrm{d}y \mathrm{d}s, \tag{1946}$$

where Φ is the fundamental solution of the heat equation, i.e.,

$$\Phi(x,t) := (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t} \right), \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1947}$$

We claim ϕ forms a contraction for T sufficiently small. Because V is complete and ϕ is a contraction for T sufficiently small, the Banach fixed point theorem asserts, for T sufficiently small, there exists a unique fixed point u of ϕ. However, this fixed point u satisfies the implicit equation

$$u(x,t) = (\phi \circ u)(x,t)) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \left[f(y,s) - u^2(y,s) \right] \, \mathrm{d}y \mathrm{d}s, \tag{1948}$$

from which Duhamel's Principle asserts u forms a solution to the inhomogeneous PDE (1944). The uniqueness and existence of the fixed point u of ϕ (for T sufficiently small) establishes the result.

All that remains is to verify ϕ is a contraction. Observe, for all $v_1, v_2 \in V$,

$$\begin{aligned}
\|\phi(v_1) - \phi(v_2)\| &= \sup_{(x,t)} \left| \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \left[v_2(y,s)^2 - v_1^2(y,s) \right] \ \mathrm{d}y\mathrm{d}s \right| \\
&\leq \|v_2^2 - v_1^2\| \cdot \sup_{(x,t)} \int_0^t \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t-s) \ \mathrm{d}y}_{=1} \mathrm{d}s \\
&= \|v_2^2 - v_1^2\| \cdot \sup_{(x,t)} \int_0^t \ \mathrm{d}s \\
&= T\|v_2^2 - v_1^2\| \\
&\leq 2MT\|v_2 - v_1\|.
\end{aligned} \tag{1949}$$

Fixing $T < 1/2M$, we see ϕ forms a contraction on V, and the proof is complete. □

F00.5. Consider the eigenvalue problem in the interval $[0,1]$,

$$-y''(t)+p(t)y(t)=\lambda y(t), \quad y(0)=y(1)=0. \tag{1950}$$

a) Prove all eigenvalues λ are simple.

b) Prove there is at most a finite number of negative eigenvalues.

Solution:

a) First observe this problem may be rewritten in regular Sturm-Liouville form

$$\begin{cases} [-1y']'+py=\lambda y \\ 1y(0)+0y'(0)=0 \\ 1y(1)+0y'(1)=0. \end{cases} \tag{1951}$$

Define the differential operator L by $Ly:=-y''+py$. Suppose y_1 and y_2 are eigenfunctions with a common eigenvalue λ. Then

$$\begin{aligned} 0 &= \lambda(y_1y_2-y_1y_2) \\ &= (Ly_1)y_2-y_1(Ly_2) \\ &= [-y_1''+py_1]y_2-y_1[-y_2''+py_2] \\ &= y_1y_2''-y_1''y_2 \\ &= (y_1y_2')'-(y_1'y_2)' \\ &= \left(y_1y_2'-y_1'y_2\right)'. \end{aligned} \tag{1952}$$

This shows $y_1y_2'-y_1'y_2$ is constant. But, from our boundary conditions, we see

$$y_1(0)y_2'(0)-y_1'(0)y_2(0)=0y_2'(0)-y_1'(0)0=0, \tag{1953}$$

which implies the Wronskian satisfies

$$W(y_1,y_2)(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}=y_1(x)y_2'(x)-y_1'(x)y_2(x)=0. \tag{1954}$$

This shows y_1 and y_2 are linearly dependent, from which it follows that each eigenvalue λ corresponds to a single linearly independent solution of the ODE, as desired.

b) See S09.2b.

□

F00.6. Consider the initial boundary value problem

$$\begin{cases} u_t - u_{xx} + au = 0 & \text{in } \{t > 0\} \times \{x > 0\}, \\ u = 0 & \text{on } \{x > 0\} \times \{t = 0\}, \\ u = g & \text{on } \{x = 0\} \times \{t > 0\}, \end{cases} \tag{1955}$$

where $g(t)$ is a continuous function with compact support and a is a constant. Find the explicit solution of this problem.

Solution:

See F18.05. □

1999 Fall

F99.1. Suppose $\Delta u = 0$ in the weak sense in $\mathbb{R}^n$ and there is a constant C such that

$$\int_{\{|x-y|<1\}} |u(y)| \, \mathrm{d}y < C, \quad \text{for all } x \in \mathbb{R}^n. \tag{1956}$$

Show u is constant.

Solution:

Let Ω be a compact subset of $\mathbb{R}^n$. Then the collection $\{B(x,1)\}_{x\in\mathbb{R}^n}$ forms an open cover of Ω. By compactness, there exists a finite subcover, i.e., there are $\{x^i\}_{i=1}^m$ such that

$$\Omega \subseteq \bigcup_{i=1}^m B(x_i, 1). \tag{1957}$$

By our hypothesis, it follows that

$$\int_\Omega |u(y)| \, \mathrm{d}y \le \sum_{i=1}^m \int_{B(x_i,1)} |u(y)| \, \mathrm{d}y \le \sum_{i=1}^m C = Cm < \infty. \tag{1958}$$

Because Ω is an arbitrary compact subset of $\mathbb{R}^n$, it follows that $u \in L^1_{loc}(\mathbb{R}^n)$. Together with the fact u is a weak solution of Laplace's equation, the assumptions of Weyl's lemma are satisfied. Whence, up to redefinition on a set of measure zero, $u \in C^\infty(\Omega)$ is smooth and satisfies $\Delta u = 0$ pointwise in Ω.

Fix $x \in \mathbb{R}^n$ and set[53]

$$\phi(r) := \fint_{\partial B(x,r)} u(y) \, \mathrm{d}\sigma(y) = \fint_{\partial B(0,1)} u(x+rz) \, \mathrm{d}\sigma(z). \tag{1959}$$

Then

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz) \cdot z \, \mathrm{d}\sigma(z) = \fint_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} \, \mathrm{d}\sigma(y) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial n} \, \mathrm{d}\sigma(y). \tag{1960}$$

Upon integrating by parts, we then see

$$\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y) \, \mathrm{d}y = 0. \tag{1961}$$

[53] This follows the approach in the proof of Theorem 2 on page 25 of Evans' PDE text.

This implies ϕ is constant, and so

$$\phi(r) = \lim_{t\to 0^+} \phi(t) = \lim_{t\to 0^+} \fint_{\partial B(x,t)} u(y)\ \mathrm{d}\sigma(y) = u(x). \tag{1962}$$

In particular,

$$|u(x)| = \left| \fint_{B(x,1)} u(y)\ \mathrm{d}y \right| \leq \alpha(n) \int_{B(x,1)} |u|\ \mathrm{d}y \leq C\alpha(n), \tag{1963}$$

where $\alpha(n)$ is the measure of the unit ball in $\mathbb{R}^n$. Because $x \in \mathbb{R}^n$ was arbitrarily chosen, it follows that u is bounded. Thus, Liouville's theorem[54] asserts u is constant. $\square$

[54] See Theorem 8 on page 30 of Evans' PDE text.

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